The Karenina Principle: A Decision Rule for Contradicting Requirements

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We propose the Generalized Karenina Principle as a decision rationale for allocating a finite resource among essential requirements. Beyond competing for resources, requirements may be contradictory: gratifying one can directly impair another. Maximizing a multiplicative measure of success yields an adaptive rule that restores the optimal balance among gratification levels. In the two-requirement model, interior Karenina-optimal allocation remains locally stable as contradiction increases, whereas need-responsive allocation can generate destabilizing overcompensation. With interacting decision makers, Karenina-optimal reallocation also raises the threshold for collective instability. The principle bases adaptive allocation on restoring optimal gratification ratios rather than responding to individual deficits.
JEL: D01, D91, C61, C62

One Sentence Summary

The Generalized Karenina Principle guides the allocation of a finite resource by restoring the optimal balance among essential requirements that compete for resources and may directly impair one another.

I. Introduction

Leo Tolstoy's observation that "Happy families are all alike; every unhappy family is unhappy in its own way" suggests that success requires several conditions to be satisfied simultaneously, whereas failure can arise from a deficiency in any one of them (Tolstoy, 1878). Diamond termed this logic the Anna Karenina principle and applied it to animal domestication and societal collapse, emphasizing how societies respond to interacting challenges (Diamond, 1997, 2005). This paper studies the corresponding allocation problem: how should a decision maker distribute scarce resources among essential requirements when an action that gratifies one can directly impair another?

Examples include households maintaining work, family life, and health, and firms balancing current operations, innovation, and workforce satisfaction. These requirements compete for resources, but their interaction can extend beyond ordinary opportunity costs. An additional hour of work leaves less time for family life; fatigue or stress can also reduce the gratification obtained from the remaining family time. We distinguish the resource constraint from these direct cross-effects and ask how repeated allocation decisions affect stability.

The framework treats gratification levels as stocks maintained by resource allocation and reduced by losses. A leverage matrix maps each allocation into its effects on all requirements. This defines the dynamics independently of how overall success is evaluated. We then introduce a multiplicative Karenina function and derive an adaptive decision rule by maximizing its next-period value. The central comparison is between responding to current need and restoring an optimal balance among gratifications while accounting for the cross-effects of allocation.

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A. *Multiple requirements and essentiality*

Conventional utility theory already accommodates multidimensional consequences within a scalar preference representation (Debreu, 1954). Multiattribute utility theory identifies conditions for utility independence and additive or multiplicative representations (Keeney, 1971, 1974; Fishburn, 1974*b*; Keeney and Raiffa, 1976). Our contribution concerns the allocation dynamics that produce these consequences: the decision maker chooses resource shares, while the leverage structure and gratification losses determine the attainable state.

The Karenina function, $\kappa = \prod_i g_i^{\gamma_i}$ with $\gamma_i > 0$, represents essentiality rather than strictly non-compensatory choice. It permits trade-offs at positive gratification levels, but approaches zero when any essential gratification approaches zero while the others remain fixed and finite. This differs from threshold screening, lexicographic choice, and elimination procedures (Einhorn, 1970; Fishburn, 1974*a*; Tversky, 1972; Gilbride and Allenby, 2004). It also differs from Arrow's problem of aggregating individual preference orderings into a social ordering (Arrow, 1951). The model requires neither a rejection of ordinal choice theory nor a new social aggregation theorem; it gives the individual components of success explicit state dynamics.

Product objectives themselves have familiar precedents in Cobb–Douglas production, Nash bargaining, and multiplicative wealth growth (Cobb and Douglas, 1928; Nash, 1950; Kelly, 1956). Kremer's O-ring theory is especially close in emphasizing complementary tasks whose failures reduce overall output (Kremer, 1993). Here the additional question is how an agent repeatedly maintains essential gratifications when the activities sustaining them can interfere. Complementarity in the objective can therefore coexist with inhibition in the allocation technology.

B. *Scarcity and interacting activities*

Becker's theory of time allocation establishes the importance of finite time alongside expenditure constraints (Becker, 1965). Evidence on temporal flexibility, preferences over work arrangements, and nonlinear productivity effects of working hours further shows why the consequences of time allocation depend on how activities are organized (Goldin, 2014; Mas and Pallais, 2017; Pencavel, 2015). These findings motivate distinguishing resource scarcity from the effects that one activity has on other outcomes.

Several economic literatures make such interdependence explicit. Markowitz evaluates assets jointly through return covariances (Markowitz, 1952); our leverage coefficients instead represent marginal effects of resource allocation on gratification. Multitask principal–agent theory shows how incentives on one task redistribute effort across activities (Holmstrom and Milgrom, 1991). March examines competing demands for exploration and exploitation (March, 1991), while Milgrom and Roberts analyze complementarities among organizational choices (Milgrom and Roberts, 1995). Our framework represents facilitating and inhibiting effects within a single matrix, $\boldsymbol{\alpha}$: its entry α_{ij} measures the effect of allocating resource to activity j on gratification i.

The constraint $\sum_j \tau_j = 1$ captures competition for the resource; a negative off-diagonal coefficient captures direct inhibition in addition to that competition. When allocation responds to current gratification, these cross-effects create feedback. An attempt to correct one deficit can worsen another, which then attracts resources in the following decision period. The question is whether that repeated adjustment converges, even when the environment remains unchanged.

C. *Behavioral motivation and goal conflict*

Bounded rationality motivates decision procedures that economize on information and computation (Simon, 1955). Sparse decision models formalize selective attention to consequences (Gabaix, 2014); experiments show that scarcity can concentrate attention on pressing deficits and that financial concerns can consume cognitive resources (Shah, Mullainathan and Shafir, 2012; Mani et al., 2013). Evidence and computational models of multiple-goal pursuit likewise connect discrepancies between goals and performance to subsequent effort allocation (Schmidt and DeShon, 2007; Vancouver, Weinhardt and Schmidt, 2010; Neal, Ballard and Vancouver, 2017). These findings motivate our need-responsive rule, which directs resources toward current deficiencies without evaluating their complete cross-effects. Karenina optimization also operates one period at a time, but incorporates the allocation technology when choosing the next action.

Psychological evidence supports distinguishing scarcity from direct conflict. Emmons and King associate conflicting personal strivings with poorer well-being (Emmons and King, 1988); Riediger and Freund distinguish competition for limited resources from incompatible attainment strategies and also document facilitation between goals (Riediger and Freund, 2004). These distinctions correspond to the resource constraint and the signs of the leverage coefficients. The framework adds an explicit dynamical account of how responses to goal imbalance change subsequent gratification.

Work-related burnout provides a possible application, rather than an interpretation established by the model. Research links exhaustion and disengagement to chronic stressors, job demands, insufficient resources, and work–nonwork conflict (Maslach, Schaufeli and Leiter, 2001; Demerouti et al., 2001; Reichl, Leiter and Spinath, 2014). Our mechanism concerns the temporal pattern of allocation: responding strongly to a deficit may produce alternating over-correction when activities inhibit one another. Time-use observations could distinguish this process from a stable low-gratification state with similar average resource use. Establishing its relationship to burnout requires empirical evidence beyond the mathematical instability result.

D. Dynamic allocation and stability

The distinction between a plausible individual decision and its repeated consequences also appears in dynamic choice. Strotz analyzes dynamically inconsistent intertemporal plans (Strotz, 1955); our instability instead arises with fixed preferences from repeated state-responsive allocation. Grandmont demonstrates endogenous economic cycles, and Brock and Hommes show how adaptive predictor choice can generate complex dynamics (Grandmont, 1985; Brock and Hommes, 1997). Behavioral dynamic programming considers decision makers who attend only partially to states and consequences (Gabaix, 2023). These precedents motivate studying allocation rules themselves as sources of endogenous dynamics.

Here the first instability requires neither changing expectations nor nonlinear gratification gains or losses. Linear gains and proportional losses suffice: the interaction between the need-responsive rule and inhibiting cross-effects destabilizes the stationary state. Applying a different adaptive rule to the same technology can remove this mechanism. The comparison therefore identifies how the decision rationale affects stability, rather than attributing instability to contradictory requirements alone.

E. Contribution and organization

The paper's central analytical result characterizes interior Karenina-optimal allocation. For a nonsingular leverage matrix, maximizing next-period success fixes all pairwise gratification ratios at values determined by the leverage matrix and Karenina elasticities. Each interior decision places gratification on this optimal ray in one period. Under common proportional loss and maintained interiority, subsequent allocation becomes constant and coincides with the allocation maximizing stationary success. Heterogeneous loss rates generally break this equivalence between myopic and stationary optimization.

The stability analysis distinguishes three strategies. Fixed allocation provides a stable but non-adaptive benchmark for an isolated decision maker under the stated loss-rate restrictions. Need-responsive allocation is adaptive but can lose stability through an alternating local instability as contradiction increases. Interior Karenina-optimal allocation restores the optimal ratios and excludes this leverage-induced flip mechanism in the two-requirement setting. The supplemental analysis explores a quadratic map's period-doubling dynamics, while distinguishing its mathematical continuation from trajectories with positive gratifications. Interaction between decision makers introduces a further source of instability: even individually stable fixed allocations can become collectively unstable, while Karenina-optimal reallocation raises the critical coupling threshold in the setting studied.

The resulting empirical distinction concerns responses to disturbances. Fixed allocation leaves resource shares unchanged; need-responsive allocation follows current deficits; Karenina-optimal allocation restores the leverage-adjusted gratification ratios whenever feasible. Observations of households, firms, or national economies after shocks could discriminate among these mechanisms, provided their gratification measures and allocation technology can be identified. Such tests are beyond this paper's scope.

Section II states the Generalized Karenina Principle, and Section III defines the dynamical environment. Sections IV–VII derive the allocation rules and compare their dynamics. Section VIII examines interacting decision makers; the conclusion discusses empirical implications and limitations.

II. The Generalized Karenina Principle

In his opening sentence of Anna Karenina Tolstoy suggests a particular structure of human well-being: a successful outcome requires several distinct conditions to be satisfied simultaneously, whereas failure in any one of them can substantially impair the whole. In his later non-fictional book "What I Believe" he gives a concrete list of requirements that are necessary for a person to be happy: (i) contact with nature, (ii) physical labour, (iii) family, (iv) free and friendly intercourse with other people, and (v) bodily health (Tolstoy, 1884). The idea that human flourishing depends on more than a single dimension has much older antecedents. In the Nicomachean Ethics, Aristotle argues that happiness (eudaimonia) cannot be identified with any single good and that flourishing also requires a sufficient provision of external goods (Aristotle, 1999). Here we focus on the decision problem that arises when essential conditions can be maintained through scarce resource allocation.

DEFINITION 1 (Generalized Karenina decision problem): *A decision problem is subject to the Generalized Karenina Principle if achieving its objective requires the simultaneous gratification of several distinct requirements, while the resources available for their gratification are limited and the gratification of these requirements cannot in general be pursued independently.*

Essentiality means that failure of one requirement cannot be fully offset by success elsewhere, although trade-offs remain possible when all gratifications are positive. A multiplicative objective captures this property: it vanishes when any essential gratification vanishes while the others remain finite.

A firm provides an economic example. Its continued success may depend jointly on five requirements: (i) revenue, (ii) customer satisfaction, (iii) worker satisfaction, (iv) shareholder satisfaction, and (v) product advancement. A company that pursues current revenue at the expense of its customers, workers, or future products may undermine its continued success. These requirements compete for finite resources, and pursuing one can directly impair another. Their relative importance can be represented by the elasticities of the Karenina function; their ordering here does not indicate priority.

For family well-being, an illustrative set of five requirements comprises (i) mutual affection, (ii) health, (iii) financial security, (iv) children, and (v) social acceptance. The particular choice is illustrative rather than essential to the theory: financial security, for example, need not compensate for the absence of mutual affection or severe ill health. Medical treatment provides a particularly direct example of potentially contradictory requirements. Maintaining a healthy body depends on the functioning of several systems, including the (i) cardiovascular system, (ii) nervous system, (iii) respiratory system, (iv) digestive system, and (v) urinary system. A treatment intended to improve one system may adversely affect another. It therefore does not merely consume scarce resources but can directly impair another essential requirement. The same structure can also describe public policy, where livelihood, peace, freedom, and institutional integrity may require simultaneous attention.

Scarcity and contradiction are distinct. Scarcity means that allocating resources to one use leaves less for others. Contradiction means that the allocation itself has adverse cross-effects. The Karenina principle does not require such inhibition, but inhibition makes the choice of allocation rule especially consequential: correcting one deficit may create another.

PRINCIPLE (Generalized Karenina Principle): *When achieving an objective requires the simultaneous gratification of several essential requirements that compete for limited resources and cannot in general be pursued independently, resources should be allocated so as to maintain or restore the optimal balance among their gratifications, taking into account how resource allocation affects all requirements.*

The classical principle concerns conditions for success; the generalized principle prescribes how resources should respond to gratification imbalance. Section III formalizes the decision environment, and Section IV derives the corresponding allocation rule.

III. A Karenina-Principle-based decision theory for contradictory requirements

A. *General concept*

Let $g_{i,t} > 0$ denote the gratification of requirement $i = 1, \ldots, R$ at decision time t, and let $\tau_{i,t}$ denote the fraction of the available resource allocated to that requirement during the following decision period. Throughout the paper we use time as the example resource and therefore refer to $\tau_{i,t}$ as a time allocation. The theory, however, applies to any divisible and limited resource.

The total available resource is normalized to unity, such that

$$\sum_{i=1}^{R} \tau_{i,t} = 1, \qquad \tau_{i,t} \geq 0. \tag{1}$$

Allocation is chosen at discrete decision times and held fixed during each period. The resulting gratification determines the state for the next decision.

The framework developed in this section specifies how gratification responds to resource allocation and how the different requirements combine into overall gratification and need. It does not yet specify how the decision maker chooses the resource allocation. Alternative decision rationales are introduced in Section IV.

B. *The leverage matrix*

We assume a linear gain function,

$$G_i(\boldsymbol{\tau}_t) = \sum_{j=1}^{R} \alpha_{ij} \tau_{j,t}, \tag{2}$$

where $\boldsymbol{\alpha}$ is the leverage matrix. The element α_{ij} measures the effect on the gratification of requirement i of allocating one unit of the resource to requirement j. Thus, the column indexed by j describes the complete effect of allocating resources to requirement j on all gratification levels.

Diagonal coefficients describe own effects. Off-diagonal coefficients describe co-benefits when positive and direct inhibition when negative. The latter represents contradiction beyond the resource constraint in equation (1).

Gratification does not persist indefinitely without further resource allocation. We therefore introduce a loss function $L_i(\mathbf{g}_t, t)$, which may represent both intrinsic depreciation of gratification and external perturbations. The gratification dynamics is

$$g_{i,t+1} = g_{i,t} + \sum_{j=1}^{R} \alpha_{ij}\tau_{j,t} - L_i(\mathbf{g}_t, t), \qquad i = 1, \ldots, R. \tag{3}$$

The loss function may also include external disturbances, with negative contributions representing positive shocks. We first isolate allocation-induced dynamics and introduce interactions later.

We restrict the analysis to states with positive gratification,

$$g_{i,t} > 0 \qquad \forall\, i, t,$$

because both the Karenina function and the need measures introduced below are defined on this domain.

C. *The Karenina function and need*

The Karenina principle states that overall success requires several requirements to be gratified simultaneously. We represent this logical AND conjunction by the multiplicative Karenina function

$$\kappa_t \equiv \prod_{i=1}^{R} g_{i,t}^{\gamma_i}, \qquad \gamma_i > 0, \tag{4}$$

where γ_i is the requirement-specific Karenina elasticity. As discussed in Section I, the product represents essentiality while permitting trade-offs at positive gratification levels.

The need associated with requirement i is defined as the inverse of gratification raised to a potentially different elasticity,

$$n_{i,t} \equiv g_{i,t}^{-\beta_i}, \qquad \beta_i > 0, \tag{5}$$

and total need is the sum of the individual needs,

$$\nu_t \equiv \sum_{i=1}^{R} n_{i,t} = \sum_{i=1}^{R} g_{i,t}^{-\beta_i}. \tag{6}$$

The parameters β_i determine how strongly the perceived need associated with a requirement increases as its gratification decreases. They need not be identical to the Karenina elasticities γ_i. The two sets of elasticities describe different objects: γ_i determines how gratification of requirement i contributes to overall gratification, whereas β_i determines how strongly a low gratification level generates need.

D. *The loss function*

The general loss function $L_i(\mathbf{g}_t, t)$ in equation (3) may depend on the complete gratification vector and on time. It can therefore represent both endogenous depreciation and external disturbances.

For the main analytical results we use a proportional loss function,

$$L_i(\mathbf{g}_t, t) = \phi_i \cdot g_{i,t}, \qquad \phi_i > 0. \tag{7}$$

In this case the gratification dynamics becomes

$$g_{i,t+1} = (1 - \phi_i) g_{i,t} + \sum_{j=1}^{R} \alpha_{ij} \tau_{j,t}. \tag{8}$$

The proportional loss has a straightforward interpretation. A given level of gratification has to be maintained by continued resource allocation; without such allocation, a fraction ϕ_i of the current gratification is lost during each decision period. For $0 < \phi_i < 1$, gratification therefore decays gradually in the absence of compensating gains.

This specification also provides a useful analytical baseline. Under a fixed resource allocation, equation (8) generates a stable affine relaxation over the parameter range considered below, i.e. the problem is linear. Any nonlinearities and instabilities that arise after the resource allocation itself becomes endogenous can therefore be attributed to the decision rule rather than to nonlinearities in either the gain or the loss function.

For comparison, we will occasionally consider a time-independent loss,

$$L_i(\mathbf{g}_t, t) = \Phi_i,$$

which removes the intrinsic relaxation toward a stationary gratification level. Because the proportional-loss case is the main case investigated in the paper, the constant-loss limit will only be used where it provides additional analytical insight.

Equations (1), (3), and (4) define the basic decision environment. The remaining element required to close the model is the rule by which the decision maker chooses $\boldsymbol{\tau}_t$. We now compare three such resource-allocation strategies.

IV. Competing resource-allocation strategies

We compare fixed allocation, allocation proportional to current need, and myopic maximization of next-period Karenina gratification. These rules differ in their response to the current state and their use of the leverage matrix.

A. *Time-constant resource allocation*

The simplest strategy is a time-independent allocation,

$$\tau_{i,t} = \tau_i^*, \qquad i = 1, \ldots, R, \qquad \forall t \tag{9}$$

with

$$\sum_{i=1}^{R} \tau_i^* = 1, \qquad \tau_i^* \geq 0.$$

Fixed resource shares provide a benchmark without feedback from gratification to allocation. Examples include a firm's fixed staffing assignments or an individual's regular division of time across activities.

Not all time-constant allocations are equally desirable. For a given leverage matrix and loss function, one can ask which constant allocation maximizes the stationary Karenina function. We derive this *optimal time-constant resource allocation* in Appendix A.A1. Under proportional loss with a common loss rate, the resulting allocation coincides with the time-constant allocation reached by Karenina-optimal resource allocation once the optimal gratification ratios have been established. This will provide an important benchmark below.

B. Need-responsive resource allocation

A natural adaptive response to unequal gratification is to direct more resources towards requirements that are currently less well gratified. Need-responsive allocation has a direct analogue in the literature on multiple-goal pursuit, where goal-performance discrepancies have been found to guide the allocation of limited time and effort: greater resources are directed towards goals that are currently further from attainment (Schmidt and DeShon, 2007; Vancouver, Weinhardt and Schmidt, 2010). We represent this principle in continuous form by allocating resources in proportion to current need as introduced in equation (5)

$$\tau_{i,t}^{(n)} = \frac{n_{i,t}}{\sum_{j=1}^{R} n_{j,t}} = \frac{g_{i,t}^{-\beta_i}}{\sum_{j=1}^{R} g_{j,t}^{-\beta_j}}. \tag{10}$$

The need elasticities β_i determine how strongly the allocation reacts to low gratification. For equal $\beta_i = \beta$, increasing β makes the allocation increasingly sensitive to gratification differences. In the limit $\beta \to \infty$, essentially the entire resource is allocated to the currently least gratified requirement. For finite β, all requirements continue to receive resources, but with different urgency.

Importantly, this allocation rule also contains a natural marginal-value rule as a special case. From the Karenina function of equation (4) the marginal contribution of requirement i is

$$\frac{\partial \kappa}{\partial g_i} = \gamma_i \frac{\kappa}{g_i}. \tag{11}$$

Allocating resources in proportion to these marginal contributions gives the marginal Karenina allocation,

$$\tau_{i,t}^{(\nabla\kappa)} = \frac{\partial \kappa_t / \partial g_{i,t}}{\sum_{j=1}^{R} \partial \kappa_t / \partial g_{j,t}} = \frac{\gamma_i / g_{i,t}}{\sum_{j=1}^{R} \gamma_j / g_{j,t}}. \tag{12}$$

This rule directs resources towards the requirement for which an additional unit of gratification would generate the largest marginal increase in the Karenina function. In this sense, the decision maker follows the current gradient of the objective function in gratification space. For equal Karenina elasticities, $\gamma_i = \gamma$, the common elasticity cancels and the marginal Karenina allocation becomes

$$\tau_{i,t}^{(\nabla\kappa)}\Big|_{\gamma_i=\gamma} = \frac{g_{i,t}^{-1}}{\sum_{j=1}^{R} g_{j,t}^{-1}} = \tau_{i,t}^{(n)}\Big|_{\beta_i=1}. \tag{13}$$

Hence, for equal Karenina-, $\gamma_i = \gamma$, and need elasticities of one, $\beta_i = 1$, need-responsive allocation coincides exactly with marginal Karenina allocation. The canonical inverse-gratification rule can therefore be motivated in two independent ways: as a direct response to current need and as an allocation proportional to the marginal value of gratification under the Karenina objective.

Crucially, however, neither motivation accounts for how resources are transformed into gratification. Both rules use the current gratification vector to determine where resources should be directed, but they do not account for how the chosen allocation changes the gratification vector through the leverage matrix. If allocating resources to one requirement also affects the gratification of others, directing resources towards the greatest current need—or equivalently, in the special case above, following the Karenina gradient in gratification space—need not move the system in the intended direction.

C. Myopic Karenina-optimal resource allocation

Karenina-optimal allocation takes precisely this additional effect into account. The leverage matrix determines how a change in resource allocation changes the different gratification levels. Given the current gratification vector $\mathbf{g}_t$, the decision maker therefore chooses the allocation that maximizes the Karenina function one decision period ahead,

$$\boldsymbol{\tau}_t^{(\kappa)} = \arg\max_{\boldsymbol{\tau}_t} \kappa_{t+1}\left(\boldsymbol{\tau}_t, \mathbf{g}_t\right), \tag{14}$$

subject to

$$\mathbf{1}^\top \boldsymbol{\tau}_t = 1, \qquad \boldsymbol{\tau}_t \geq 0.$$

The strategy is myopic in the sense that it maximizes the Karenina function only one decision period ahead and does not optimize an intertemporal objective. Unlike need-responsive allocation, however, it accounts explicitly for the effect of the current allocation on next-period gratification.

The distinction from marginal Karenina allocation can be expressed by the chain rule,

$$\nabla_{\boldsymbol{\tau}} \kappa_{t+1} = \boldsymbol{\alpha}^\top \nabla_{\mathbf{g}} \kappa_{t+1}. \tag{15}$$

Marginal Karenina allocation responds to the gradient of the objective in gratification space. Karenina-optimal allocation instead responds to the gradient with respect to the actual decision variable, the resource allocation. The leverage matrix therefore translates the marginal value of gratification into the marginal value of resource use.

For an interior solution, maximizing κ_{t+1} is equivalent to maximizing $\ln \kappa_{t+1}$. The first-order condition is

$$\frac{\partial \ln \kappa_{t+1}}{\partial \tau_{j,t}} = \sum_{i=1}^{R} \frac{\gamma_i \alpha_{ij}}{g_{i,t+1}} = \lambda_t, \qquad j = 1, \ldots, R, \tag{16}$$

where λ_t is the Lagrange multiplier associated with the resource constraint. It may naturally vary in each decision cycle, since the optimization only occurs within one decision cycle.

The condition has a straightforward interpretation. At an interior optimum, all active uses of the resource must generate the same weighted marginal increase in the logarithm of the Karenina function. Otherwise resources could be shifted towards a use with a larger marginal return and the Karenina function would increase. In vector notation, the first-order condition is

$$\boldsymbol{\alpha}^\top \begin{pmatrix} \gamma_1 / g_{1,t+1} \\ \vdots \\ \gamma_R / g_{R,t+1} \end{pmatrix} = \lambda_t \mathbf{1}. \tag{17}$$

For a nonsingular leverage matrix, this condition determines all pairwise ratios of next-period gratification. Define

$$q_i \equiv \gamma_i \left[\left(\boldsymbol{\alpha}^\top \right)^{-1} \mathbf{1} \right]_i^{-1}, \qquad i = 1, \ldots, R. \tag{18}$$

Every interior optimum then satisfies

$$\frac{g_{i,t+1}}{g_{j,t+1}} = \frac{q_i}{q_j}. \tag{19}$$

Hence an interior Karenina-maximizing decision maps the gratification vector onto a one-dimensional ray whose direction depends on the Karenina elasticities and the leverage matrix, but not on the current gratification levels. There exists a scalar $\sigma_t > 0$ such that

$$\mathbf{g}_{t+1} = \sigma_t \mathbf{q}. \tag{20}$$

To obtain the corresponding resource allocation explicitly, define the pre-allocation gratification

$$\widetilde{\mathbf{g}}_t \equiv \mathbf{g}_t - L(\mathbf{g}_t, t), \tag{21}$$

that is, the gratification remaining after the loss during decision period t but before the newly allocated resources affect gratification.

The gratification dynamics can then be written as

$$\mathbf{g}_{t+1} = \widetilde{\mathbf{g}}_t + \boldsymbol{\alpha}\boldsymbol{\tau}_t = \sigma_t \mathbf{q}. \tag{22}$$

For a nonsingular leverage matrix,

$$\boldsymbol{\tau}_t^{(\kappa)} = \boldsymbol{\alpha}^{-1}\left(\sigma_t \mathbf{q} - \widetilde{\mathbf{g}}_t\right). \tag{23}$$

Using the resource constraint,

$$\mathbf{1}^\top \boldsymbol{\tau}_t^{(\kappa)} = 1,$$

determines the scale of the optimal gratification vector,

$$\sigma_t = \frac{1 + \mathbf{1}^\top \boldsymbol{\alpha}^{-1}\widetilde{\mathbf{g}}_t}{\mathbf{1}^\top \boldsymbol{\alpha}^{-1}\mathbf{q}}. \tag{24}$$

The interior Karenina-optimal resource allocation is therefore

$$\boldsymbol{\tau}_t^{(\kappa)} = \boldsymbol{\alpha}^{-1}\left[\frac{1 + \mathbf{1}^\top \boldsymbol{\alpha}^{-1}\widetilde{\mathbf{g}}_t}{\mathbf{1}^\top \boldsymbol{\alpha}^{-1}\mathbf{q}}\mathbf{q} - \widetilde{\mathbf{g}}_t\right]. \tag{25}$$

This expression applies whenever all components of $\boldsymbol{\tau}_t^{(\kappa)}$ are positive. If one or more components lie outside the resource-allocation simplex, the constrained optimum lies on the corresponding boundary. We return to these boundary cases when analyzing the Karenina-optimal dynamics.

D. *The Generalized Karenina principle*

Equation (19) implements the Generalized Karenina Principle: an interior decision restores the optimal gratification ratios rather than the previous absolute gratification levels.

Under common proportional loss, $L_i = \phi g_i$, the loss preserves these ratios. Substituting a state on the optimal ray into equation (25) gives

$$\boldsymbol{\tau}^{(\kappa)*} = \frac{\boldsymbol{\alpha}^{-1}\mathbf{q}}{\mathbf{1}^\top \boldsymbol{\alpha}^{-1}\mathbf{q}}. \tag{26}$$

Provided this allocation remains interior and gratifications remain positive, subsequent allocation is constant even while gratification levels continue to adjust. Appendix A.A1 shows that it also maximizes stationary Karenina gratification.

Following a disturbance, fixed allocation retains these shares, whereas Karenina optimization reallocates to restore the optimal ratios whenever feasible. Once the balance is restored in an otherwise unchanged environment, allocation returns to the same constant shares. The following sections compare the resulting dynamics.

V. Time-constant allocation is stable but non-adaptive

We begin with time-constant resource allocation

$$\tau_{i,t} = \tau_i^*, \qquad i = 1, \dots, R.$$

as the stability benchmark. Under proportional gratification loss, a fixed allocation generates affine gratification dynamics with a unique stable stationary state. Thus, contradiction between requirements does not by itself generate instability when the allocation is held fixed.

A. *Stable gratification dynamics*

For an arbitrary number R of requirements, the gratification dynamics under a fixed allocation with proportional loss function follows from equation (3),

$$g_{i,t+1} = (1-\phi_i) g_{i,t} + \sum_{j=1}^{R} \alpha_{ij} \tau_j^*. \tag{27}$$

The stationary gratification

$$g_i^* = \frac{\sum_{j=1}^{R} \alpha_{ij} \tau_j^*}{\phi_i}. \tag{28}$$

is admissible if

$$\sum_{j=1}^{R} \alpha_{ij} \tau_j^* > 0 \qquad \text{and} \qquad \phi_i > 0 \qquad \forall i, \tag{29}$$

so that all stationary gratification levels remain positive. The constant gain function can be replaced by the stationary state to rewrite equation (27)

$$g_{i,t+1} - g_i^* = (1-\phi_i)\left(g_{i,t} - g_i^*\right). \tag{30}$$

This decouples the individual equations of the requirements even though their steady state solutions remain interlinked via the leverage matrix. The dynamics is easily solved to

$$g_{i,t} = g_i^* + (1-\phi_i)^t \left(g_{i,0} - g_i^*\right). \tag{31}$$

with the corresponding eigenvalues $\lambda_i = 1 - \phi_i$ for each requirement. The stationary state is asymptotically stable whenever

$$0 < \phi_i < 2 \qquad \forall i. \tag{32}$$

For the economically relevant range $0 < \phi_i < 1$, all eigenvalues lie between zero and one. Gratification therefore converges monotonically towards its stationary value.

B. The two-requirement representation

To prepare the comparison with the endogenous allocation strategies, we now specialize to two requirements, denoted by $+$ and $-$. We define total gratification

$$\bar{g}_t \equiv g_{+,t} + g_{-,t}, \tag{33}$$

and the gratification difference

$$\Delta g_t \equiv g_{+,t} - g_{-,t}. \tag{34}$$

The two variables have direct interpretations. $\bar{g}_t$ measures the overall level of gratification across the two requirements, while Δg_t measures their imbalance. The corresponding stationary quantities are denoted $\bar{g}^*$ and Δg^*. For equal proportional loss rates $\phi_+ = \phi_- \equiv \phi$, we obtain

$$\begin{aligned} \bar{g}_{t+1} &= \bar{g}^* + (1-\phi)\left(\bar{g}_t - \bar{g}^*\right), \\ \Delta g_{t+1} &= \Delta g^* + (1-\phi)\left(\Delta g_t - \Delta g^*\right). \end{aligned} \tag{35}$$

Thus also total gratification and gratification imbalance decouple and relax towards their stationary values with the same eigenvalue $\lambda = 1 - \phi$. This decomposition will become useful for the analysis that follows. Under time-constant allocation, an imbalance between the two requirements simply decays towards its stationary value. Under need-responsive allocation, by contrast, the imbalance changes the next resource allocation. That additional feedback turns Δg_t into a nonlinear dynamical variable and, as shown in Section VI, can destabilize the stationary state.

C. Stability does not imply adaptability

Stability under fixed allocation does not ensure that the allocation remains desirable after a disturbance. The resource shares do not respond to changed gratification levels or to changes in the allocation technology.

Interaction can also overturn individual stability. Section VIII considers two decision makers whose gratification shortfalls reinforce one another. With a common loss rate ϕ and symmetric coupling c, their fixed-allocation equilibrium becomes unstable when $c > \phi$. Karenina-optimal reallocation raises this threshold in the setting studied there. This motivates an adaptive rule capable of counteracting disturbances that fixed shares transmit without adjustment.

VI. Myopic need-responsive allocation can become unstable

We now analyze the dynamics generated by the myopic need-responsive allocation strategy introduced and motivated in Section IV.B. In contrast to time-constant allocation, the resource allocation now responds to the current gratification state and can thereby adapt to external shocks and interactions with other decision makers. This introduces a feedback from gratification to allocation and back to gratification. We show that, when contradictory requirements interact sufficiently strongly, this feedback can destabilize an otherwise stable gratification dynamics.

A. Fully symmetric two-requirement case

To understand its behavior analytically, we restrict the analysis to two requirements under the proportional loss function. For two requirements, denoted by the indices $+$ and $-$, the need-responsive resource allocation of equation (10) becomes

$$\tau_{\pm,t} = \frac{g_{\pm,t}^{-\beta_\pm}}{g_{+,t}^{-\beta_+} + g_{-,t}^{-\beta_-}} = \frac{g_{\mp,t}^{\beta_\mp}}{g_{+,t}^{\beta_+} + g_{-,t}^{\beta_-}}. \tag{36}$$

For convenience we define $\tau_t \equiv \tau_{+,t}$ and eliminate $\tau_{-,t} = 1 - \tau_t$. In order to isolate the effect of contradictory requirements, we first consider the most symmetric case possible. We assume identical loss rates $\phi \equiv \phi_+ = \phi_i$ and identical responsiveness $\beta \equiv \beta_+ = \beta_-$ of the two requirements. Furthermore we assume a leverage matrix fully symmetric under exchange of the two requirements, i.e.

$$\alpha_{++} = \alpha_{--}, \qquad \alpha_{+-} = \alpha_{-+}.$$

For interpretation purposes we define

$$\begin{aligned} S_\alpha &\equiv \alpha_{++} + \alpha_{-+} = \alpha_{--} + \alpha_{+-} && \text{(total gratification gain)}, \\ \delta\alpha &\equiv \alpha_{++} - \alpha_{+-} = \alpha_{--} - \alpha_{-+} && \text{(relocation imbalance)}, \\ \Delta\alpha &\equiv \frac{(\alpha_{++} + \alpha_{+-}) - (\alpha_{--} + \alpha_{-+})}{2} && \text{(leverage imbalance)}. \end{aligned} \tag{37}$$

Symmetry against exchange of requirements implies $\Delta\alpha = 0$. The governing equations can then be written as

$$g_{\pm,t+1} = (1-\phi)g_{\pm,t} + \frac{S_\alpha}{2} \pm \delta\alpha \left(\tau_t - \frac{1}{2}\right), \tag{38}$$

Using the total gratification and gratification difference introduced in equations (33) and (34) the dynamics becomes

$$\bar{g}_{t+1} = (1-\phi) \cdot \bar{g}_t + S_\alpha, \tag{39}$$

$$\Delta g_{t+1} = (1-\phi) \cdot \Delta g_t + \delta\alpha \left(2\tau_t - 1\right). \tag{40}$$

We restrict attention throughout the paper to $\delta\alpha > 0$. This means that shifting resources towards a requirement increases its gratification relative to that of the competing requirement. In other words, a requirement is gratified predominantly by spending resources on that requirement rather than predominantly by spending resources on the other. The opposite case, $\delta\alpha < 0$, would be counterintuitive in the applications considered here and would require separate analysis. The stationary state is

$$\bar{g}^* = \frac{S_\alpha}{\phi}, \qquad \Delta g^* = 0, \tag{41}$$

Equation (39) shows directly why S_α is the total gratification gain: stationary total gratification is determined by the balance between the gratification gain S_α and the proportional loss ϕ. The corresponding symmetric eigenvalue is

$$\lambda_{\bar{g}} = 1 - \phi.$$

The asymmetric mode governs the gratification difference and has eigenvalue

$$\lambda_{\Delta g} = 1 - \phi - \beta a, \tag{42}$$

where the dimensionless parameter

$$a \equiv \frac{\phi}{S_\alpha} \cdot \delta\alpha = \frac{\delta\alpha}{\bar{g}^*}. \tag{43}$$

measures the strength of the contradiction between the two requirements relative to the stationary gratification level. It increases with the relocation imbalance $\delta\alpha$. At the threshold the eigenvalue passes through -1, corresponding to a flip bifurcation. Thus, even under perfect symmetry, $\Delta\alpha = 0$, sufficiently contradictory requirements lead to alternating overcompensation between the two gratifications. The symmetric case therefore provides the simplest illustration of the instability that is investigated more generally below. The stationary state and the asymmetric eigenvalue are derived explicitly in Appendix B.B1.

B. *Behavioral interpretation of the first instability*

The transition at the critical value a_{crit} has a natural behavioral interpretation. Below the threshold, directing more resources towards the relatively undergratified requirement reduces the gratification imbalance, and the decision process converges towards a stationary gratification distribution, a stationary Karenina function value and a stationary resource allocation.

Above the threshold, however, reallocating resources towards one requirement not only withdraws resources from the competing requirement but also impairs it through the negative off-diagonal elements of the leverage matrix. The correction therefore generates an even larger imbalance in the opposite direction. In the following decision period, resources are shifted back towards the now undergratified requirement, again overshooting the balanced state. The decision maker thereby enters a self-generated oscillation between competing allocation patterns.

Importantly, this oscillation does neither require stochastic forcing nor external shocks nor any other interaction with the environment. It is an entirely deterministic consequence of repeatedly applying the same myopic decision rule to mutually inhibitive requirements. The instability arises because the rule responds to the current gratification state without accounting for the consequences of the resulting allocation through the leverage matrix.

The mechanism can arise whenever a decision maker allocates a limited resource among objectives that directly interfere with one another. Examples include managerial attention allocated between innovation and current operations, household time divided between work and family, or public resources distributed among competing policy objectives. When progress in one dimension directly reduces gratification in another, repeated need-responsive corrections can generate endogenous oscillations even in the absence of external shocks.

C. *Instability cascade for increasingly contradictory requirements*

The decomposition into total gratification and gratification difference does not require full symmetry. In order to show the universality of the instability and the richness of the dynamics, we therefore relax the symmetry conditions and allow

$$\Delta\alpha \neq 0, \tag{44}$$

In fact without loss of generality we restrict our analysis to cases with $\Delta\alpha > 0$, since the labels of the two requirements can otherwise be exchanged. We retain equal column sums, $S_\alpha = \alpha_{++} + \alpha_{-+} = \alpha_{--} + \alpha_{+-}$. Thus, allocating one unit of either resource has the same total effect on gratification. This condition also implies equal row differences, so that the relocation imbalance remains $\delta\alpha = \alpha_{++} - \alpha_{+-} = \alpha_{--} - \alpha_{-+}$.

Long-run reduction to a one-dimensional problem

With this remaining symmetry, total gratification continues to evolve independently following equation (39), i.e. it relaxes towards $\bar{g}^*$ on a time scale of approximately ϕ^{-1}. If the system is initialized on the invariant manifold $\bar{g}_0 = \bar{g}^*$, or once this transient has decayed, the dynamics is entirely described by

$$\Delta g_{t+1} = (1-\phi)\Delta g_t + \delta\alpha(2\tau_t - 1) + \Delta\alpha. \tag{45}$$

On this manifold of $\bar{g}_0 = \bar{g}^*$,

$$2\tau_t - 1 = -\frac{(\bar{g}^* + \Delta g_t)^\beta - (\bar{g}^* - \Delta g_t)^\beta}{(\bar{g}^* + \Delta g_t)^\beta + (\bar{g}^* - \Delta g_t)^\beta}. \tag{46}$$

We normalize the gratification difference by stationary total gratification and define

$$h_t \equiv \frac{\Delta g_t}{\bar{g}^*}. \tag{47}$$

The normalization has the advantage that because $g_{\pm,t} = (\bar{g}^* \pm \Delta g_t)/2 = \bar{g}^*(1 \pm h_t)/2$ positivity of both gratification levels is equivalent to $-1 < h_t < 1$. For $\Delta\alpha > 0$, the stationary state lies on the positive branch, and the relevant stationary solution therefore satisfies $0 < h^* < 1$.

In addition to the previously defined dimensionless parameter $a = \delta\alpha/\bar{g}^*$, we introduce the dimensionless

$$\epsilon \equiv \frac{\Delta\alpha}{\bar{g}^*}. \tag{48}$$

The parameter a measures the strength of the relocation imbalance relative to the stationary gratification scale and therefore measures the contradictoriness between the two requirements. The parameter ϵ measures the leverage imbalance relative to the same gratification scale. The need-responsive allocation yields the one-dimensional dynamics

$$h_{t+1} = (1-\phi)h_t - a\frac{(1+h_t)^\beta - (1-h_t)^\beta}{(1+h_t)^\beta + (1-h_t)^\beta} + \epsilon. \tag{49}$$

Stationary state and emergence of instability

To investigate the stationary state and its stability, define

$$R_\beta(h) \equiv \frac{(1+h)^\beta - (1-h)^\beta}{(1+h)^\beta + (1-h)^\beta} = \tanh(\beta \operatorname{artanh} h). \tag{50}$$

The stationary state satisfies

$$\phi h^* + aR_\beta(h^*) = \epsilon,$$

To follow the stationary state as contradictoriness a is varied, it is useful to parametrize the stationary branch by h^*

$$a(h^*) \equiv \frac{\epsilon - \phi h^*}{R_\beta(h^*)}. \tag{51}$$

For the admissible positive stationary state, $\partial a(h^*)/\partial h^* < 0$, so increasing contradictoriness decreases the corresponding steady state h^*. The stationary-state multiplier that is relevant for the stability

$$\left.\frac{\partial h_{t+1}}{\partial h_t}\right|_{h_t=h^*} = 1-\phi-aR'_\beta(h^*). \tag{52}$$

reaches the instability boundary -1 at

$$a_1 = \frac{2-\phi}{R'_\beta(h_1^*)}, \tag{53}$$

where h_1^* satisfies

$$\phi h_1^* + (2-\phi)\frac{R_\beta(h_1^*)}{R'_\beta(h_1^*)} = \epsilon. \tag{54}$$

For every $\beta > 1$, $0 < \phi < 1$, and $\epsilon > 0$, there exists a unique physical solution $0 < h_1^* < 1$ and a corresponding finite critical value $a_1 > 0$, as shown in Appendix B.B2. Thus sufficiently strong contradictoriness destabilizes the stationary need-responsive allocation also away from perfect symmetry. The limiting case $\beta = 1$ undergoes the same first flip instability but is dynamically exceptional beyond it. Since $R_1(h) = h$ and the map becomes affine,

$$h_{t+1} = (1-\phi-a)h_t + \epsilon.$$

and the stationary state loses stability at $a_1 = 2-\phi$ following the same equation (53). Above this threshold, deviations alternate with growing amplitude; no nonlinear saturation exists to generate a stable period-two orbit. For $\beta > 1$, by contrast, the nonlinear need response allows the initial flip bifurcation to develop into further period doublings.

Period-doubling route into chaos

For the quadratic need-response rule, $\beta = 2$, equation (49) simplifies to

$$h_{t+1} = (1-\phi)h_t - 2a\frac{h_t}{1+h_t^2} + \epsilon \equiv F_{a,\phi,\epsilon}(h_t). \tag{55}$$

The normalization separates two economically distinct properties of the leverage matrix. Increasing a strengthens the relocation imbalance and hence the contradiction between the requirements while leaving the intrinsic leverage imbalance unchanged. Increasing ϵ, by contrast, strengthens the intrinsic advantage of requirement + while leaving the relocation imbalance unchanged. We therefore hold ϕ and ϵ fixed and increase a, isolating the effect of increasing contradictoriness.

Increasing a destabilizes the stationary state at the threshold derived above. The subsequent dynamics depend on the nonlinear response and on whether trajectories remain in the positive-gratification domain. The Supplemental Appendix provides additional calculations for the quadratic rule and a bifurcation diagram of the associated rational-map continuation. That example exhibits successive period doublings, but its later cascade leaves $-1 < h_t < 1$ and therefore does not establish admissible economic chaos. The analytical result established here is that sufficiently strong contradiction destabilizes need-responsive adjustment through an alternating local instability.

The instability identified above does not require stochastic disturbances, changing preferences, or imperfect information. The gain and loss functions are linear, and the environment can remain constant. It arises from the interaction between mutually inhibitive requirements and the repeated application of the need-responsive rule: a statically plausible allocation rule need not generate stable dynamics when applied repeatedly. The next section shows that this instability is not an unavoidable consequence of adaptive allocation.

VII. Myopic Karenina-optimal allocation is stable and adaptive

We now analyze the dynamics generated by myopic Karenina-optimal allocation. Unlike need-responsive allocation, which directs resources toward the largest current deficits, this strategy chooses the allocation that restores the optimal balance among gratifications while accounting for all direct and cross-effects of resource use. Economically, it therefore responds not to low gratification as such, but to deviations from the balance appropriate to the decision problem. We again consider two requirements, denoted by $+$ and $-$, and use the leverage-matrix notation introduced in the preceding section. The key dynamical consequence is that every interior decision places the gratification vector on the Karenina-optimal ray defined by equation (19). The gratification difference is therefore no longer an independent dynamical degree of freedom, allowing the stability analysis to be reduced to the evolution of gratification along that ray.

A. The two-requirement case

We retain the equal-column-sum leverage structure of Section VI.C, with parameters defined in equation (37), and write $\tau_{+,t} = \tau_t$, $\tau_{-,t} = 1 - \tau_t$. We assume $S_\alpha > 0$ and $\delta\alpha > 0$, and allow the proportional loss rates ϕ_+ and ϕ_- to differ. Equation (3) then becomes

$$g_{\pm,t+1} = (1 - \phi_\pm) g_{\pm,t} \pm \delta\alpha \left(\tau_t - \frac{1}{2}\right) + \frac{S_\alpha \pm \Delta\alpha}{2}. \tag{56}$$

This extends the governing equation (38) to unequal loss rates and nonzero leverage imbalance. We also retain positive direct effects, $\alpha_{++}, \alpha_{--} > 0$. Since $\delta\alpha > 0$, shifting resources toward either requirement raises its next-period gratification while reducing that of the other. This trade-off arises both under mutual inhibition and when cross-effects are positive but weaker than direct effects. An interior Karenina optimum additionally requires a maximizing allocation with $0 < \tau_t < 1$ and positive next-period gratifications. The leverage matrix can be rewritten

$$\boldsymbol{\alpha} = \frac{1}{2} \begin{pmatrix} S_\alpha + \delta\alpha + \Delta\alpha & S_\alpha - \delta\alpha + \Delta\alpha \\ S_\alpha - \delta\alpha - \Delta\alpha & S_\alpha + \delta\alpha - \Delta\alpha \end{pmatrix}, \tag{57}$$

with $\det \boldsymbol{\alpha} = S_\alpha \delta\alpha$.

B. Karenina-optimal resource allocation

Under the equal-column-sum structure, the general optimal-ratio condition (19) becomes

$$\frac{g_{+,t+1}}{g_{-,t+1}} = \frac{\gamma_+}{\gamma_-}. \tag{58}$$

At an interior optimum, the gratification ratio therefore depends only on the Karenina elasticities. For convenience we define the ratio

$$\chi \equiv \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-} \tag{59}$$

Equal column sums imply that reallocation leaves total next-period gratification unchanged; optimization distributes this total in proportion to the elasticities. The leverage parameters determine the allocation required to achieve this balance and whether it is attainable within the resource constraint. Using the pre-allocation gratification from equation (21), let $\widetilde{\bar{g}}_t \equiv \widetilde{g}_{+,t} + \widetilde{g}_{-,t}$ and $\Delta\widetilde{g}_t \equiv \widetilde{g}_{+,t} - \widetilde{g}_{-,t}$. The general allocation rule (25) then yields

$$\tau^{(\kappa)}_{+,t} = \frac{1}{2} - \frac{\Delta\widetilde{g}_t + \Delta\alpha - \chi \cdot \left(S_\alpha + \widetilde{\bar{g}}_t\right)}{2\delta\alpha}, \tag{60}$$

with $\tau^{(\kappa)}_{-,t} = 1 - \tau^{(\kappa)}_{+,t}$. This expression applies when both allocations lie in $(0, 1)$ and next-period gratifications are positive. For equal Karenina elasticities, the optimal allocation simplifies to

$$\tau^{(\kappa)}_{+,t} = \frac{1}{2} - \frac{\Delta\widetilde{g}_t + \Delta\alpha}{2\delta\alpha}. \tag{61}$$

The decision maker thus compensates for both the gratification imbalance remaining after losses and the leverage imbalance. Equal importance of the requirements does not generally imply equal resource shares: more resources may be needed to offset a disadvantage in how allocation generates gratification.

For equal loss rates, $\phi_+ = \phi_- = \phi$, proportional losses preserve the optimal ratio for arbitrary Karenina elasticities. As shown in Section IV.D, subsequent allocation is therefore time independent, provided the implied allocation remains interior and next-period gratifications remain positive:

$$\tau^{(\kappa)}_{+} = \frac{1}{2} - \frac{\Delta\alpha - \chi \cdot S_\alpha}{2\delta\alpha}, \tag{62}$$

with $\tau^{(\kappa)}_{-} = 1 - \tau^{(\kappa)}_{+}$. This expression holds for arbitrary Karenina elasticities; for equal elasticities it reduces to $\tau^{(\kappa)}_{+} = \frac{1}{2} - \frac{\Delta\alpha}{2\delta\alpha}$. Constant allocation therefore emerges from repeated optimization once the optimal balance is established, even while absolute gratification levels continue to adjust.

C. *Reduced dynamics, stationary state, and stability*

The central dynamical consequence of Karenina-optimal allocation is that a single interior decision places the gratification state on a fixed line, even if the initial state lies away from it. This follows from equation (58): the chosen allocation makes next-period gratifications satisfy the optimal ratio exactly, rather than merely moving them closer to it. Using total gratification and the gratification difference defined in equations (33) and (34), this condition yields

$$\Delta g_{t+1} = \chi \bar{g}_{t+1}. \tag{63}$$

From any positive initial state for which the maximizing allocation satisfies $0 < \tau^{(\kappa)}_{+,t} < 1$ and next-period gratifications are positive, the first decision maps the system onto the optimal ray $\Delta g = \chi\bar{g}$, $\bar{g} > 0$. Every subsequent interior decision keeps it there. The optimal ratio is therefore attained in one decision period, even while gratification levels continue to adjust toward their stationary values.

Economically, the allocation restores the optimal balance between requirements while leaving their overall gratification level to adjust over time. The gratification difference consequently ceases to be an independent dynamical variable: after the first interior decision, it is determined entirely by total gratification. As will be shown below this removes the repeated overcompensation of imbalance that can destabilize need-responsive allocation.

Adding the two component equations in (56) cancels the allocation terms. On the optimal line, the resulting dynamics becomes

$$\bar{g}_{t+1} = (1 - \phi_\gamma)\bar{g}_t + S_\alpha \tag{64}$$

The effective loss rate

$$\phi_\gamma \equiv \frac{\gamma_+\phi_+ + \gamma_-\phi_-}{\gamma_+ + \gamma_-}. \tag{65}$$

is the average of the two loss rates weighted by their shares in optimal gratification. For equal loss rates $\phi_\gamma|_{\phi_+=\phi_-=\phi} = \phi$ and for equal gratification elasticities it becomes the average loss rate, $\phi_\gamma|_{\gamma_+=\gamma_-=\gamma} = (\phi_+ + \phi_-)/2$. This generalizes equation (39) to unequal loss rates. Total gratification therefore follows an affine adjustment process: resource use contributes the constant gain S_α, while losses are proportional to the current total.

For $\phi_\gamma > 0$, the reduced map has the unique stationary state

$$\begin{aligned} \bar{g}^* &= \frac{S_\alpha}{\phi_\gamma} = \frac{\gamma_+ + \gamma_-}{\gamma_+\phi_+ + \gamma_-\phi_-} \cdot S_\alpha \\ \Delta g^* &= \chi\frac{S_\alpha}{\phi_\gamma} = \frac{\gamma_+ - \gamma_-}{\gamma_+\phi_+ + \gamma_-\phi_-} \cdot S_\alpha \end{aligned} \tag{66}$$

or

$$g^*_\pm = \frac{\gamma_\pm}{\gamma_+\phi_+ + \gamma_-\phi_-} \cdot S_\alpha. \tag{67}$$

This state is an interior equilibrium of the original decision problem provided its gratifications are positive and its allocation from equation (60) lies strictly between zero and one. The stationary total balances gains against losses, while its distribution between requirements reflects their Karenina elasticities.

Local stability follows from the same reduction. Every interior decision sets the ratio deviation $\Delta g_{t+1} - \chi\bar{g}_{t+1}$ to zero, while perturbations along the optimal ray have multiplier $1 - \phi_\gamma$. The full gratification dynamics therefore has a zero transverse eigenvalue, $\lambda_\perp = 0$, and

$$\lambda_\parallel = 1 - \phi_\gamma. \tag{68}$$

Here "transverse" refers to deviations from the optimal ratio, not necessarily to a geometrically perpendicular eigenvector. The interior equilibrium is locally asymptotically stable if and only if $0 < \phi_\gamma < 2$. Under the economically natural restriction $0 < \phi_\pm < 1$, one has $0 < \phi_\gamma < 1$. After the first interior decision, gratification levels therefore converge monotonically to their stationary values, provided the trajectory remains interior.

The stability result identifies the central difference from need-responsive allocation. In Section VI, stronger contradiction can amplify successive reallocations until the stationary state loses stability. Under interior Karenina-optimal allocation, equation (68) shows that the eigenvalues depend only on the loss rates and Karenina elasticities. Changes in the leverage parameters can alter the allocation required to maintain the optimal balance and its feasibility, but cannot generate the leverage-induced flip bifurcation. Restoring the optimal ratio after each decision removes the independent imbalance dynamics responsible for repeated overcompensation.

For equal loss rates, $\phi_+ = \phi_- = \phi$, the effective loss rate reduces to $\phi_\gamma = \phi$, and equation (64) coincides with equation (39). The distinction between the strategies therefore concerns the adjustment of gratification imbalance, rather than total gratification. As shown in equation (62), Karenina-optimal allocation becomes constant after the first interior decision, while gratification levels continue to adjust. The decision maker thus achieves a stable allocation through adaptive optimization, without committing to fixed resource shares in advance. With heterogeneous loss rates, however, the stationary allocation generated by myopic Karenina optimization need not maximize the stationary Karenina function; this distinction is derived in Appendix A.A2.

D. Scope of the stability result

The preceding result applies while the optimum remains interior, $0 < \tau^{(\kappa)}_{+,t} < 1$, and gratifications remain positive. If the constrained optimum assigns the entire resource to one requirement, the optimal gratification ratio need not be attainable within a single decision period.

Within either boundary regime, the allocation is fixed at $\boldsymbol{\tau} = (1,0)^\top$ or $(0,1)^\top$. The dynamics and its Jacobian are therefore

$$\mathbf{g}_{t+1} = \begin{pmatrix} 1-\phi_+ & 0 \\ 0 & 1-\phi_- \end{pmatrix} \mathbf{g}_t + \boldsymbol{\alpha}\boldsymbol{\tau}, \qquad J = \begin{pmatrix} 1-\phi_+ & 0 \\ 0 & 1-\phi_- \end{pmatrix}. \tag{69}$$

Thus perturbations contract within each fixed boundary regime when $0 < \phi_\pm < 2$. A boundary fixed point is locally asymptotically stable if it has positive gratifications and lies strictly within the region where that allocation remains optimal.

These regime-specific results do not by themselves establish global convergence when allocations switch between regimes. The analysis establishes local stability of the interior equilibrium and excludes the leverage-induced flip mechanism there; it does not provide a global stability proof for the complete constrained dynamics.

VIII. Collective instability of interacting decision makers

Section V.C showed that interaction can destabilize individually stable decision makers under constant allocation. Such dependence may arise when a deterioration in one firm's access to finance affects another through shared lenders or trading relationships. We now examine how Karenina-optimal reallocation raises the threshold for this collective instability.

A. Interaction between decision makers

Consider two decision makers, A and B, each with requirements $+$ and $-$. They interact only through requirement $+$. For $(i,j) \in \{(A,B),(B,A)\}$, their dynamics is

$$\begin{aligned} g^i_{+,t+1} &= (1-\phi^i_+)g^i_{+,t} + \alpha^i_{++}\tau^i_{+,t} + \alpha^i_{+-}\tau^i_{-,t} + c_{ij}\left(g^j_{+,t} - g^{j*}_+\right), \\ g^i_{-,t+1} &= (1-\phi^i_-)g^i_{-,t} + \alpha^i_{-+}\tau^i_{+,t} + \alpha^i_{--}\tau^i_{-,t}. \end{aligned} \tag{70}$$

Here $c_{ij} > 0$ transmits deviations with the same sign, so a shortfall in one decision maker's gratification reduces the other's. The reference levels g^{i*}_+ are the uncoupled stationary levels; the interaction therefore changes their stability without shifting the reference equilibrium.

B. Collective instability under constant allocation

Hold each allocation fixed at the value sustaining its uncoupled stationary state. With $x^i_t \equiv g^i_{+,t} - g^{i*}_+$,

$$\begin{pmatrix} x^A_{t+1} \\ x^B_{t+1} \end{pmatrix} = \begin{pmatrix} 1-\phi^A_+ & c_{AB} \\ c_{BA} & 1-\phi^B_+ \end{pmatrix} \begin{pmatrix} x^A_t \\ x^B_t \end{pmatrix}, \tag{71}$$

with eigenvalues

$$\lambda_{\pm} = 1 - \frac{\phi_+^A + \phi_+^B}{2} \pm \sqrt{\left(\frac{\phi_+^A - \phi_+^B}{2}\right)^2 + c_{AB}c_{BA}}. \tag{72}$$

The $-$ requirements evolve independently with fixed resource inputs, giving the remaining eigenvalues $1-\phi_-^A$ and $1-\phi_-^B$. For $0 < \phi_-^A, \phi_-^B < 2$, collective instability can therefore arise only through the coupled $+$ block.

For identical decision makers with $\phi_+^A = \phi_+^B \equiv \phi$ and symmetric coupling $c_{AB} = c_{BA} \equiv c$, the coupled eigenvalues become

$$\lambda_s = 1 - \phi + c, \qquad \lambda_a = 1 - \phi - c, \tag{73}$$

corresponding to symmetric deviations $x_t^A = x_t^B$ and antisymmetric deviations $x_t^A = -x_t^B$. For $0 < \phi < 1$, the symmetric eigenvalue reaches $+1$ first, at

$$c_{\text{crit}}^{(\tau^*)} = \phi. \tag{74}$$

Unlike the flip bifurcation in Section VI, this instability amplifies a common deviation through mutual reinforcement rather than alternating overcompensation.

C. *Stabilization through Karenina-optimal reallocation*

Each decision maker now maximizes their own next-period Karenina function, taking the other's current state as given. We consider identical decision makers with a common loss rate $0 < \phi < 1$ for both requirements, symmetric coupling c, and the equal-column-sum leverage structure of Section VII. Fixing the constant-allocation benchmark at the uncoupled interior Karenina optimum in equation (62) ensures that both strategies sustain the same reference equilibrium.

Every interior decision restores the optimal ratio in equation (58). Define

$$\chi_+ \equiv \frac{\gamma_+}{\gamma_+ + \gamma_-} = \frac{1+\chi}{2}, \tag{75}$$

where χ is defined in equation (59). After the first interior decision,

$$g_{+,t}^i = \chi_+ \bar{g}_t^i, \qquad g_{-,t}^i = (1-\chi_+)\bar{g}_t^i, \qquad i = A, B. \tag{76}$$

Adding each decision maker's component equations eliminates the allocation terms:

$$\bar{g}_{t+1}^i = (1-\phi)\bar{g}_t^i + S_\alpha + c\left(g_{+,t}^j - g_+^*\right), \qquad (i,j) \in \{(A,B),(B,A)\}. \tag{77}$$

The common reference equilibrium is $\bar{g}^* = S_\alpha/\phi$, with $g_+^* = \chi_+\bar{g}^*$. For $y_t^i \equiv \bar{g}_t^i - \bar{g}^*$, substitution of equation (76) yields

$$\begin{pmatrix} y_{t+1}^A \\ y_{t+1}^B \end{pmatrix} = \begin{pmatrix} 1-\phi & c\chi_+ \\ c\chi_+ & 1-\phi \end{pmatrix} \begin{pmatrix} y_t^A \\ y_t^B \end{pmatrix}. \tag{78}$$

The symmetric and antisymmetric eigenvalues are

$$\lambda_s^{(\kappa)} = 1 - \phi + c\chi_+, \qquad \lambda_a^{(\kappa)} = 1 - \phi - c\chi_+. \tag{79}$$

The other two eigenvalues are zero because each interior decision eliminates deviations from the optimal rays. The symmetric mode again loses stability first, now at

$$c_{\mathrm{crit}}^{(\kappa)} = \frac{\phi}{\chi_+} = \frac{\gamma_+ + \gamma_-}{\gamma_+} c_{\mathrm{crit}}^{(\tau^*)} > c_{\mathrm{crit}}^{(\tau^*)}. \tag{80}$$

For equal Karenina elasticities, $\chi_+ = 1/2$, so reallocation doubles the threshold from ϕ to 2ϕ. Thus, for $\phi < c < \phi/\chi_+$, the common equilibrium is unstable under constant allocation but locally stable under Karenina-optimal reallocation.

The mechanism is that reallocation distributes a transmitted disturbance across both requirements to restore their optimal ratio. Only the fraction χ_+ remains in requirement $+$, reducing the effective coupling to $c\chi_+$. Reallocation cannot, however, remove the disturbance to total gratification: sufficiently strong interaction still destabilizes motion along the optimal rays.

This result concerns local stability around a positive interior equilibrium. The reduction requires positive gratifications and $0 < \tau_{+,t}^{i(\kappa)} < 1$ for both decision makers. Large disturbances may violate these conditions even below the threshold; the boundary qualification in Section VII.D then applies.

IX. Conclusion and Discussion

This paper proposes the Generalized Karenina Principle for allocating scarce resources among interdependent essential requirements. The principle directs allocation toward the gratification balance that maximizes their joint contribution to success, accounting for the direct and cross-effects of resource use.

The analysis distinguishes adaptation from stability. Under the stated proportional-loss restrictions, time-constant allocation is stable for an isolated decision maker but cannot adjust resource shares to disturbances. Need-responsive allocation is adaptive, yet sufficiently strong contradiction can turn its corrective response into destabilizing overcompensation. Interior Karenina-optimal allocation restores the optimal gratification ratios and excludes this leverage-induced instability in the two-requirement setting. Under common proportional loss, it reaches the same stationary allocation as optimally chosen constant shares, while responding differently away from equilibrium. For interacting decision makers, Karenina-optimal reallocation also raises the threshold for collective instability, although sufficiently strong interaction can still destabilize the common equilibrium.

A. *Testable empirical implications*

The allocation strategies imply different responses to a shock that disproportionately affects some requirements. Under time-constant allocation, gratification levels and ratios gradually return toward their stationary values. An interior Karenina-optimal decision instead restores the optimal ratios in one decision period, followed by adjustment along the optimal ray. Need-responsive allocation may generate overshooting, reversals, or alternating imbalances when contradiction is sufficiently strong. These restrictions on joint outcome trajectories can be tested without directly observing resource shares, although they do not by themselves uniquely identify the underlying decision rule.

The financial crisis of 2007/2008 offers a possible application at the level of a national economy. Candidate requirements include functioning credit provision, employment, gross domestic product, fiscal capacity, and social cohesion. Their joint recovery could be examined for gradual rebalancing, rapid restoration of characteristic gratification ratios, or corrective overshooting. A corresponding firm-level analysis could examine liquidity, production continuity, workforce capacity, customer retention, and product development following demand, credit, or supply shocks.

Such tests require economically meaningful positive measures of gratification and reference ratios defined independently of the recovery trajectories being tested. Normalization to pre-crisis values alone does not establish that the pre-crisis balance was optimal. The interpretation must also account for changes in requirements, leverage, or Karenina elasticities, which may shift the relevant balance after a shock. Multiple episodes, comparable firms, or held-out periods would provide stronger tests than a single recovery trajectory. The empirical object is the joint adjustment of levels and ratios; rapid recovery or oscillation alone is insufficient to distinguish the strategies.

The interaction model adds a prediction concerning the transmission of disturbances. Stronger positive dependence increases the persistence and amplification of common deviations, while Karenina-optimal reallocation attenuates their transmission. Variation in interaction strength across firms or organizations could therefore be used to test whether adaptive restoration of internal balance moderates this amplification.

B. *Limitations and extensions*

The framework treats requirements, Karenina elasticities, and the leverage matrix as fixed and known, and assumes that gratification is observed accurately. Learning, imperfect information, and endogenous changes in these objects may affect both the feasible allocation rules and their relative performance. In particular, responding to an observed deficit may require less information than evaluating the consequences of alternative allocations for all requirements.

Karenina-optimal allocation is also myopic: it maximizes the next-period Karenina function without adjustment costs. Forward-looking decision makers may tolerate temporary imbalance to improve future outcomes, while reallocation costs may make immediate restoration inefficient. Introducing expectations, discounting, and adjustment costs would clarify when the myopic rule approximates an intertemporal optimum.

The principal stability results concern linear gains, proportional losses, two requirements, and positive interior equilibria. Large disturbances may force boundary allocations, where the optimal ratio need not be attainable in one period; global convergence across interior and boundary regimes has not been established. More requirements, nonlinear technologies, heterogeneous decision makers, and network interactions remain important extensions.

Within these limits, the central finding is that the basis for adaptive allocation matters for stability. Responding to current deficits can amplify imbalance, whereas accounting for the consequences of allocation across requirements can restore balance and stabilize adjustment. This distinction provides both a decision-theoretic explanation of instability and a set of restrictions for empirical investigation.

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Appendix A: Optimal time-constant allocation

A1. Karenina-optimal time-constant resource allocation

Under common proportional loss, $L_i(g_i) = \phi g_i$ with $0 < \phi < 1$, a constant allocation $\boldsymbol{\tau}$ yields the stationary gratification vector

$$\mathbf{g}^*(\boldsymbol{\tau}) = \frac{1}{\phi}\boldsymbol{\alpha\tau}. \tag{A1}$$

We maximize the stationary Karenina function over allocations satisfying $\mathbf{1}^\top\boldsymbol{\tau} = 1$, $\boldsymbol{\tau} \geq 0$, and $\boldsymbol{\alpha\tau} > 0$. Equivalently, we maximize

$$\ln \kappa^*(\boldsymbol{\tau}) = \sum_{i=1}^{R} \gamma_i \ln(\boldsymbol{\alpha\tau})_i - \left(\sum_{i=1}^{R} \gamma_i\right) \ln \phi. \tag{A2}$$

For positive γ_i and nonsingular $\boldsymbol{\alpha}$, this objective is strictly concave on the feasible domain. Its interior first-order conditions therefore identify the unique maximizer:

$$\sum_{i=1}^{R} \frac{\gamma_i \alpha_{ij}}{(\boldsymbol{\alpha\tau})_i} = \mu, \qquad j = 1, \ldots, R. \tag{A3}$$

Using the vector $\mathbf{q}$ defined in equation (18), these conditions give $\boldsymbol{\alpha\tau} = \mathbf{q}/\mu$. The resource constraint then yields

$$\boldsymbol{\tau}^{\mathrm{TC}*} = \frac{\boldsymbol{\alpha}^{-1}\mathbf{q}}{\mathbf{1}^\top\boldsymbol{\alpha}^{-1}\mathbf{q}}, \tag{A4}$$

provided this allocation is strictly positive and generates strictly positive gratification. Its stationary gratification vector is

$$\mathbf{g}^{\mathrm{TC}*} = \frac{\mathbf{q}}{\phi\, \mathbf{1}^\top\boldsymbol{\alpha}^{-1}\mathbf{q}}. \tag{A5}$$

Thus $g_i^{\mathrm{TC}*}/g_j^{\mathrm{TC}*} = q_i/q_j$: the stationary optimum lies on the same gratification ray as the interior myopic Karenina optimum. Under common proportional loss, substituting a state on this ray into equation (25) gives $\boldsymbol{\tau}_t^{(\kappa)} = \boldsymbol{\tau}^{\mathrm{TC}*}$. Repeated interior Karenina optimization therefore selects the optimal constant allocation once the optimal ratios have been established. The strategies differ following a disturbance: fixed allocation leaves resource shares unchanged, whereas Karenina optimization reallocates resources to restore these ratios whenever feasible.

A2. Heterogeneous loss rates

With heterogeneous proportional losses, $L_i(g_i) = \phi_i g_i$ and $0 < \phi_i < 1$, a constant allocation yields $g_i^* = (\boldsymbol{\alpha\tau})_i/\phi_i$. Consequently,

$$\ln \kappa^*(\boldsymbol{\tau}) = \sum_{i=1}^{R} \gamma_i \ln(\boldsymbol{\alpha\tau})_i - \sum_{i=1}^{R} \gamma_i \ln \phi_i. \tag{A6}$$

The loss rates enter only through an allocation-independent term. The stationary first-order conditions and the optimal constant allocation therefore remain those in equations (A3) and (A4), subject to the same interiority and positivity conditions.

By contrast, at an interior stationary state of myopic Karenina optimization, equation (17) requires

$$\sum_{i=1}^{R} \frac{\gamma_i \phi_i \alpha_{ij}}{(\boldsymbol{\alpha\tau})_i} = \lambda, \qquad j = 1, \dots, R. \tag{A7}$$

For nonsingular $\boldsymbol{\alpha}$ and positive gratifications, the stationary and myopic first-order conditions can hold at the same allocation only if $\phi_i = \lambda/\mu$ for every i, where μ is the multiplier in equation (A3). Thus the two interior optima coincide under common loss rates, but differ when the loss rates are heterogeneous. Myopic maximization of next-period gratification therefore need not maximize the stationary Karenina function.

Appendix B: Instability of need-responsive resource allocation

B1. Fully symmetric case

Under the assumptions of Section VI.A, with $\beta > 0$, $0 < \phi < 1$, and $\delta\alpha > 0$, stationarity requires

$$\phi \Delta g^* = \delta\alpha(2\tau^* - 1). \tag{B1}$$

The need-responsive rule makes $2\tau^* - 1$ have the opposite sign to Δg^* whenever $\Delta g^* \neq 0$. Consequently, the unique stationary state satisfies $\Delta g^* = 0$, $\tau^* = 1/2$, and $g_+^* = g_-^* = S_\alpha/(2\phi)$.

Writing the allocation as

$$\tau = \frac{(\bar{g} - \Delta g)^\beta}{(\bar{g} + \Delta g)^\beta + (\bar{g} - \Delta g)^\beta} \tag{B2}$$

gives $\partial\tau/\partial\Delta g|_* = -\beta/(2\bar{g}^*)$. Since total gratification evolves independently, the two eigenvalues are

$$\lambda_{\bar{g}} = 1 - \phi, \qquad \lambda_{\Delta g} = 1 - \phi - \beta a, \qquad a = \frac{\delta\alpha}{\bar{g}^*}. \tag{B3}$$

The stationary state is therefore locally asymptotically stable for $0 \leq a < (2 - \phi)/\beta$ and loses stability through the multiplier -1 at

$$a_{\text{crit}} = \frac{2 - \phi}{\beta}. \tag{B4}$$

B2. Asymmetric case

On the invariant manifold $\bar{g} = S_\alpha/\phi$, use the normalized variables of Section VI.C. The map is

$$h_{t+1} = (1 - \phi)h_t - aR_\beta(h_t) + \epsilon, \qquad R_\beta(h) = \tanh[\beta \operatorname{artanh}(h)], \tag{B5}$$

with $-1 < h < 1$, $a > 0$, and $\epsilon > 0$. A stationary state satisfies $\phi h^* + aR_\beta(h^*) = \epsilon$. The left-hand side is strictly increasing in h, so there is at most one admissible stationary state, and it is positive. Along this branch,

$$\frac{\mathrm{d}h^*}{\mathrm{d}a} = -\frac{R_\beta(h^*)}{\phi + aR'_\beta(h^*)} < 0, \qquad \Lambda = 1 - \phi - aR'_\beta(h^*). \tag{B6}$$

For $\beta > 1$ and $0 < h < 1$,

$$R'_\beta(h) = \frac{\beta[1 - R_\beta(h)^2]}{1 - h^2} > 0, \qquad R''_\beta(h) = \frac{2\beta[1 - R_\beta(h)^2][h - \beta R_\beta(h)]}{(1 - h^2)^2} < 0. \tag{B7}$$

The second inequality follows from $R_\beta(h) > h$. Thus Λ decreases strictly as a increases along the stationary branch. Since $\Lambda < 1$, its stability boundary is $\Lambda = -1$, or

$$a_1 = \frac{2 - \phi}{R'_\beta(h_1^*)}, \qquad H(h_1^*) = \epsilon, \qquad H(h) = \phi h + (2 - \phi)\frac{R_\beta(h)}{R'_\beta(h)}. \tag{B8}$$

Here

$$H'(h) = \phi + (2 - \phi)\left[1 - \frac{R_\beta(h) R''_\beta(h)}{R'_\beta(h)^2}\right] > 0. \tag{B9}$$

Moreover, $H(0) = 0$ and $H(h) \to \infty$ as $h \to 1^-$. Hence every $\epsilon > 0$ determines exactly one $h_1^* \in (0, 1)$ and a finite $a_1 > 0$. The multiplier crosses -1 transversally, establishing the first local instability. This result does not, by itself, establish an admissible period-doubling cascade beyond the threshold.

For $\beta = 1$, the map is affine: $h_{t+1} = (1 - \phi - a)h_t + \epsilon$. Its multiplier reaches -1 at $a_1 = 2 - \phi$, where $h^* = \epsilon/2$ is admissible if $0 < \epsilon < 2$. Beyond this threshold, deviations alternate with increasing amplitude; the affine map has no nonlinear saturation producing an attracting period-two orbit.

Supplementary Material

By Anders Levermann*

This is the supplementary material for "The Karenina Principle: A Decision Rule for Contradicting Requirements".

I. Quadratic need response: additional map analysis

This section provides additional calculations for the quadratic need-response rule and illustrates the dynamics of its rational-map continuation. The economic model requires $-1 < h_t < 1$, because $g_{\pm,t} = \bar{g}^*(1 \pm h_t)/2$ with $\bar{g}^* > 0$. Although the rational map below is defined for every real h_t, trajectories leaving this interval no longer represent strictly positive gratification levels. Results for the continued map must therefore be distinguished from results for the admissible economic model.

For the quadratic need-response rule, $\beta = 2$,

$$R_2(h) = \frac{2h}{1+h^2}, \tag{1}$$

and the reduced dynamics becomes

$$h_{t+1} = F_{a,\phi,\epsilon}(h_t) \equiv (1-\phi)h_t - 2a\frac{h_t}{1+h_t^2} + \epsilon. \tag{2}$$

The stationary state satisfies

$$\phi h^* + 2a\frac{h^*}{1+h^{*2}} = \epsilon, \tag{3}$$

or equivalently

$$a = A(h^*) \equiv \frac{(\epsilon - \phi h^*)(1+h^{*2})}{2h^*}. \tag{4}$$

Its derivative is

$$\frac{\mathrm{d}A}{\mathrm{d}h} = \frac{1}{2}\left(\epsilon - \frac{\epsilon}{h^2} - 2\phi h\right) < 0 \tag{5}$$

throughout the admissible stationary branch.

The stationary-state multiplier is

$$\Lambda = 1 - \phi - 2a\frac{1-h^{*2}}{(1+h^{*2})^2}. \tag{6}$$

At the first flip bifurcation, $\Lambda = -1$, and hence

$$a_1 = \frac{(2-\phi)(1+h_1^{*2})^2}{2(1-h_1^{*2})}. \tag{7}$$

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Combining this condition with equation (3) yields

$$\epsilon = \phi h_1^* + (2-\phi)\frac{h_1^*(1+h_1^{*2})}{1-h_1^{*2}}. \tag{8}$$

Multiplication by $1-h_1^{*2}$ gives

$$Q(h) \equiv \epsilon - \epsilon h^2 - 2h - 2(1-\phi)h^3 = 0. \tag{9}$$

For $0<\phi<1$ and $\epsilon>0$,

$$Q(0)=\epsilon>0, \qquad Q(1) = -2(2-\phi)<0, \tag{10}$$

while

$$Q'(h) = -2-2\epsilon h - 6(1-\phi)h^2 < 0 \qquad (h>0). \tag{11}$$

Consequently, there is exactly one admissible stationary state at the first instability, $0<h_1^*<1$. For the numerical example below, the bifurcation diagram shows an attracting period-two orbit emerging beyond this threshold. Subsequent periodic orbits must be checked separately for both stability and admissibility.

A period-two orbit consists of two distinct points h_1 and h_2 satisfying

$$h_2 = F_{a,\phi,\epsilon}(h_1), \qquad h_1 = F_{a,\phi,\epsilon}(h_2), \qquad h_1 \neq h_2. \tag{12}$$

Its multiplier is

$$\Lambda_2 = F'_{a,\phi,\epsilon}(h_1)F'_{a,\phi,\epsilon}(h_2), \tag{13}$$

and the period-two orbit is stable for $|\Lambda_2|<1$. Its next period doubling occurs at

$$\Lambda_2 = -1. \tag{14}$$

More generally, a stable orbit of period 2^n loses stability when

$$\prod_{k=1}^{2^n} F'_{a,\phi,\epsilon}(h_k) = -1. \tag{15}$$

Figure 1 illustrates this progression for

$$\phi = 0.2, \qquad \epsilon = 0.5, \tag{16}$$

with a as the control parameter. For each value of a, the map in equation (2) is iterated until the initial transient has decayed, after which the remaining values of h_t are plotted. Below a_1, trajectories converge to the stationary branch. At a_1, the stationary state loses stability and a period-two orbit emerges. Further increases in contradictoriness produce successive period doublings and, eventually, irregular deterministic dynamics interrupted by periodic windows.

The numerical example illustrates the nonlinear dynamics of the rational-map continuation beyond its first instability. Establishing a period-doubling route to chaos within the economic model additionally requires showing that the relevant trajectories remain entirely within $-1<h_t<1$. The present example does not establish that stronger result.

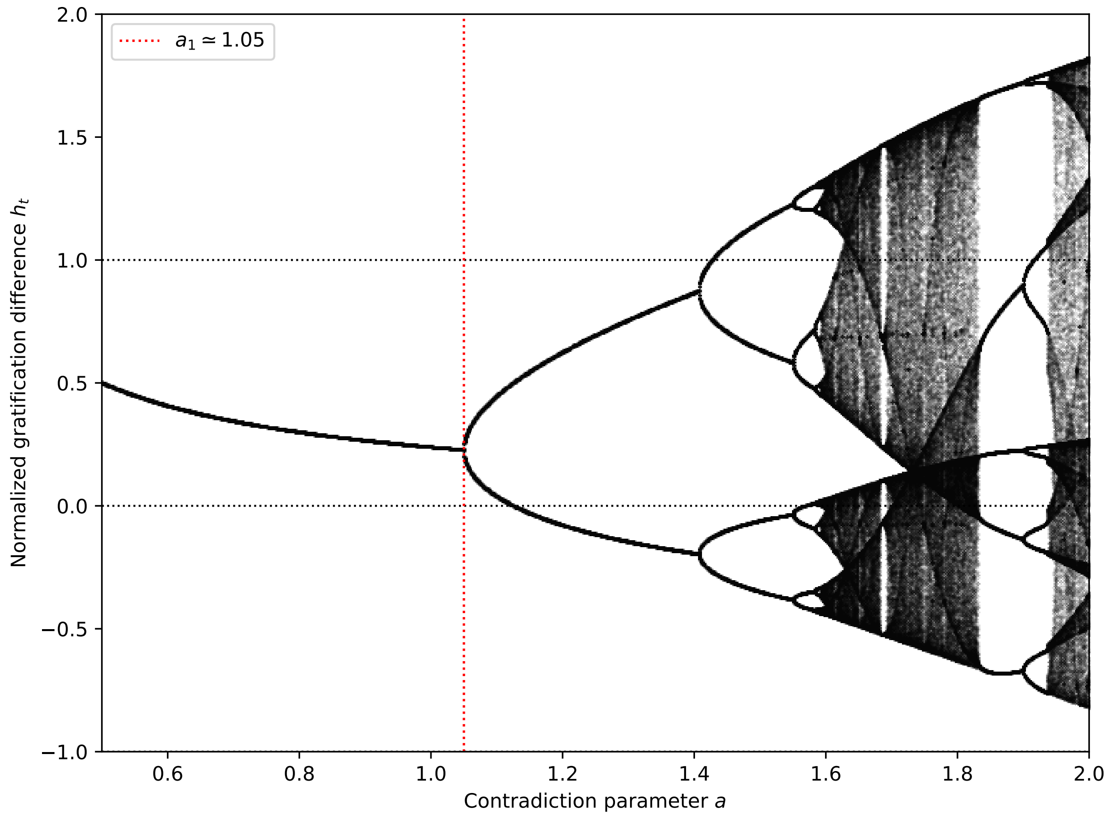

FIGURE 1. BIFURCATION DIAGRAM OF THE RATIONAL MAP $F_{a,\phi,\epsilon}$ FOR $\beta = 2$, $\phi = 0.2$, AND $\epsilon = 0.5$, INCLUDING ITS CONTINUATION OUTSIDE THE ECONOMIC DOMAIN. THE STATIONARY STATE FIRST LOSES STABILITY NEAR $a_1 = 1.05$, FOLLOWED BY PERIOD DOUBLINGS AND IRREGULAR DETERMINISTIC DYNAMICS. STRICTLY POSITIVE GRATIFICATIONS REQUIRE $-1 < h_t < 1$; THE LATER CASCADE SHOWN HERE INCLUDES VALUES OUTSIDE THIS INTERVAL. THE FIGURE THEREFORE ILLUSTRATES THE CONTINUED MAP AND DOES NOT ESTABLISH A CHAOTIC ATTRACTOR CONFINED TO THE ADMISSIBLE ECONOMIC DOMAIN.