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Fractional calculus and its applications

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Introduction

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One contribution of 14 to a Theme Issue 'Fractional calculus and its applications'.

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Fractional calculus was formulated in 1695, shortly after the development of classical calculus. The earliest systematic studies were attributed to Liouville, Riemann, Leibniz, etc. [1,2]. For a long time, fractional calculus has been regarded as a pure mathematical realm without real applications. But, in recent decades, such a state of affairs has been changed. It has been found that fractional calculus can be useful and even powerful, and an outline of the simple history about fractional calculus, especially with applications, can be found in Machado *et al.* [3].

Now, *fractional calculus and its applications* is undergoing rapid developments with more and more convincing applications in the real world [4,5]. This Theme Issue, including one review article and 12 research papers, can be regarded as a continuation of our first special issue of *European Physical Journal Special Topics* in 2011 [4], and our second special issue of *International Journal of Bifurcation and Chaos* in 2012 [5]. These selected papers were mostly reported in *The Fifth Symposium on Fractional Derivatives and Their Applications* (FDTA'11) that was held in Washington DC, USA in 2011.

The first paper presents an overview of chaos synchronization of coupled fractional differential systems. A list of coupling schemes are presented, including one-way coupling, Pecora–Carroll coupling, active–passive decomposition coupling, bidirectional coupling and other unidirectional coupling configurations. Meanwhile, several extended concepts of synchronizations are introduced, namely projective synchronization, hybrid projective synchronization, function projective synchronization, generalized synchronization and generalized projective synchronization. Corresponding to different

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kinds of synchronization schemes, various analysis methods are presented and discussed [6]. The rest of the papers can be roughly grouped into three parts: three papers for fundamental theories of fractional calculus [7–9], five papers for fractional modelling with applications [10–14] and four papers for numerical approaches [15–18].

In the theory part, three papers focus on the existence of the solutions to the considered classes of nonlinear fractional systems, the equivalence system of the multiple-rational-order fractional system, and the reflection symmetry with applications to the Euler–Lagrange equations [7–9]. Baleanu *et al.* [7] use fixed-point theorems to prove the existence and uniqueness of the solutions to a class of nonlinear fractional differential equations with different boundary-value conditions. Li *et al.* [8] apply the properties of the fractional derivatives to change the multiple-rational-order system into the fractional system with the same order. Such a reduction makes it convenient for stability analysis and numerical simulations. The reflection symmetry and its applications to the Euler–Lagrange equations in fractional mechanics are investigated in Klimek [9], where an illustrative example is presented.

The part on fractional modelling with applications consists of five papers [10–14]. Chen *et al.* [10] establish a fractional variational optical flow model for motion estimation from video sequences, where the experiments demonstrate the validity of the generalization of derivative order. Another fractional modelling in heat transfer with heterogeneous media is studied in Sierociuk *et al.* [11]. In the following paper, two-particle dispersion is explored in the context of the anomalous diffusion, where two modelling approaches related to time subordination are considered and unified in the framework of self-similar stochastic processes [12]. The last two papers in this part emphasize the applications of fractional calculus [13,14], where a novel method for the solution of linear constant coefficient fractional differential equations of any commensurate order is introduced in the former paper, and where the CRONE control-system design toolbox for the control engineering community is presented in the latter paper.

The last four papers in part three are attributed to numerical approaches [15–18]. Sun *et al.* [15] construct a semi-discrete finite-element method for a class of temporal-fractional diffusion equations. On the other hand, an implicit numerical algorithm for the spatial- and temporal-fractional Bloch–Torrey equation is established, where stability and convergence are also considered [16]. In Fukunaga & Shimizu [17], a high-speed scheme for the numerical approach of fractional differentiation and fractional integration is proposed. In the last paper, Podlubny *et al.* [18] further develop Podlubny's matrix approach to discretization of non-integer derivatives and integrals, where non-equidistant grids, variable step lengths and distributed orders are considered.

We try our best to organize this Theme Issue in order to offer fresh stimuli for the fractional calculus community to further promote and develop cutting-edge research on fractional calculus and its applications.

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