Heterogenous Capital and the Production Function

A Contribution to Energy Economics, Multiple Optima Growth and the Cambridge Controversy

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Nico Bauer*

Potsdam Institute for Climate Impact Research (PIK)

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Abstract

The marginal productivity of capital is determined from a model with heterogenous capital and it is shown that it can be increasing for a succession of technologies. We derive a non-linear macro-economic production function from a bottom-up optimisation model of linear production technologies and show how assumptions about available technologies translate into qualitative properties of the production function. We find that the concept of the elasticity of substitution can only be derived from the bottom-up model, if the value of capital is applied and not the physical capacity. Applying the economic instead of the technical concept of capital we show that non-convex production iso-quants can be derived, if the variety of technological alternatives satisfies a certain curvator condition. Multiple optima implied by the non-convexity in

* *Corresponding author:* Telegraphenberg A31, 14412 Potsdam, Germany. Tel.: ++49 +331 288 2540. E-mail: nico.bauer@hotmail.com.

the macro-economic framework are equivalent to leap-frogging of technologies in the bottom-up model. Deriving the macro-economic production function we show that the marginalist theory of distribution is fully determined because all combinations of production factors will always be fully exhausted. The marginal product of capital is shown to be increasing in capital over an interval as the iso-quants are non-convex. Consequently, an increasing marginal product of capital is equivalent to leap-frogging of technologies. This line of argumentation is different to the argument of re-switching of technologies that is prominent in the theoretical Cambridge-capital controversy.

1 Introduction

We attend to the relationship between two modeling approaches: (i) the bottom-up approach focuses on technology choice in a world of heterogenous capital stocks and (ii) the top-down approach deals with the production of income and its distribution among aggregate production factors. The heterogeneity of capital is a striking consequence of technological variety, but the reward to financial capital – the interest rate – applies to a homogenous aggregate. The theoretic essence is that the macro-economic production function represents the detailed technological variety in aggregate entities and summarizes the response of aggregate factor allocation induced by factor price changes. Hence, the aggregation of heterogenous capital stocks is an indispensable link between the two model approaches. Results on this very basic theoretic level have important implications for many other fields of economic research, in particular production, distribution and gorwth.

The present paper first provides an overview to the literature to which it is related; see Sec. 2. Next, we give a short introduction to macroeconomic production functions in Sec. 3. The following Sec. 4 introduces the model of heterogenous capital and discusses the relationship with the macro-economic model. Finally, we discuss the results and show directions of future research in Sec. 5.

2 Literature Overview

Interest into the issue comes from at least three fields of economic research that will be discussed below. First, we have a recent discussion in the applied field of energy economics; see Sec. 2.1. Second, the work is related to the discussion on growth models with poverty traps; see Sec. 2.2. Third, there are long-lasting debates – know as the Cambridge-capital controversy – on capital stock aggregation, production functions and theories of distribution; see Sec. 2.3.

2.1 Energy Economics

The subject this paper is dealing with is essential in energy economics and discussed as the bottom-up/top-down dichotomy. The issue is highly relevant in the debate about the costs of global climate change mitigation because the representation of energy technologies and energy carriers has significant influence, when it comes to the reduction of greenhouse gas emissions. The term dichotomy is mainly motivated from the differences in results. The theoretical relationship is a different matter that will be discussed below.

Bottom-up refers to a modeling approach in which technological choice is explicit and the capital stock is heterogenous. The input-output relationship of each single technology is linear. Changes in relative input and output prices induce re-evaluation of technology choices that leads to changes in the heterogenous capital stock and thus to changes in the input-output relationships of the overall system. The term top-down refers to a modeling approach that ignores explicit technology choice. Instead the changes of input-output relationships due to price changes are parameterized in a production function. In the bottom-up approach it is possible that a single technology takes over a whole market due to small changes in prices. In the top-down approach such qualitative behavior – similar to revolutionary changes – are usually not considered..

The difference in the modeling approaches comes with two ways of determining the parameters. In the bottom-up approach data on technologies – existing and speculative – are used. The top-down approach is usually based on econometric estimations from statistical data. Hence, the bottom-up approach represents technologies that are not already available, that could replace all existing technologies. The top-down approach applies parameters derived from historic data, which eventually improve smoothly over time.

Emphasising the dichotomy of the two approaches Richels and Robinson (1996, p. 286), Markandya and Halsnaes (2001, p. 489) and Hourcade et al. (2006, p. 2) pointed out that bottom-up models exhibit high technological flexibility, but top-down models instead do not consider technological flexibility beyond current practice. Therefore, studies on climate change mitigation using bottom-up models generally lead to lower mitigation costs than top-down models; see Edenhofer et al. (2006). Wing (2006b) provides a different view based on a modeling study for the US.

Wing (2006a) points out the "deep duality" (p. 542) between both approaches, since they are interrelated in a theoretical sense: a production function represents a set of technologies regarding the economically meaningful information. For illustration he distinguishes two production factors that he named "clean" representing a composite of capital, labour etc. and "dirty" that is fossil fuels. Different factor ratios are feasible within a given set of technologies. He argues that a macro-economic iso-quant is simply the smooth approximation of the input combinations. He discusses changes of the iso-quant as new technologies become available, but he implicitly assumes that the iso-quant always has a convex shape. In this paper it will be shown that the shape of the iso-quant depends on the techno-economic properties of the set of available technologies.

In the literature of energy economics *hybrid models* become increasingly important. The term hybrid model used here refers to a class of models that embed a bottom-up energy system model into a top-down macro-economic growth model. The energy system model follows the bottom-up approach with a great variety of technologies and energy carriers; the macro-economic growth model follows the top-down approach based on a macro-economic production function. The two models are connected via energy and capital markets that assures simultaneous intertemporal equilibrium on all markets; see Bauer et al. (2008). Hybrid models have been identified as state-of-the-art models that are worth to develop further; see e.g. Edenhofer et al. (2006) and Hourcade et al. (2006). There is a number of models like *MERGE* by Manne et al. (1995) and Kypreos and Bahn (2003), *MIND/ ReMIND* by Edenhofer et al. (2005), Bauer (2005) and Leimbach et al. (2009), and *WITCH* by Bosetti et al. (2006).

The hybrid approach brings both model approaches together in order to combine the technological explicitness of bottom-up models with the macro-economic completeness and realism of top-down models; see Hourcade et al. (2006, p. 4). Unfortunately, the relationship of coupling both approaches remains unclear. A contribution to its clarification would improve our understanding of climate change mitigation cost assessments. The question is still open why hybrid models tend to emphasis energy sector restructuring in the bottom-up part in the long run, but suggest energy substitution in top-down part in the short run; see Edenhofer et al. (2006, p. 105).

2.2 Multiple Optima in Growth Models

Most studies on economic growth are based on neo-classical production functions that exhibit diminishing marginal productivity of capital. A distinct part of the theoretical literature on economic growth is based on production functions that are assumed to have increasing marginal productivity on a limited interval of capital input; see Skiba (1978), Dechert and Nishimura (1983), Azariadis (1996), Haunschmied et al. (2003) and Semmler and Ofori (2007).

These studies reveal that the steady-state growth path needs not to be unique, showing that multiple optima can exist. The long-run development depends on initial conditions and possibly also on near-term behaviour; see espec. Haunschmied et al. (2003, p. 711-2). The studies provide useful insights into the dynamics of economic growth and why some economies remain stucked in a poverty trap and others experience accelerating growth rates.

However, these studies never asked for the theoretical foundation of such production functions. The problem can be pinned down to the point that the studies mentioned above assume only capital as the single production factor. However, it is impossible to derive a production function with increasing returns reasonably, for the case of a single production factor. The problem is that each marginal unit of capital can be identified with a separate investment project. A production function could be considered as a rationally ordered sequence of investment opportunities. The sequence would place the investment project with the highest profitability first and then in descending order investment projects with the next highest profitability, which is equivalent to diminishing returns. This paper improves the theoretical basis of convex-concave production functions by providing conditions for increasing returns to capital. The analysis below reveals that two production factors are a minimum ingredient for the derivation of increasing returns.

2.3 Cambridge-Capital Controversy

The Cambridge capital controversy was a heated debate during the 50ies until the 70ies and much has been written about it. Felipe and Fisher (2003) and Cohen and Harcourt (2003) provide fruitful summaries of the literature. For the purpose of this paper we distinguish two strands of literature that differ with respect to the representation of the bottom-up technolologies and therefore, how the macroeconomic porperties are derived.

The common starting point has been the factor price frontier that results from a model where capital and labour are used either to produce a consumption or the capital good; see e.g. Sraffa (1960), Samuelson (1962), Garegnani (1970) and Hahn (1982). The main dispute was a problem of summarising heterogenous capital goods into a homogenous capital aggregate in the Clarke-Ramsey sense, which is essential for the neo-classical theory of production and distribution. Clarke-Ramsey capital simultaneously fulfills three properties. First, the price of capital is the interest rate that – in market equilibrium – equals the marginal productivity. Second, the marginal productivity of capital is monotonically decreasing in capital and the output is completely distributed to the production factors capital and labour rewarded according to their marginal products. This is – finally – equivalent with a monotonous succession of using technologies implied by changes in the wage-interest ratio; see Cohen and Harcourt (2003, p. 201). Re-switching of a technology – a technology once decommissioned at a factor price ratio becomes competitive at a very different wage-interest ratio – implies increasing marginal productivity of capital, also known as capital reversing; see e.g. Samuelson (1966, p. 582). The significance of re-switching was highly disputed; see e.g. Ferguson (1971) and Robinson (1975). It was abandoned mainly because of its empirical irrelevance; see Stiglitz (1974, p. 897).

A smaller number of publications in the second strand of literature applied the class of bottom-up models that will be analysed below; see Solow (1962), Kurz and Manne (1963) and Phelps (1963). Here only one output is produced from one input by using one capital good chosen from a great variety of alternatives. The idea is that all technological designs are collected in a book of blueprints that describes the technological and economic characteristics of each alternative. In the simplest case a technology is characterized by investment costs and a fixed input-output transformation coefficient. Around this idea a bottom-up model can be formulated that is then summarised in a macro-economic production function.

The contributions of the smaller group, however, left the job undone to work out the conditions under which capital stock aggregation is valid and under which conditions the properties are inconsistent with the neo-classical theory of production and distribution. This paper tries to fill part of this gap showing that the three properties of Clarke-Ramsey capital need not be fulfilled simultaneously. We will show that capital reversing is not implied by re-switching. If the set of technologies implies capital-reversing it would be optimal to leap-frog some technologies as relative prices are around a critical value. Some technologies that are potentially competitive would be left out in the succession of technologies, because more expensive technologies being much more efficient become competitive. It will be shown that given amounts of production factors are always fully exhausted therefore the shadow prices of the production factors equals to the marginal factor productivities.

Therefore, increasing returns to capital are uniquely derived from a model of heterogenous capital. However, this is at odds with the three properties of Clarke-Ramsey capital given above.

3 Allocation and Distribution in Top-Down Models

In this section we repeat the standard textbook example of a cost-minimizing firm in a particular variant by Acemoglu (2002, Ch. 3). The allocation of aggregate factors is determind by a cost-minimizing sector, who takes prices as given parameters. The technological variety is parameterised in the production function that implies a theory of distribution of income based on marginal factor productivities.

The model is a cost-minimisation problem of a firm that takes factor prices p_1 and p_2 as given and demands the cost minimising quantities of production factors x_1 and x_2 in order to produce fixed output $\overline{y}$. It is worth to underline here that the prices p_1 and p_2 are independently chosen parameters. The production function $f(x_1, x_2)$ is of the constant elasticity of substitution (CES) type; see Arrow et al. (1961):

$$\overline{y} = \left[(a_1 x_1)^\rho + (a_2 x_2)^\rho\right]^{\frac{1}{\rho}} . \tag{1}$$

The parameter ρ is related to the elasticity of substitution by the well-known definition $\sigma = \frac{1}{1-\rho}$. The parameters a_1 and a_2 are the efficiency levels of the production factors. The objective is to minimise the costs $c = p_1 x_1 + p_2 x_2$.

Applying standard methods of constraint optimisation we find that changes in the factor prices and the productivities imply reallocation of production factors. For the sake of brevity the tilde will denote ratios in the following. The substitution effect is induced by changes in factor prices:

$$\frac{\partial \tilde{x}}{\partial \tilde{p}} \frac{\tilde{p}}{\tilde{x}} = -\sigma. \tag{2}$$

The substitution effect is equivalent to an iso-quant function that is convex relative to the origin, representing a convex production possibility set. The change of the factor ratio could also be triggered by biased technological change:

$$\frac{\partial \tilde{x}}{\partial \tilde{a}} \frac{\tilde{a}}{\tilde{x}} = \sigma - 1. \tag{3}$$

The model – though very parsimonious – provides insights into allocation and distribution: production factor's remuneration equals its marginal product. Although more general neoclassical functions than the CES are available, it is sufficient for the purposes of the present study.

4 Aggregating Heterogenous Capital

In this section we ask how a macro-production function with aggregate capital can be derived from a bottom-up model with heterogenous capital. The interrelationship between the macro-economic and the bottom-up model will be analysed and we will give insight into the duality between the techno-economic characteristics of available technologies and the shape of the production function implied by the optimal technology choice. Before the details are elaborated we provide an overview.

We consider a model where an homogenous output good Y can be produced using one input good X in combination with the capacity of one technology K_i that is chosen from a continuum of alternatives[1] $i = 1, ..., n$ being different in their techno-economic characteristics (η, ι), expressing the transformation coefficient η in units of output per input and the investment costs ι in units of monetary costs per unit capacity. For each ι it is only necessary to consider the technology with the highest η. Hence, we can express all techno-economic combinations using a function $\eta = \eta(\iota)$. This function is monotonously increasing and we assume that it is continuously differentiable. It is interpreted as the *book of blueprints* of all available technological designs worth to consider.

The total production costs are defined as the sum of costs for capital $p^k K$ and input $p^x X$. Here p^k denotes the capital costs being the annualised costs of the investment ι. They are computed with the following equation $p^k = \iota \cdot r$, where the interest rate r is exogenously given.[2] The variable p^x represents the costs of the non-reproducible input. The input costs can also be expressed in terms of output units, if we consider the transformation coefficient $\frac{p^x}{\eta}$.

In a first step, we assume that the output demand $\bar{Y}$ is given exogenously. The firm minimizes the costs by choosing a technology from the set of alternatives. Hence, we can derive the minimal production costs p and capital costs

[1]We will skip the sub-script in the following as it eases readyability.
[2]We do not consider depreciation.

p^k for all combinations (p^x, r). This will be done in Sec. 4.1.

In the second step, we have available all information on quantities (Y, X, K) and prices (p, p^x, r, ι) in order to interpret the bottom-up model in macro-economic terms. However, the crucial scientific challenge is to choose between two concepts of capital stock aggregation: technical capacities $\mathcal{K}^t$ in units of output potentials or economic capital $\mathcal{K}^e$ in units of monetary values. For instance, we will show in Sec. 4.2 that only economic capital is compatible with the concept of the elasticity of substitution, which can have any positive value depending on the techno-economic characteristics.

In the third step of Sec. 4.3, we apply the economic capital concept and derive qualitative properties of the macro-production function that are determined by the bottom-up model's optimal solution. We will give a condition for the function $\eta(\iota)$ that leads to non-convex iso-quants in the $\mathcal{K}^e - X$-space. This implies that non-convex iso-quants are equivalent to leap-frogging technology alternatives as the input price ratio passes a critical value.

Finally, we ask in Sec. 4.4 whether the macro-economic production function is determined, if we apply the concept of economic capital. The question is all but trivial because assuming a fixed amount of capital in money units does not determine the capacity nor the transformation coefficient. We shall see that maximizing output both factors are always fully exhausted in the bottom-up model. Hence, we can derive marginal productivities and a theory of income distribution based on production factors' shadow prices that supposed to be represented in the production function. Finally, we will show that increasing marginal productivity of capital is equivalent with leap-frogging of technologies due to the multiple optima implied by non-convex iso-quants. Hence, Clarke-Ramsey capital is only a special case because the marginalist theory of factor remuneration does not exclude increasing returns and technology leap-frogging.

4.1 Cost Functions of the Bottom-Up Model

We will derive a unique function θ mapping every p^x to p^k as a result of the cost-minimal choice between technologies characterised by (η, ι). Since $\bar{Y}$ is given and as soon as η is determined we also know K and X according to the linear-limitational production function:[3]

$$X = \frac{1}{\eta}\bar{Y}, \quad (4)$$

$$K = \bar{Y}. \quad (5)$$

The first equation is simply the technical conversion of one input unit into η output units. X and Y are both measured in physical units. The second equation is the constraint on capacity, where K is the production potential for a period measured in output units.[4]

Fig. 1 illustrates the idea that serves as the basis for the function $p^k = \theta(\cdot)$. Both graphs show on the abscissa the price p^x and on the ordinate the costs per unit of output p. Assuming market equilibrium cost minimisation of an atomistic firm implies that p equals the sum of $\frac{p^x}{\eta}$ and $p^k = r \cdot \iota$. Hence, the total revenue would be distributed among the two production factors.

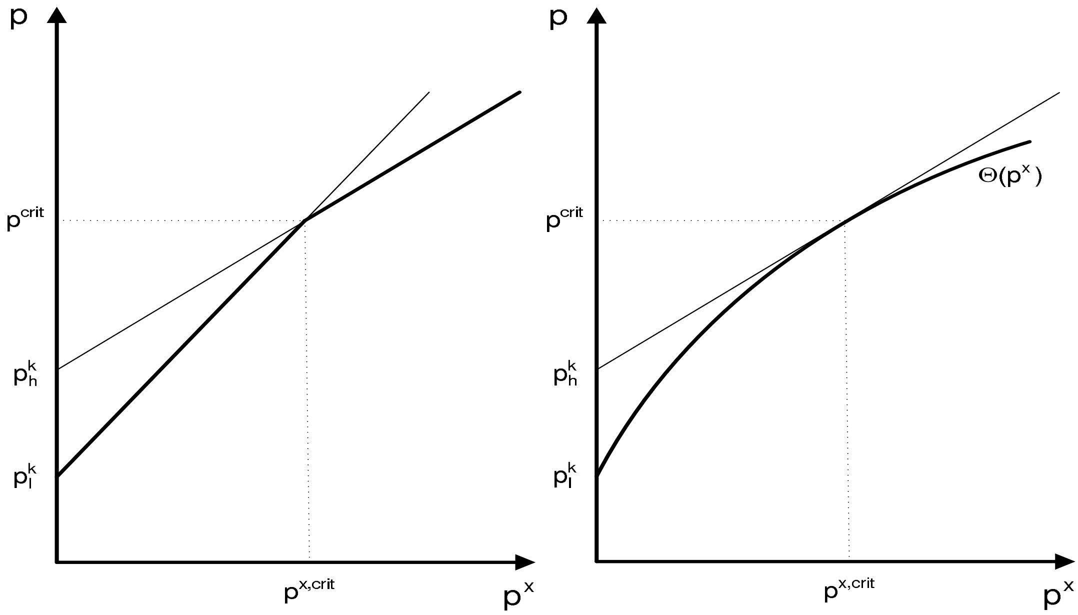
Fig. 1(a) compares two linear production technologies. One technology is characterised by low transformation coefficient η_l and low capital costs p^k_l and the second technology is characterised by high transformation coefficient η_h and high capital costs p^k_h. The differences in technical transformation lead to the different slopes of the lines: the higher the transformation coefficient, the flatter is the curve. p^k is measured by the intersect with the ordinate.

Technology choice depends on the input price p^x. If it is low – i.e. $p^x < p^{x,crit}$ – the $l-$technology is chosen because it has lower production costs. If the input price is higher than $p^{x,crit}$ then the $h-$technology is chosen. At the critical input price $p^{x,crit}$ the firm would be indifferent between the two technologies producing the output at the same price p^{crit}. The bold parts of the two linear functions connected at the critical input price indicate the cost-minimal choice of technologies.

Generalising this idea to n technologies, we assume that there is a continuum of technologies with differing combinations of (η, ι) such that the cost minimal choices are approximated by a continuous function $\Theta : p^x \mapsto p$ shown in Fig. 1(b). It has the following properties, provided that it is differentiable

[3]Usually the linear-limitational production function is formalized in terms of a min-operator. This is redundant, if we assume that unnecessary factors will not be demanded.

[4]We assume the same capital avaiability factor for technologies. Differing capital availability factors can be considered by re-scaling the investment costs ι so that lower capital availability would imply higher investment costs.



(a) Two technologies (b) Continuum of n technologies

Figure 1: Illustration of the rational for the function $\theta(\cdot)$.

at every p^x:[5]

$$\frac{\partial\Theta}{\partial p^x} > 0; \tag{6}$$

$$\frac{\partial\Theta^2}{\partial^2 p^x} < 0. \tag{7}$$

Tentatively, we assume that Θ is continuously differentiable. Below (Sec. 4.2) we will work out the conditions for that in detail.

For reasons of orientation we kept some information in Fig. 1(b) that explains the rational of the n technology case. It suggests that the l-technology is the one having the lowest possible p^k. The linear line of the h-technology is now tangent to $\Theta(\cdot)$. The function suggests higher η are attainable at higher ι:

$$\frac{\partial\eta}{\partial\iota} > 0. \tag{8}$$

In the following we say that $\eta = \eta(\iota) = \eta(\frac{p^k}{r})$ and we assume throughout the paper that it is differentiable. The function represents the book of blue-

[5] We omit linear pieces in the function, which would imply to consider additionally an equality sign in Eq. 7.

prints available to the economy. *Nota bene*: we do not impose conditions on the sign of the second order derivative. In this point we deviate from Solow (1962).

We now come to the function $\theta : p^x \mapsto p^k$. For every p^x we know the production costs $p = \Theta(p^f)$ and the slope Θ'. With this information and the tangent argument we can compute the corresponding capital costs p^k:

$$p^k = \theta(p^x) = \Theta(p^x) - \frac{\partial\Theta}{\partial p^x} p^x. \tag{9}$$

From the properties of the function $\Theta(\cdot)$ given in Eq. 6 – 7 we can conclude that $\theta(\cdot)$ has the following property, provided that $\Theta(\cdot)$ is continuously differentiable at p^x:

$$\frac{\partial\theta}{\partial p^x} > 0. \tag{10}$$

Note that we did not derive a condition on the second order derivative of $\theta(\cdot)$, as we did for $\Theta(\cdot)$.

Finally, we can say that given the function $\eta(\cdot)$ the price p^x determines η via the function Eq. 9:

$$\eta = \eta(\theta(p^x)) = \tilde{\eta}(p^x). \tag{11}$$

The optimal solution is to buy more efficient, but also more expensive technologies as the input price increases.

4.2 Substitution and Capital Stock Aggregation

In this section we discuss capital stock aggregation and the relationship with the elasticity of substitution. In a first step we show that the concept of the elasticity of substitution is only reasonable, if the concept of economic capital is applied. In the second step we show that any positive value of the elasticity of substitution can be justified by the bottom-up model's solution.

Since the elasticity of substitution is related to distribution, we examine how the output value is distributed among the two production factors. Under conditions of competitive markets we have:

$$p\bar{Y} = p^f X + p^k K. \tag{12}$$

We insert the essential information from the above analysis, considering now $\eta(\cdot)$ as a function of p^x and normalise $\bar{Y} = 1$:

$$p = \frac{p^x}{\tilde{\eta}(p^x)} + \theta(p^x). \tag{13}$$

The total derivative is:

$$\mathrm{d}p = \left(\frac{1}{\eta} - \frac{\frac{\partial \eta}{\partial p^x}}{\eta} \frac{p^x}{\eta} + \frac{\partial \theta}{\partial p^x} \right) \mathrm{d}p^x. \tag{14}$$

We see that three effects work on p as p^x changes. First, there is a direct *input cost effect* due to the transformation coefficient: the smaller η is, the higher is the output price increase. Second, it can be counteracted by the *efficiency effect*: the relative change of η is valued with the input price per output unit. The sign of this effect is unambiguously negative because it reduces the pressure of increasing input prices on output prices. However, this reduction is not for free, since it induces the *capital cost effect*; the derivative of θ with respect to p^x is always greater than zero as was already stated in Eq. 11.

The relative magnitudes of the three effects give insights into the technological choice and its implications on distribution. First, the sum of the efficiency and the capital cost effect – the *technology choice effect* – needs to be negative, otherwise the technological choice would not have reduced the costs. Second, the technology choice effect can reduce the input cost effect, if technological alternatives with reasonably higher efficiency are available. However, the input cost effect will always exceed the technology choice effect because otherwise the slope of Θ would be negative, which would be in contradiction with Eq. 6; thus:

$$\frac{1}{\eta} > \frac{\frac{\partial \eta}{\partial p^x}}{\eta} \frac{p^x}{\eta} - \frac{\partial \theta}{\partial p^x}. \tag{15}$$

We are now ready to translate the result of the technology choice problem into the concepts and terminology of macroeconomics. However, the heterogenous capital stock has to be aggregated and there are two concepts:

1. ***Technical capacity:*** Consider the sum of all technological alternatives $\mathcal{K}^t = \sum_i K_i$ in terms of physical output potentials. In the example at hand K_i will be zero for all but one i. The price of this capacity is p_i^k.

2. ***Economic capital:*** Aggregate all technological alternatives valued with their investment costs resulting in an aggregate capital stock $\mathcal{K}^e = \sum_i \iota_i K_i$ in value units. Since only one capital stock will be chosen, K_i will be non-zero for only one i. The price of capital is the interest rate r.

For both concepts p^k and therefore the capital share would be the same. The difference is: Following the technical capacity approach we keep to the classical distinction between physical quantities and prices. The price of capital would be the user costs, but it depends on p^x via cost minimal technology choice; see Eq. 11. In the economic capital case the input argument is an aggregate value – a sum of valued quantities – and hence a deviation of the paradigm of the duality of physical quantities and prices; the latter would now be the interest rate. Now, the value of capital input changes with p^x due to cost minimising technology choice.

Technical capacity: Considering technical capacities as input to the production process, it was shown above that the capital costs p^k and the output production costs p depend on p^x. Hence the top-down macro-economic cost-minimisation problem would be essentially different from the one introduced in Sec. 3, because input prices would now be interdependent.

This has implications on the elasticity of substitution that characterizes the corresponding macro-economic production function. Employing a simple special case we will show that a change of the factor price ratio would result in a change of technology choice that is equivalent to BTC. We write down the condition for BTC for the problem at hand:

$$\frac{\frac{\partial p^k}{p^k}}{\frac{\partial p^x}{p^x}} = 0 \text{ and } \frac{\partial \mathcal{K}^t}{\partial X}\frac{X}{\mathcal{K}^t} \neq 0. \tag{16}$$

The first condition requires that the relative change of p^x leads to the same relative change of p^k. This is possible since the partial derivative of $\theta(\cdot)$ can have any positive value. Thus, X decreases since a technology with improved conversion efficiency is employed, but $\mathcal{K}^t$ remains constant because the output is given. This is exactly BTC. Hence, the concept of technical capacities implies that the elasticity of substitution cannot be sensibly applied.

Economic capital: The prices considered now are p^x and r that are independent of each other. Now the quantity of capital $\mathcal{K}^e$ changes with p^x due to optimal technology choice. The share of capital is defined as $s^k \equiv \frac{r\iota}{p} = \frac{p^k}{p}$. Due to the definition of total costs we can say that the elasticity of substitution between input and capital $\sigma_{X,\mathcal{K}^e}$ is related to changes in p^x and p^k in the following way:

$$\sigma_{X,\mathcal{K}^e} \equiv \frac{\partial \frac{X}{\mathcal{K}^e}\frac{\mathcal{K}^e}{X}}{\partial \frac{p^x}{r}\frac{r}{p^x}} \lesseqgtr 1 \qquad \Leftrightarrow \qquad \frac{dp^x}{\eta} \lesseqgtr dp^k. \tag{17}$$

This condition is combined with Eq. 15:

$$\frac{dp^x}{\eta} \lesseqqgtr dp^k \quad \Leftrightarrow \quad 1 - \frac{\partial \eta}{\partial p^x}\frac{p^x}{\eta} \lesseqqgtr \frac{\partial \theta}{\frac{\partial p^x}{\eta}}. \tag{18}$$

It says that the distribution among the factors depends only on the input price per output unit and the capital price (indicated right to the double arrow). Both change with p^x, but the relative magnitudes are not given *a priori.* for example: if the capital costs increase more significantly than the input costs per output unit, the capital suppliers gain share as the input price increases. This requires that the partial derivative of capital costs with respect to input costs per output unit are higher than one minus the price elasticity of the transformation coefficient. Combined with condition Eq. 17 this implies an elasticity of substitution smaller than one.

In summary, the elasticity of substitution is only with the economic concept of capital. The concept of capital as physical capacities and the user costs of capital lead to an interrelationship between input and capital prices that includes absurd cases. The elasticity of substitution between input and economic capital may have any value greater zero due to technology choice, although each technological alternative is characterised by fixed coefficients.

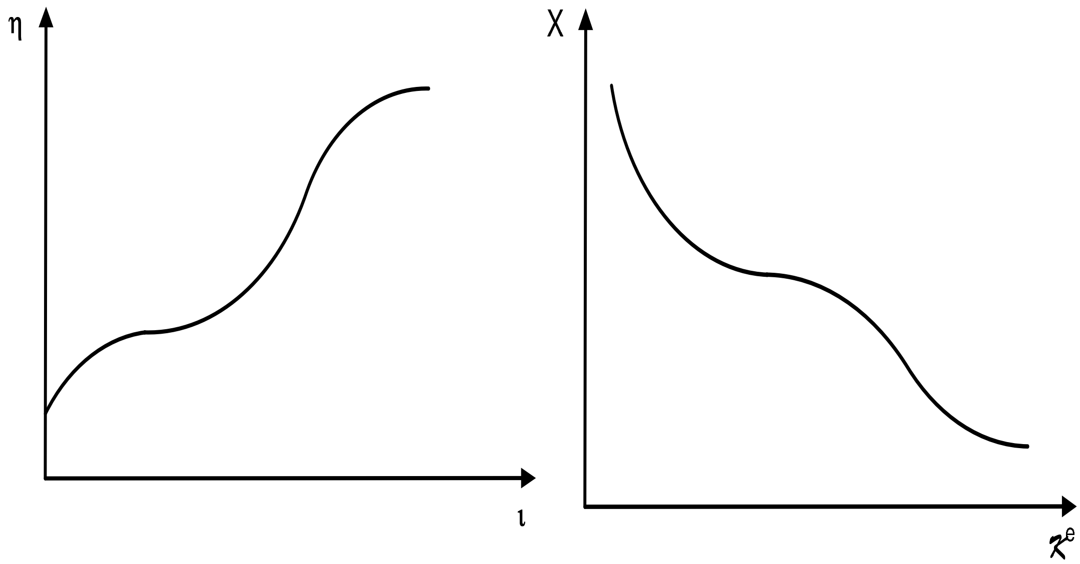
4.3 Non-Convex Iso-quants

In this section the iso-quants will be derived from the book of blueprints function $\eta = \eta(\iota)$ applying the concept of economic capital. The iso-quants can be non-convex depending on the shape of the available technologies. We proceed in two steps of which the first one entails the derivation of an iso-quant from the book of technologies and deriving conditions on $\eta(\cdot)$ for the iso-quant being non-convex. In the second step we will show that the non-convexity of the iso-quant implies that the function $\Theta(\cdot)$ – and thus $\theta(\cdot)$ – are not continuously differentiable everywhere. The kink will be located at the point where an increase in the input price leads to a leap-frog of technologies.

We assume that a fixed output $\bar{Y} = 1$ has to be produced. This implies that the input is $X = \frac{1}{\eta(\iota)}$ and the capital requirement in monetary terms is $\mathcal{K}^e = \iota$.[6] Thus, we can write a simple relationship between X and $\mathcal{K}^e$:

$$X = \frac{1}{\eta(\mathcal{K}^e)}. \tag{19}$$

[6] *Nota bene*: it could be complained that the units of $\mathcal{K}^e$ and ι are different, however the units are implicitly converted by considering $\bar{Y} = 1$.



(a) Book of blueprints $\eta(\iota)$ (b) Implied iso-quant $(X, \mathcal{K}^e)$

Figure 2: Illustration of the book of blueprints function and the implied iso-quant.

It represents all combinations $(X, \mathcal{K}^e)$ that are sufficient to produce one unit of output, which is simply the textbook definition of an iso-quant. The curvator depends on the shape of the function $\eta(\cdot)$:

$$\frac{\partial^2 X}{\partial \mathcal{K}^{e2}} \gtreqless 0 \quad \Leftrightarrow \quad 2\frac{\eta'}{\eta}\iota \gtreqless \frac{\eta''}{\eta'}\iota. \tag{20}$$

The condition says that if twice the elasticity of the transformation coefficient with respect to the investment costs is less than the corresponding marginal elasticity, then the curvature of the iso-quant is negative. Literally, this describes a situation where small changes in investments improve the conversion factor at increasing rates; however, the effect needs to be sufficiently strong so that the iso-quant's curvature is negative. In this case the iso-quant is the fringe of a non-convex production possibility set, which is at odds with the common neo-classical theory of production. Fig. 2(a) illustrates the book of blueprints function $\eta(\cdot)$ and Fig. 2(b) gives the iso-quant implied by such function. In the following we assume that the curvature of the iso-quant is positive for low and high $\mathcal{K}^e$.

It is well-known in the theory of non-linear programming that optimisation of a function over a non-convex set can lead to multiple local optima. Consequently, changes of exogenous parameters do not always induce smooth changes in endogenous variables anymore as is suggested by the implicit function theorem that is valid for convex problems; see Samuelson (1974). In the following we show that the non-smooth solution behaviour of the non-convex top-down problem is equivalent to leap-frogging technologies in the bottom-up model.

We form the first and second order derivative of the total costs with respect to ι:

$$\frac{\partial p}{\partial \iota} = r - \frac{p^x}{\eta(\iota)}\eta'; \tag{21}$$

$$\frac{\partial^2 p}{\partial \iota^2} = -\frac{p^x}{\eta^2(\iota)}\left(2\frac{\eta'}{\eta}\eta + \eta''\right). \tag{22}$$

Obviously, the sufficient condition for a *cost maximum* is the same as for the non-convexity of the iso-quant as can be seen in the term in brackets on the right hand side of Eq. 22. Given the additional assumption for the curvator at very small and great ι the maximum would separate two minima; we call the minimum with lower (higher) ι the low-tech (high-tech) solution. There is some critical level $\tilde{p}^x$ where the two minima are equally good. For input prices lower than this critical value the low-tech solution is superior *et vice versa*. Increasing the input price crossing $\tilde{p}^x$ the technology choice jumps from the low-tech minimum to the high-tech minimum leap-frogging a whole set of technologies. The point is that the input cost effect (see previous section) is more severe for the low-tech solution than for the high-tech soltion. This means that the function $p = \Theta(\cdot)$ is not continuously differentiable at $\tilde{p}^x$:

$$\frac{\partial \Theta}{\partial \check{p}^x} > \frac{\partial \Theta}{\partial \hat{p}^x}, \quad \text{for any } \check{p}^x < \tilde{p}^x < \hat{p}^x. \tag{23}$$

Leap-frogging changes the cost structure and therefore the distribution of income discontinuously. The input suppliers receive significantly less income because the demand will decrease as the technology choice switches from the low to the high-tech solution. Formally, this is due to the function $\theta(\cdot)$ that is not continuous at $\tilde{p}^x$ and because the investment costs increase whereas the input decreases as input price increases induce a switch from the low to the high-tech solution.

4.4 Unique Determination of the Production Function

In this section we derive the top-down production function $F(X, \mathcal{K}^e)$ implied the by bottom-up model. The production function being determined uniquely implies marginal productivities for the two production factors. We show that the marginal productivity of economic capital is diminishing, if and only if the book of blueprints function $\eta(\cdot)$ satisfies the same curvature condition as Eq. 20.

The production function represents the bottom-up system that now has available given positive amounts of financial capital and input quantity. The derivation of the production function is all but trivial because each factor combination needs to be fully employed, otherwise the marginal productivity of the non-exhausted factor would be zero. We will show that such cases are excludeable in the bottom-up model and therefore each increase of one production factor increases the output for every level of availability of the other factor. Hence, marginal productivities are always positive.

The production function $F : \mathcal{K}^e \times X \rightarrow Y$ is the maximum Y that can be produced with any combination $(\mathcal{K}^e, X)$ given the book of blue-prints $\eta(\iota)$. This depends on technology choice, but now it needs to be determined by output maximization not cost minimization as we did so far.

Using the bottom-up model we can ask for the highest possible Y attainable with given quantities $\bar{X}$ and $\bar{\mathcal{K}}^e$ by choosing a technology and employing the available quantities:

$$\underset{\mathcal{K}^t, X \geq 0, \iota}{\text{Max}} \; Y = \eta(\iota)X, \tag{24}$$

$$\text{s.t. } \mathcal{K}^t = \eta(\iota)X, \tag{25}$$

$$\bar{\mathcal{K}}^e \geq \iota \mathcal{K}^t, \tag{26}$$

$$\bar{X} \geq X. \tag{27}$$

Nato bene: the problem does not contain prices r or p^k, which are therefore separated from the production function. The Lagrangian $\mathcal{L}$ is:

$$\mathcal{L} = \eta(\iota)X + \lambda(\mathcal{K}^t - \eta(\iota)X) + \mu(\bar{\mathcal{K}}^e - \iota\mathcal{K}^t) + \nu(\bar{X} - X) + \psi^X X + \psi^K \mathcal{K}^t. \tag{28}$$

For this Lagrangian we have to provide the necessary and sufficient conditions for a maximum. Hence, from the necessary conditions of the Lagrangian we will firstly show that both production factors are always completely exhausted. Then we formulate a simplified maximization problem in which technology choice is the only variable to chose. For this maximization problem we will show that the necessary and sufficient condition for the maximum are satisfied and thus the maximum Y is unique. The necessary conditions for an optimum and the Karush-Kuhn-Tucker conditions are:

$$\frac{\partial \mathcal{L}}{\partial \iota} = \eta' X(1-\lambda) - \mu \mathcal{K}^t = 0; \tag{29}$$

$$\frac{\partial \mathcal{L}}{\partial \mathcal{K}^t} = \lambda - \mu\iota + \psi^K = 0, \qquad \psi^K \mathcal{K}^t = 0; \tag{30}$$

$$\frac{\partial \mathcal{L}}{\partial X} = \eta(1-\lambda) - \nu + \psi^X = 0, \quad \psi^X X = 0; \tag{31}$$

$$0 = \mu(\bar{\mathcal{K}}^e - \iota\mathcal{K}^t); \tag{32}$$

$$0 = \nu(\bar{X} - X). \tag{33}$$

We proof the complete exhaustion of production factors by contradiction. For this purpose we solve for the two multipliers (μ, ν) and show that they must be positive. Eq. 32 – 33 imply that both production factors are completely exhausted. We simplify the system by standard manipulations and substitutions assuming that ψ^K and ψ^X are zero:

$$\mu = \frac{1}{\frac{\eta}{\eta'} + \iota}; \tag{34}$$

$$\nu = \eta(1 - \mu\iota). \tag{35}$$

The parameters on the right hand side of Eq. 34 are all positive and therefore μ is positive. In Eq. 35 ν can only be zero, if $\mu\iota = 1$. Insert this information after multiplying Eq. 34 with ι:

$$1 = \mu\iota = \frac{\iota}{\frac{\eta}{\eta'} + \iota}. \tag{36}$$

Few standard manipulations lead us to $\eta = 0$, which would make the whole problem not worth to discuss. We can conclude that both production factors are always completely exhausted. Therefore, we can write down a much simpler maximization problem:

$$\underset{\iota}{\text{Max}}\ Y = \eta(\iota)\bar{X}, \tag{37}$$

$$\text{s.t. } \bar{\mathcal{K}}^e = \iota\eta(\iota)\bar{X}. \tag{38}$$

The Lagrangian is:

$$\mathcal{L} = \eta(\iota)\bar{X} + \lambda(\bar{\mathcal{K}}^e - \iota\eta(\iota)\bar{X}). \tag{39}$$

The necessary condition for the maximum is:

$$\frac{\partial \mathcal{L}}{\partial \iota} = \eta' - \lambda(\eta + \iota\eta) = 0. \tag{40}$$

The sufficient condition for a maximum requires that the determinant of the bordered Hessian $\mathcal{B}$ is negative:

$$\mathcal{B} = \begin{pmatrix} 0 & -\eta - \iota\eta' \\ -\eta - \iota\eta' & \eta'' - \mu(2\eta' + \iota\eta'') \end{pmatrix}. \tag{41}$$

The determinant of $\mathcal{B}$ is:

$$|\mathcal{B}| = 0 \cdot \left[\eta'' - \mu(2\eta' + \iota\eta'')\right] - (\eta - \iota\eta')^2. \tag{42}$$

The determinant is unambiguously non-positive since the first term is zero and the second is never positive. Hence, the necessary condition could be fulfilled at an inflexion point. However, in that case the sign of $|\mathcal{B}|$ would need to change sign. However, this is not possible because it cannot get positive. We conclude that the maximum is unique. This means that the function $F(\cdot)$ is uniquely determined for every combination $(\mathcal{K}^e, X)$ and therefore marginal increases in inputs increase output. Therefore, the two production factors are used up to their limits implying non-zero multipliers μ and ν that can now be interpreted as marginal productivites $\frac{\partial Y}{\partial \mathcal{K}^e} = \mu$ and $\frac{\partial Y}{\partial X} = \nu$.

We check whether the output Y is exhaustibly distributed among the two production factors:

$$Y = \mu\mathcal{K}^e + \eta(1 - \mu\iota)X. \tag{43}$$

Rearranging terms and using the linear production relation $\eta = \frac{\mathcal{K}^t}{X}$ results in:

$$Y = \eta X + \mu\left(\mathcal{K}^e - X\frac{\mathcal{K}^t}{X}\iota\right). \tag{44}$$

The parenthesis equals zero, leaving the linear production production relationship. Hence, the production function $F(\cdot)$ that represents the bottom-up system provides a theory of distribution based on marginal productivities that equal the corresponding multipliers of the bottom-up model. Since there are not any economies of scales assumed and the conversion technology is linear, the production function can also be written in intensive form:

$$F(\mathcal{K}^e, X) = XF(k, 1) = Xf(k), \quad \text{with } k \equiv \frac{\mathcal{K}^e}{X}. \tag{45}$$

Up to this point we have shown that the bottom-up system can be represented with a top-down production function that is in line with the marginalist theory of distribution. The final point in this section is to provide the condition for the slope of F.

We take the inverse of Eq. 34 and substitute ι:

$$\frac{1}{\mu} = \frac{\eta\left(\frac{\mathcal{K}^e}{\mathcal{K}^t}\right)}{\eta'\left(\frac{\mathcal{K}^e}{\mathcal{K}^t}\right)} + \frac{\mathcal{K}^e}{\mathcal{K}^t}. \tag{46}$$

Forming the partial derivative with respect to $\mathcal{K}^e$ leads to:

$$\frac{\partial 1/\mu}{\partial \mathcal{K}^e} = \frac{1}{\mathcal{K}^t}\left(2 - \frac{\eta}{\eta'}\frac{\eta''}{\eta'}\right). \tag{47}$$

The sign of the derivative depends on the following condition:

$$\frac{\partial \mu}{\partial \mathcal{K}^e} \lesseqgtr 0 \quad \Leftrightarrow \quad \frac{\eta''}{\eta'}\iota \lesseqgtr 2\frac{\eta'}{\eta}\iota. \tag{48}$$

The quest for increasing or diminishing marginal product of capital simply depends on the very same condition on the book of blueprint function as the distinction for convex and concave parts of iso-quants given in Eq. 20. Hence, the marginal product of capital increases with capital, if and only if twice the investment elasticity of the improvement of the transformation coefficient is less than the investment elasticity of the marginal improvement *et vice versa*. Since both conditions are the same and both are necessary and sufficient, we can conclude that increasing marginal products of capital appear, if and only if non-convex iso-quants are present. And from our discussion above it is clear that increasing returns are equivalent with leap-frogging technologies. This shows that the three properties of Clarke-Ramsey need not be fulfilled simultaneously: the marginal productivity of capital is determined and increasing for a succession of technologies.

5 Discussion and Further Research

The final discussion of the results shall point to further research that is related to the discussions provided in Sec. 2 and beyond. There is interesting research to be done in numeric modeling studies and theoretical research, but also in empirical research that was not mentioned explicitly above.

Application to numerical models shall study the macro-economic properties that characterize detailed bottom-up models. This is very much in the vein of Solow (1962, p. 211-4), who proposed such "empirical" studies, but never – up to my knowledge – carried them out. The results of prices and quantities computed endogenously with a model are interpreted as data that can be analysed with standard econometric tools. Thus, macro-economic indicators help to summarize the result of the top-down model. For example, increasing fuel prices (or carbon prices) over time that lead to changes in technological choice. It would be of great interest to know how the shares of primary energy and capital develop in order to deal with increasing resource scarcity.

Theoretical research on production and distribution could be improved by applying the kind of analysis to different settings of the bottom-up model. For example, two outputs produced by a single input using two different technologies. Demand functions shall value the two outputs so that an aggregate output can be computed. The interesting point will be how techno-economic and demand parameters interact in the determination of the elasticity of substitution.

As Solow (1967, p. 28) mentioned theories on production and distribution always are related to theories of economic growth. In Sec. 2.2 we referred to theories of economic growth, but the present paper lacked the integration of such a production function into a growth framework, in which the production function is combined with a household sector that supplies capital. The quest for economic growth may become even more interesting, if it were possible to integrate the bottom-up model into the growth framework allowing for flexible utilisation of the available capital stocks. This would be in the tradition of Phelps (1963).

Finally, some notes on empirical research are made. As has been said in Sec. 2.3 the critique of neo-classical production functions based on the argument of re-switching of technologies faded away from the center of economic debates partially due to the lack of empirical evidence of re-switching. The present study emphasised the role of technology leap-frogging, which might be more accessible for empirical research. Good examples from history might be technology choice in manufacturing like the transition from hand looms to weaving machines. In energy economics the household choice in developing countries of using inefficient and polluting equipement (traditional stoves, kerosen lamps, etc.) instead of replacing it by a equipment that may require relatively little capital, but is much more efficient and cleaner. Another example from developing countries is the immediate use of mobile phones omitting the telephone set. In studies related to poverty an important issue should be the role of access to capital; see Stiglitz and Weiss (1981). If economic actors are subject to credit rationing, this may may select between multiple optima and leap-frogging is prevented.

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