Pinning Distributed Synchronization of Stochastic Dynamical Networks: A Mixed Optimization Approach

Yang Tang, *Member, IEEE*, Huijun Gao, *Fellow, IEEE*, Jianquan Lu, *Member, IEEE*, and Jürgen Kurths

***Abstract*— This paper is concerned with the problem of pinning synchronization of nonlinear dynamical networks with multiple stochastic disturbances. Two kinds of pinning schemes are considered: 1) pinned nodes are fixed along the time evolution and 2) pinned nodes are switched from time to time according to a set of Bernoulli stochastic variables. Using Lyapunov function methods and stochastic analysis techniques, several easily verifiable criteria are derived for the problem of pinning distributed synchronization. For the case of fixed pinned nodes, a novel mixed optimization method is developed to select the pinned nodes and find feasible solutions, which is composed of a traditional convex optimization method and a constraint optimization evolutionary algorithm. For the case of switching pinning scheme, upper bounds of the convergence rate and the mean control gain are obtained theoretically. Simulation examples are provided to show the advantages of our proposed optimization method over previous ones and verify the effectiveness of the obtained results.**

***Index Terms*— Complex networks, evolutionary algorithms (EAs), multiagent systems, neural networks, stochastic disturbances, synchronization.**

I. Introduction

OVER the past few years, synchronization and consensus problems have been widely investigated in networked dynamic systems [1]–[8] and diverse applications have been found in the fields of coupled oscillators, heart beating, neuronal networks, flocking, formation control, gossip algorithms, rendezvous in space, artificial intelligence, distributed sensor fusion in sensor networks, and belief propagation [9]–[14].

In realistic systems, the agent systems or complex networks often suffer from noisy environment and, therefore, the stochastic modeling approach has been used to represent stochastic events in many branches of science, such as neuroscience, communication networks, and networked systems. To model the dynamics of networks in a more realistic way, many efforts have been devoted to problems of external stochastic disturbances for the synchronization of stochastic linear/nonlinear dynamical networks [15]–[17]. However, the corresponding research for the impacts of stochastic disturbances on synchronization performance of networks/multiagent systems, i.e., mean control gain and convergence rate, has received much less attention, and this constitutes the first motivation for this paper.

For many biological, physical, and social networks, there exists a common requirement to regulate the behavior of large ensembles of interacting units using a small fraction of inputs [7], [16]–[18], which can strongly reduce the control gain compared with a control with more inputs. Recently, different kinds of effective approaches, including adaptive controllers [17], [19], impulsive controllers [16] and pinning state feedback controllers [20]–[22], have been proposed for the coordination and synchronization of complex dynamical networks.

In networked circumstances, multiagent systems/complex dynamical networks are often subjected to a switching environment [1], [5]. One important modeling approach is random switching using Bernoulli stochastic variables, which can be used to represent random switching in topology [23], nonlinearity [15] and control failure [17], and so on. However, the concept of switching pinning scheme has been overlooked widely and the intrinsic connection between the Bernoulli switching pinning scheme and the usual pinning strategy has not been studied, which is the second incentive of this paper.

Usually, the statistical properties of networks are used to set the pinned nodes at the beginning and several criteria should be satisfied to ensure the synchronization of dynamical networks [17], [19], [24]. However, such kinds of methods lead to unavoidable conservativeness, since the selection of driver nodes is a combinatorial optimization problem and thus is a natural NP-hard problem. To improve the selection of driver nodes, several initial attempts have been made to convert the selection of driver nodes into a single objective optimization

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problem [10], [25], a constraint optimization problem [7], and a multiobjective optimization problem [14]. Nevertheless, the optimization of control gains is a continuous optimization problem, which greatly imposes a serious burden on computational resources. Therefore, natural questions arise: is it possible to narrow such a gap using evolutionary algorithms (EAs) to select driver nodes without designing control gains and find feasible solutions for obtained criteria in terms of convex optimization methods? Is it feasible to take advantage of convex optimization methods for matrix computation and EAs for combinatorial optimization problems? The third purpose of this paper is to pave the way for dealing with the pinning distributed synchronization of stochastic complex networks using a novel mixed optimization method.

Summarizing the above discussions, the focus of this paper is on the pinning distributed synchronization problem of nonlinear networked systems with multiple stochastic disturbances using a novel mixed optimization method. The main contributions of this paper lie in the study on the relationship of two kinds of pinning schemes and a novel optimization method, which can be summarized as follows: 1) two kinds of pinning schemes are investigated and their relationship is figured out; 2) the effects of multiple stochastic disturbances on synchronization performance are illustrated; and 3) the optimization problem of pinning distributed synchronization is investigated by a newly developed mixed optimization method.

This paper is organized as follows. Section II presents the model. In Section III, several conditions are presented to guarantee the pinning distributed synchronization of stochastic networked systems in mean square sense. Section IV presents the developed optimization method. Several examples are given in Section V to verify the effectiveness of the proposed method. Concluding remarks are given in Section VI.

II. Preliminaries

In this section, the model of nonlinear dynamical networks with multiple stochastic disturbances as well as some basic concepts of constraint optimization problems are presented.

Before presenting the results, the following notations are required.

Notations: In this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ indicate, respectively, the n-dimensional Euclidean space and the set of all real $n \times m$-dimensional matrices. The superscript T represents matrix transposition and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). I_m is the identity matrix of order m. The Kronecker product of matrices $X \otimes Y \in \mathbb{R}^{mp\times nq}$, where $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{p\times q}$. $\|\cdot\|$ is the Euclidean vector norm in $\mathbb{R}^n$. Let a graph be $\mathcal{G} = [\mathcal{V}, \mathcal{E}]$, where $\mathcal{V} = \{1, \ldots, N\}$ stands for the vertex set and $\mathcal{E} = \{e(i, j)\}$ is the edge set. $\mathcal{N}_i$ represents the neighborhood of vertex i in the sense $\mathcal{N}_i = \{j \in \{\mathcal{V} : e(i, j) \in \mathcal{E}\}$. The graph $\mathcal{G}$ is supposed to be connected and undirected, ($e(i, j) \in \mathcal{E}$ implies $e(j, i) \in \mathcal{E}$) and simple (without self-loops and multiple edges). Let $L = [a_{ij}]_{i,j=1}^N$ be the Laplacian matrix of graph $\mathcal{G}$, which is defined as: for any pair $i \neq j, a_{ij} = a_{ji} = -1$ if $e(i, j) \in \mathcal{E}$; otherwise, $a_{ij} = a_{ji} = 0$. $a_{ii} = -\sum_{j=1, j\neq i}^N a_{ij}$ is the degree of vertex i ($i \in \mathcal{V}$). The symbol $\#\mathcal{Q}$ stands for the number of elements in finite set $\mathcal{Q}$ composed of the vertices to be controlled. For $\mathcal{Q} \subset \mathcal{V}$ and $\forall i \in \mathcal{V}\backslash\mathcal{Q}$, there exists at least one vertex $j \in \mathcal{Q}$ such that a path between the vertices i and j exists. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space, where Ω represents a sample space, $\mathcal{F}$ is a σ-algebra and $\mathcal{P}$ is a probability measure. $\mathbb{E}\{a\}$ stands for the expectation of a. Prob{.} denotes the probability of one event.

A. Nonlinear Dynamical Networks With Multiple Stochastic Disturbances

Here, we consider complex networks, which are composed of identical nodes with multiple stochastic disturbances as follows:

$$dx_i(t) = \left[f(x_i, t) + k\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))\right]dt + \sum_{m\in\mathcal{A}}\sigma_m(x_i, t)dw_m(t), \quad i \in \mathcal{V} \tag{1}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n (i \in \mathcal{V})$ denotes the state vector of the ith node; $f(x_i, t) = [f_1(x_i, t), \ldots, f_n(x_i, t)]^T$ is a continuous nonlinear function to stand for the node's dynamics; k is the global coupling strength of the network; $\sigma_m(\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the noise intensity function, where $\mathcal{A}$ is the set $\mathcal{A} = \{1, \ldots, q\}$ for different noise intensities; $w_m(t) (m \in \mathcal{A})$ is 1-D Brownian motion defined on $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying $\mathbb{E}\{dw_m(t)\} = 0$ and $\mathbb{E}\{[dw_m(t)]^2\} = dt$. According to Gershgorin disk theorem [26], all the eigenvalues of L corresponding to graph $\mathcal{G}$ satisfy the following relationship $0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_N(L)$. In addition, $\mathcal{G}$ is connected if and only if $\lambda_2(L) > 0$. Since $\mathcal{G}$ is connected, we conclude that $(L \otimes I_n)x = 0$ if and only if $e_{ij} = 0$ holds for $\forall i, j = 1, \ldots, N$, where $e_{ij} = x_i - x_j$ is the difference between the states of nodes i and j. Note that N equals to the cardinality of $\mathcal{V}$.

To achieve global synchronization of the complex network in (1) in mean square, distributed controllers $u_i(t)$ are added into the set of driver nodes (pinned nodes)

$$\begin{aligned} dx_i(t) &= \left[f(x_i, t) + k\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t)) + u_i(t)\right]dt \\ &\quad + \sum_{m\in\mathcal{A}}\sigma_m(x_i, t)dw_m(t), \quad i \in \mathcal{Q} \\ dx_i(t) &= \left[f(x_i, t) + k\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))\right]dt \\ &\quad + \sum_{m\in\mathcal{A}}\sigma_m(x_i, t)dw_m(t), \quad i \notin \mathcal{Q}. \end{aligned} \tag{2}$$

In the following, we consider two kinds of pinning mechanisms, which are formulated as follows:

$$\mathcal{Q} = \begin{cases} \mathcal{R}_1, & \mathcal{R}_1 \text{ is a fixed pinning set} \\ \mathcal{R}_2(t), & \mathcal{R}_2(t) \text{ is a time-varying pinning set} \end{cases} \tag{3}$$

where $\mathcal{R}_1$ stands for the fixed pinning set and $\mathcal{R}_2(t)$ is the stochastic switching pinning set according to a set of Bernoulli variables.

The distributed controllers $u_i(t)$ are written as follows:

$$u_i(t) = \sum_{j\in\mathcal{N}_i} \epsilon_i(t)(x_j(t) - x_i(t)), \qquad i \in \mathcal{Q} \tag{4}$$

where $\epsilon_i(t)$ is the control gain of vertex i, which is updated according to the following:

$$\begin{aligned} d\epsilon_i(t) = \alpha\Bigg[\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))\Bigg]^T \\ \times\Bigg[\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))\Bigg]dt, \qquad i \in \mathcal{Q} \end{aligned} \tag{5}$$

where $\alpha > 0$.

The Bernoulli switching pinning set $\mathcal{R}_2(t)$ is time-varying according to a set of Bernoulli variables $\phi_i(t)$, $(i \in \mathcal{V})$, which describe the following events:

$$\begin{cases} \text{node } i \in \mathcal{R}_2(t), & \text{if } \phi_i(t) = 1 \\ \text{node } i \notin \mathcal{R}_2(t), & \text{if } \phi_i(t) = 0 \end{cases} \tag{6}$$

where $\text{Prob}\{\phi_i(t) = 1\} = \mathbb{E}\{\phi_i(t)\} = \phi_i \in [\check{\phi}, \hat{\phi}] \subseteq [0, 1]$. Here, we assume that all Bernoulli variables $\phi_i(t)$ $(i \in \mathcal{V})$ and the Brownian motions $\omega_m(t)$ $(m \in \mathcal{A})$ are independent of each other. The Bernoulli variables follow an unknown but exponential distribution of switchings and the probabilities should be known *a priori* [15].

Remark 1: In most of the existing results regarding pinning synchronization problems, fixed pinning schemes are considered, see [7], [10], [17], [18], [20], [21], [24], [25], [27], and references therein. Different from these works, we consider two kinds of pinning mechanisms: the first one follows the idea in [7], [10], [17], [18], [20], [21], [24], [25], [27], and the second one is a time-varying pinning set to be updated according to a binary switching. Actually, the first pinning scheme is a special case of the second one. One can set some nodes in $\mathcal{V}$ to be always chosen (by setting the probabilities $\phi_i = 1, i \in \mathcal{Q}$) and some nodes to be never selected as driver nodes ($\phi_i = 0, i \notin \mathcal{Q}$). In this special case, the second pinning scheme reduces to the first pinning one.

Remark 2: In the fixed pinning set $\mathcal{R}_1$, only a small fraction of nodes are allowed to assess the distributed controllers along the time evolution. However, in the Bernoulli switching pinning set $\mathcal{R}_2(t)$, each node in set $\mathcal{V}$ has the possibility of being chosen as a pinned node. To make a connection between $\mathcal{R}_1$ and $\mathcal{R}_2(t)$, we give the following descriptions. Since $\mathcal{R}_2(t)$ is stochastic switching and time-varying, we can only calculate the expectation of the element number in $\mathcal{R}_2(t)$ as $\mathbb{E}\{\#\mathcal{R}_2(t)\}$. Assuming that $\mathbb{E}\{\#\mathcal{R}_2(t)\} = \#\mathcal{R}_1$ holds, the equality shows that the expectation of the number of pinned nodes equals to that of in the fixed pinning set $\mathcal{R}_1$. More precisely, we define the pinning portion $p_{\mathcal{Q}}$ for $\mathcal{R}_1$ and $\mathcal{R}_2(t)$, which has the following relationship if $\mathbb{E}\{\#\mathcal{R}_2(t)\} = \#\mathcal{R}_1$:

$$\begin{aligned} p_{\mathcal{Q}} &= \frac{\#\mathcal{R}_1}{N} \\ &= \frac{\mathbb{E}\{\#\mathcal{R}_2(t)\}}{N} \\ &= \frac{\sum_{i\in\mathcal{V}} \text{Prob}\{\phi_i(t) = 1\}}{N} \\ &= \frac{\sum_{i\in\mathcal{V}} \phi_i}{N}. \end{aligned} \tag{7}$$

Remark 3: Different from the usage of Bernoulli variables in [17], the Bernoulli variables here are used to characterize a pinning set in networked systems, while the Bernoulli variables in these works are used to characterize the failure of distributed controllers. In [17], only a fixed pinning scheme is considered. That is, for the nodes in $\mathcal{R}_1$, the controllers might suffer from the failure to activate. Therefore, only a part of nodes $i \in \hat{\mathcal{R}}_2(t) \subseteq \mathcal{R}_1$ can be activated in $\mathcal{R}_1$. Here, Bernoulli variables are just used to characterize the switching pinning set $\mathcal{R}_2(t)$. In this paper, two kinds of pinning strategies are considered for investigating the problem of pinning distributed synchronization of stochastic networked systems.

Based on the above expression and for the sake of mathematical derivation, system (2) under $\mathcal{R}_1$ and $\mathcal{R}_2(t)$ can be written as follows, respectively:

$$\begin{aligned} dx_i(t) &= \Bigg[f(x_i, t) + k\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t)) + u_i(t)\Bigg]dt \\ &\quad + \sum_{m\in\mathcal{A}} \sigma_m(x_i, t)d\omega_m(t), \qquad i \in \mathcal{R}_1 \\ dx_i(t) &= \Bigg[f(x_i, t) + k\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))\Bigg]dt \\ &\quad + \sum_{m\in\mathcal{A}} \sigma_m(x_i, t)d\omega_m(t), \qquad i \notin \mathcal{R}_1 \end{aligned} \tag{8}$$

and

$$\begin{aligned} dx_i(t) &= \Bigg[f(x_i, t) + k\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t)) + \phi_i(t)u_i(t)\Bigg]dt \\ &\quad + \sum_{m\in\mathcal{A}} \sigma_m(x_i, t)d\omega_m(t), \qquad i \in \mathcal{V}. \end{aligned} \tag{9}$$

Remark 4: The model (8) or (9) is quite general, which includes the self-nonlinear dynamics $f(.,.)$, the coupling term and multiple stochastic disturbances $\sigma_m(.,.)$. The multiple stochastic disturbances $\sum_{m\in\mathcal{A}} \sigma_m(x_i, t)d\omega_m(t)$ characterize the noises coming from different sources in a networked environment. Therefore, the stochastic perturbation term here encompasses the one in [17], which was also neglected in [24]. In addition, one of the main characteristics of this paper is to study the effects of stochastic disturbances on the synchronization performance of dynamical networks. Compared with [28], there are major differences in both research problems and optimization methods. In this paper, the optimization method is a mixed optimization method including both constraint evolutionary computation methods and semidefinite programming (SDP). In addition, two kinds of pinning schemes are considered and stochastic disturbances are included in this paper. In [28], using SDP, the synchronization was investigated for the network with multiple randomly occurring events including nonlinearities under random switchings.

For the switching pinning set $\mathcal{R}_2(t)$, we introduce two measures to characterize the average of terminal control strength and convergence rate, respectively

$$\mathcal{M}_c = \frac{1}{N}\sum_{i\in\mathcal{V}}\epsilon_{i,\infty} \tag{10}$$

and

$$\mathcal{M}_s = \mathbb{E}\left\{\int_0^\infty \frac{1}{N-1}\sum_{i\in\mathcal{V}}[x_i(t)-\bar{x}(t)]^T[x_i(t)-\bar{x}(t)]dt\right\} \tag{11}$$

where $\bar{x}(t) = (1/N)x(t)$ and $\epsilon_{i,\infty} = \lim_{t\to\infty}\mathbb{E}\{\epsilon_i(t)\}$. It is worth mentioning that a good synchronization performance indicates a high convergence rate (a small $\mathcal{M}_s$) and low mean control gain (a small $\mathcal{M}_c$), as shown in (10) and (11).

The following assumptions and definitions are necessary to derive our main results.

Assumption 1 [17]: Functions $f(x_i,t)$ and $\sigma_m(x_i,t)$ are said to be locally uniformly Lipschitz continuous with respect to t if there exists positive constants h_1 and g_i, $(m \in \mathcal{A})$ such that the following inequalities hold for all $x_i, x_j \in \mathbb{R}^n$:

$$\begin{aligned}&\|f(x_i,t)-f(x_j,t)\| \le h_1\|x_i-x_j\|\\ &\|\sigma_m(x_i,t)-\sigma_m(x_j,t)\| \le g_m\|x_i-x_j\|, \qquad i,j\in\mathcal{V}, m\in\mathcal{A}.\end{aligned} \tag{12}$$

Assumption 2 [29]: A vector-valued continuous function $f(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is said to be uniformly decreasing if there exist $\psi > 0 \in \mathbb{R}$ and $\Delta > 0 \in \mathbb{R}$ such that

$$(x-y)^T[f(x,t)-f(y,t)-\psi(x-y)] \le -\Delta(x-y)^T(x-y) \tag{13}$$

holds for all $x, y \in \mathbb{R}^n$ and $t \ge 0$.

Assumption 3: $f(0,t) = 0$ and $\sigma_m(0,t) = 0, m \in \mathcal{A}$.

Definition 1: Let $x_i(t)(1 \le i \le N)$ be a solution of the stochastic complex network in (2), where $x_i(0) = (x_1^0, x_2^0, \ldots, x_n^0)$. If for all $t \ge t_0, 1 \le i \le N$

$$\lim_{t\to\infty}\mathbb{E}\|x_i(t)-x_j(t)\|^2 = 0, \qquad i,j\in\mathcal{V}$$

then the complex network with multiple stochastic disturbances in (2) is said to achieve pinning distributed synchronization in mean square.

B. Preliminaries for Constraint Optimization Problems and Evolutionary Algorithms

In general, the constrained optimization problem can be written as follows: find the decision variables $\vec{x} = (x_1,\ldots,x_D) \in \mathbb{R}^D$ to minimize the objective function

$$\min \quad F_j(\vec{x}), \quad \vec{x}\in\Omega\subseteq\mathcal{S}$$

where Ω is the feasible region and $\mathcal{S}$ is the decision space defined by the parametric constraints $L_i \le x_i \le U_i$, $i = 1,2,\ldots,D$. $\vec{x}$ should satisfy r constraints including $\imath$ inequality constraints

$$Q_j(\vec{x}) \le 0, \qquad j = 1,2,\ldots,\imath$$

and $\eta = m - \imath$ equality constraints

$$H_j(\vec{x}) = 0, \qquad j = \imath+1,2,\ldots,r.$$

The degree of constraint violation of a vector $\vec{x}$ on the jth constraint is defined as

$$M_j(\vec{x}) = \begin{cases}\max\{0, Q_j(\vec{x})\}, & 1\le j\le\imath\\ \max\{0, |H_j(\vec{x})|\}, & \imath+1\le j\le r.\end{cases} \tag{14}$$

Then, $\Sigma(\vec{x}) = \sum_{j=1}^r M_j(\vec{x})$ shows the degree of constraint violation of the vector $\vec{x}$.

EAs are artificial intelligence search approaches inspired by natural selection and survival of the fittest in biological society [30]. Recently, constraint optimization EAs (COEAs) have been significantly improved for dealing with constraint optimization problems. COEAs are made up of two major parts: a search algorithm and a constraint-handling approach, where the search algorithm is aimed to enhance the exploration and exploitation abilities of the population and the constraint-handling approach concentrated on incorporating the constraints into EAs. In the following, more details regarding COEAs will be given.

III. Conditions and Upper Bounds for Pinning Synchronization of Stochastic Dynamical Networks

In this section, the pinning synchronization problem of the dynamical network in (8) and (9) will be investigated under two pinning sets R_1 and $R_2(t)$, respectively. Upper bounds of $\mathcal{M}_c$ and $\mathcal{M}_s$ are derived for $\mathcal{Q} = R_2(t)$. The proof of the following results are based on [17] and [21].

A. Synchronization of Dynamical Networks Under $\mathcal{Q} = R_1$

Theorem 1: For $\mathcal{Q} = R_1$, suppose that $f(.,t)$ is continuous on $(.,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and satisfies Assumptions 1–3, $\sigma_m(.,t)$, $(m \in \mathcal{A})$ satisfies Assumptions 1 and 3 and the graph $\mathcal{G}$ is connected. If there exists a positive constant a such that the following inequality holds:

$$\left[\left(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}}g_m^2\right)I_N - aL\hat{I}_N - kL\right]L \le 0 \tag{15}$$

where

$$\hat{I}_N(i,j) = \begin{cases}1, & \text{if } i=j, \delta_{\mathcal{R}_1}(i) = 1\\ 0, & \text{otherwise}\end{cases}$$

then the stochastic network in (8) under (4) and (5) will be globally synchronized in mean square and the coupling strengths will converge in mean.

Proof: Let $e_{ij} = x_i - x_j, \forall i,j \in \mathcal{V}$. Define $x = [x_1^T,\ldots,x_N^T]^T \in \mathbb{R}^{nN}$. Choose the following Lyapunov candidate

$$V(t) = \frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}e_{ij}^Te_{ij} + \sum_{i\in\mathcal{R}_1}\frac{1}{2\alpha}(\epsilon_i(t)-a)^2 \tag{16}$$

where a is a positive constant.

By the Itô-differential formula [31], the stochastic derivative of V can be obtained as follows:

$$dV(t) = \mathscr{L}V(t)dt + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{m\in\mathcal{A}} e_{ij}^T \times[\sigma_m(x_i(t),t) - \sigma_m(x_j(t),t)]dw_m(t). \quad (17)$$

To satisfy Definition 1, we need to prove that $\mathbb{E}\{dV(t)/dt\} \le 0$. Note that taking the expectation of both sides of (17), $\mathbb{E}\{dV(t)\} = \mathbb{E}\{\mathscr{L}V(t)dt\} \le 0$, since $\mathbb{E}\{dw_m(t)\} = 0$. Therefore, in the following, we only need to prove that $\mathbb{E}\{\mathscr{L}V(t)dt\} \le 0$. The operator $\mathscr{L}$ is given as follows according to (8):

$$\begin{aligned}\mathscr{L}V(t) = {} & \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T\Big\{(f(x_i,t) - f(x_j,t)) + k\sum_{k\in\mathcal{N}_i}(x_k - x_i) - k\sum_{r\in\mathcal{N}_j}(x_r - x_j)\Big\} \\ & + \frac{1}{2}\sum_{i\in\mathcal{R}_1}\sum_{j\in\mathcal{N}_i} e_{ij}^T\Big\{\epsilon_i(t)\Big[\sum_{k\in\mathcal{N}_i}(x_k - x_i)\Big] - \epsilon_j(t)\Big[\sum_{r\in\mathcal{N}_j}(x_r - x_j)\Big]\Big\} \\ & + \sum_{i\in\mathcal{R}_1}(\epsilon_i(t) - a)\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big]^T\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big] \\ & + \frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\sum_{m\in\mathcal{A}}[\sigma_m(x_i(t),t) - \sigma_m(x_j(t),t)]^T \\ & \times[\sigma_m(x_i(t),t) - \sigma_m(x_j(t),t)]. \end{aligned} \quad (18)$$

Following the definition of e_{ij}, x, L and Assumption 2, we obtain

$$\begin{aligned} & \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T e_{ij} = x^T(L\otimes I_n)x, \\ & \sum_{i\in\mathcal{V}}\Big(\sum_{j\in\mathcal{N}_i} e_{ij}\Big)^T\Big(\sum_{j\in\mathcal{N}_i} e_{ij}\Big) = x^T(L^2\otimes I_n)x, \\ & \mathbb{E}\Big\{\sum_{i\in\mathcal{R}_1}\sum_{j\in\mathcal{N}_i} e_{ij}^T\epsilon_i(t)\sum_{k\in\mathcal{N}_i} e_{ki}\Big\} = -\sum_{i\in\mathcal{R}_1}\epsilon_i(t)\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big]^T\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big] \\ & \times\frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\sum_{m\in\mathcal{A}}[\sigma_m(x_i(t),t) - \sigma_m(x_j(t),t)]^T \\ & \times[\sigma_m(x_i(t),t) - \sigma_m(x_j(t),t)] \le \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2 x^T(L\otimes I_n)x. \end{aligned} \quad (19)$$

Using Assumptions 1, (16), (18), and (19), we have

$$\begin{aligned}\mathscr{L}V(t) \le {} & \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T\{[f(x_i,t) - f(x_j,t) - \psi e_{ij}]\} \\ & + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\psi e_{ij}^T e_{ij} - k\sum_{i\in\mathcal{V}}\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big]^T\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big] \\ & - a\sum_{i\in\mathcal{R}_1}\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big]^T\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big] \\ & + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2 x^T(L\otimes I_n)x. \end{aligned} \quad (20)$$

Using Assumption 2 and (19), we yield from (20)

$$\begin{aligned}\mathscr{L}V(t) \le {} & -\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\Delta e_{ij}^T e_{ij} + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\psi e_{ij}^T e_{ij} \\ & - k\sum_{i\in\mathcal{V}}\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big]^T\Big[\sum_{j\in\mathcal{N}_i} e_{ij}\Big] \\ & + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2 x^T(L\otimes I_n)x \\ & - a\sum_{i\in\mathcal{R}_1}\Big[\sum_{j\in\mathcal{N}_i} e_{ij}]^T[\sum_{j\in\mathcal{N}_i} e_{ij}\Big] \\ = {} & -\Delta x^T(L\otimes I_n)x + x^T\Big\{\Big[(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2)I_N - aL\hat{I}_N - kL\Big]L\otimes I_n\Big\}x \\ \le {} & -\Delta x^T(L\otimes I_n)x. \end{aligned} \quad (21)$$

Therefore, pinning synchronization of the complex network in (8) will be achieved in mean square. This completes the proof. ■

Remark 5: From Theorem 1, (15) should be satisfied to ensure mean square pinning synchronization of the dynamical network in (8), in which a fixed pinning set is considered. However, $\hat{I}_N$ is a singular matrix to denote whether one node is chosen as a driver node or not, which will lead to much difficulty to satisfy this inequality. One has to determine the pinned nodes first and then use a convex optimization method to satisfy (15), as shown in [17]. Recently, a large number of methods have been proposed to select driver nodes (pinned nodes), such as degree-based methods, betweenness centrality-based methods and closeness-based methods [25]. However, it has been shown in [18] that such kinds of methods are easy to apply but fail to achieve satisfactory performance, since they inevitably induce conservativeness. In [7] and [25], we have proposed improved single objective EAs (SOEAs) and constraint optimization EAs (COEAs) to select pinned nodes for pinning synchronization of complex networks.

Remark 6: It is worth mentioning that although improved EAs have been proposed for identifying driver nodes of complex networks and biological networks [7], [10], [25], we have to design the control gains, which will greatly induce complexity of the optimization problems. Fortunately, adaptive control can avoid this problem by updating control gains according to the environment. In addition, the controllability in [7], [25], and [10] is a so-called local controllability, while in this paper, the problem of global controllability of complex networks is investigated. In the proposed method, the control gains are updated according to adaptive laws and EAs are used to identify key nodes to inject controllers, which greatly reduces complexity. Hence, our work improves the optimization methods in [7], [10], [17], and [25] by proposing a novel mixed optimization method.

B. Synchronization of Dynamical Networks Under $\mathcal{Q} = R_2(t)$

In the following, we will investigate the pinning synchronization of the dynamical network in (9) with multiple stochastic disturbances, in which the Bernoulli switching pinning set $\mathcal{R}_2(t)$ is used. It is worth mentioning that we can extend the results in Theorem 1 into Theorem 2.

Theorem 2: For $\mathcal{Q} = R_2(t)$, suppose that $f(.,t)$ is continuous on $(.,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and satisfies Assumptions 1–3, $\sigma_m(.,t), (m \in \mathcal{A})$ satisfies Assumptions 1 and 3 and the graph $\mathcal{G}$ is connected. If there exists a positive constant a such that the following inequality holds:

$$\left[\left(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2\right) I_N - aL\mathcal{I} - kL\right] L \le 0 \tag{22}$$

where $\mathcal{I} = \mathrm{diag}\{\phi_1, \ldots, \phi_N\}$, then the stochastic network in (9) under (4) and (5) will be globally synchronized in mean square and the coupling strengths will converge in mean.

Proof: Consider the following Lyapunov candidate:

$$V(t) = \frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}(i)} e_{ij}^T e_{ij} + \sum_{i\in\mathcal{V}} \frac{\phi_i}{2\alpha}(\epsilon_i(t) - a)^2 \tag{23}$$

where a is a positive constant.

Similar to the proof of Theorem 1, the operator $\mathscr{L}$ is given as follows:

$$\begin{aligned}\mathbb{E}\{\mathscr{L}V(t)\} \le \mathbb{E}\Bigg\{&\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T\Bigg\{(f(x_i,t) - f(x_j,t)) \\ &- \psi e_{ij} + k\sum_{k\in\mathcal{N}_i}(x_k - x_i) \\ &- k\sum_{r\in\mathcal{N}_j}(x_r - x_j) + \phi_i(t)\epsilon_i(t)\left[\sum_{k\in\mathcal{N}_i}(x_k - x_i)\right] \\ &- \phi_j(t)\epsilon_j(t)\left[\sum_{r\in\mathcal{N}_j}(x_r - x_j)\right]\Bigg\} \\ &+\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\psi e_{ij}^T e_{ij} \\ &+ \sum_{i\in\mathcal{V}}\phi_i(\epsilon_i(t) - a)\left[\sum_{j\in\mathcal{N}_i} e_{ij}\right]^T\left[\sum_{j\in\mathcal{N}_i} e_{ij}\right] \\ &+ \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2 x^T(L\otimes I_n)x\Bigg\}. \end{aligned} \tag{24}$$

Using Assumption 2, it can be checked from (24) that

$$\begin{aligned}\mathbb{E}\{\mathscr{L}V(t)\} &\le \mathbb{E}\Bigg\{-\Delta x^T(L\otimes I_n)x \\ &\quad + x^T\Bigg\{\Bigg[\left(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2\right) I_N \\ &\quad - aL\mathcal{I} - kL\Bigg] L\otimes I_n\Bigg\}x\Bigg\} \\ &\le \mathbb{E}\{-\Delta x^T(L\otimes I_n)x\}. \end{aligned} \tag{25}$$

Following the proof of Theorem 1, the proof is completed. ■

If $\phi_i = \phi$ $(i \in \mathcal{V})$, then one can obtain the following results.

Corollary 1: For $\mathcal{Q} = R_2(t)$, suppose that $f(.,t)$ is continuous on $(.,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and satisfies Assumptions 1–3, $\sigma_m(.,t), (m \in \mathcal{A})$ satisfies Assumptions 1 and 3 and the graph $\mathcal{G}$ is connected. If there exists a positive constant a such that the following inequality holds:

$$\left[\left(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2\right) I_N - a\phi L - kL\right] L \le 0 \tag{26}$$

then the stochastic network in (9) under (4) and (5) will be globally synchronized in mean square and the coupling strengths will converge in mean.

Following the matrix decomposition theory [26], one can further have the following results from Corollary 1.

Theorem 3: For $\mathcal{Q} = R_2(t)$, suppose that $f(.,t)$ is continuous on $(.,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and satisfies Assumptions 1–3, $\sigma_m(.,t), (m \in \mathcal{A})$ satisfies Assumptions 1 and 3 and the graph $\mathcal{G}$ is connected. If there exists a positive constant a such that the following inequality holds:

$$\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2 - a\phi\lambda_i(L) - k\lambda_i(L) < 0, \qquad i \in \mathcal{V}\backslash\{1\} \tag{27}$$

then the stochastic network in (9) under (4) and (5) will be globally synchronized in mean square and the coupling strengths will converge in mean.

Proof: According to (25) of Theorem 2

$$\begin{aligned}\mathbb{E}\{\mathscr{L}V(t)\} \le \mathbb{E}\Bigg\{&-\Delta x^T(L\otimes I_n)x + x^T\Bigg\{\Bigg[\left(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2\right) I_N \\ &- a\phi L - kL\Bigg] L\otimes I_n\Bigg\}x\Bigg\}. \end{aligned} \tag{28}$$

From the matrix decomposition theory [26], there exists a unitary matrix U such that $L = U\Lambda U^T$, where $\Lambda = \mathrm{diag}\{\lambda_1(L), \lambda_2(L), \ldots, \lambda_N(L)\} = \mathrm{diag}\{0, \lambda_2(L), \ldots, \lambda_N(L)\}$, $U = [u_1, u_2, \ldots, u_N]$, and $u_1 = 1/\sqrt{N}[1, 1, \ldots, 1]^T$. Let $z(t) = (U^T \otimes I_n)x(t) = [z_1^T(t), z_2^T(t), \ldots, z_N^T(t)]^T$, where $z_i(t) \in \mathbb{R}^n$ $(i \in \mathcal{V})$, $x(t) \in \mathbb{R}^{Nn}$, $(U^T \otimes I_n) \in \mathbb{R}^{Nn\times Nn}$. Therefore, it yields that

$$\begin{aligned}&x^T\left[\left(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2\right) L\otimes I_n\right]x \\ &\quad = z^T(U^T\otimes I_n)\left[\left(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2\right) L\otimes I_n\right](U\otimes I_n)z \\ &\quad = \sum_{i=2}^{N} z_i^T\left[\left(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2\right)\lambda_i(L)\right]z_i. \end{aligned} \tag{29}$$

Similarly, one has

$$\begin{aligned}-a\phi x^T(L^2\otimes I_n)x &= -a\phi z^T(U^T\otimes I_n)(L^2\otimes I_n)(U\otimes I_n)z\\ &= -a\phi z^T(U^TL^2U\otimes I_n)z\\ &= -a\phi\sum_{i=2}^{N}\lambda_i^2(L)z_i^Tz_i \end{aligned}\tag{30}$$

and

$$-kx^T(L^2\otimes I_n)x = -k\sum_{i=2}^{N}\lambda_i^2(L)z_i^Tz_i. \tag{31}$$

Therefore, using (29)–(31) and the inequality in Theorem 3, we have

$$\begin{aligned}&x^T\left\{\left[\left(\psi+\frac{1}{2}\sum_{m\in\mathcal{A}}g_m^2\right)I_N-a\phi L-kL\right]L\otimes I_n\right\}x\\ &=\sum_{i=2}^{N}z_i^T\lambda_i(L)\left[\psi+\frac{1}{2}\sum_{m\in\mathcal{A}}g_m^2-a\phi\lambda_i(L)-k\lambda_i(L)\right]z_i\le 0.\end{aligned}\tag{32}$$

Substituting (32) into (28), we have $\mathbb{E}\{\mathscr{L}V(t)\}\le\mathbb{E}\{-\Delta x^T(L\otimes I_n)x\}\le 0$. Therefore, it is easy to observe that $\mathbb{E}\{\mathscr{L}V(t)\}\le 0$. Following the statement below (17), pinning distributed synchronization will be achieved in mean square. This completes the proof. ■

Remark 7: Theorem 3 only presents a sufficient condition that ensures the mean square synchronization if the noise intensity is below a given upper bound. It is important to find necessary conditions that ensure the mean square synchronization.

By considering the lower bound of ϕ, one further corollary can be obtained as follows based on Theorems 2 and 3.

Corollary 2: For $\mathcal{Q}=R_2(t)$, suppose that $f(.,t)$ is continuous on $(.,t)\in\mathbb{R}^n\times\mathbb{R}^+$ and satisfies Assumptions 1–3, $\sigma_m(.,t)$, $(m\in\mathcal{A})$ satisfies Assumptions 1 and 3, and the graph $\mathcal{G}$ is connected. If there exists a positive constant a such that the following inequality holds:

$$\left[\left(\psi+\frac{1}{2}\sum_{m\in\mathcal{A}}g_m^2\right)I_N-a\check{\phi}L-kL\right]L\le 0 \tag{33}$$

then the stochastic network in (9) under (4) and (5) will be globally synchronized in mean square and the coupling strengths will converge in mean.

Remark 8: It is worth mentioning that one can easily extend the main results to the case of (9) without stochastic disturbances. The corollaries are omitted here due to the page limitation.

C. Upper Bounds of $\mathcal{M}_c$ and $\mathcal{M}_s$ of Dynamical Networks Under $\mathcal{Q}=R_2(t)$

In the following, upper bounds of the mean control gain $\mathcal{M}_c$ and the convergence rate $\mathcal{M}_s$ are derived for the case of switching pinning set $\mathcal{R}_2(t)$. Here, we further assume that $\check{\phi}$ is nonzero, where $\check{\phi}$ is given in (6).

Theorem 4: If all assumptions and conditions of Corollary 2 are satisfied, then when $\epsilon_i(0)=0$, $(\forall i\in\mathcal{V})$, an upper bound of the mean control gain $\mathcal{M}_c$ is

$$\mathcal{M}_c\le\hat{\mathcal{M}}_c=\begin{cases}\mathbb{E}\left\{2\mathscr{X}+\sqrt{\dfrac{2q_0\alpha}{\check{\phi}N}}\right\}, & \text{if } \mathscr{X}\ge 0\\ \mathbb{E}\left\{\sqrt{\dfrac{2q_0\alpha}{\check{\phi}N}}\right\}, & \text{otherwise}\end{cases}\tag{34}$$

where $q_0=\frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\|e_{ij}(0)\|^2$, $\mathscr{X}=(2\psi+\sum_{m\in\mathcal{A}}g_m^2-2k\lambda_2(L))/2\lambda_2(L)\check{\phi}+a(\tilde{\phi}-1)$, $\tilde{\phi}=\frac{\hat{\phi}}{\check{\phi}}$. An upper bound of $\mathcal{M}_s\le\hat{\mathcal{M}}_s$ is

$$\hat{\mathcal{M}}_s=\begin{cases}\mathbb{E}\left\{\dfrac{N}{(N-1)\lambda_2^2(L)\alpha}\left[2\mathscr{X}+\sqrt{\dfrac{2q_0\alpha}{\check{\phi}N}}\right]\right\}\\ \text{if } \mathscr{X}\ge 0\\ \mathbb{E}\left\{\dfrac{N}{(N-1)\lambda_2^2(L)\alpha}\sqrt{\dfrac{2q_0\alpha}{\check{\phi}N}}\right\}, & \text{otherwise.}\end{cases}\tag{35}$$

Proof: Let p_i be the eigenvector of L associated with the eigenvalue $\lambda_i(L)$ ordered by $0=\lambda_1(L)\le\lambda_2(L)\le\lambda_3(L)\le\cdots\le\lambda_N(L)$. The eigenvectors are chosen such that they correspond to the same eigenvalue with multiplicity, i.e., $p_1,\ldots,p_N$ compose an orthogonal standard basis of $\mathbb{R}^N$. Any $p\in\mathbb{R}^N$ can be written as $p=\sum_{i\in\mathcal{V}}\upsilon_ip_i$, $(i\in\mathcal{V})$. Thus, we get $p_i^Tp_j=0,\forall i\ne j$

$$\begin{aligned}&p^T\left\{\left[(a\check{\phi}+k)-\frac{2\psi+\sum_{m\in\mathcal{A}}g_m^2}{2\lambda_2(L)}\right]L^2\right.\\ &\qquad\left.-\left[(a\check{\phi}+k)L-\frac{1}{2}\left(2\psi+\sum_{m\in\mathcal{A}}g_m^2\right)I_N\right]L\right\}p\\ &=\sum_{i\in\mathcal{V}}p_i^Tp_i\left[\left(a\check{\phi}+k-\frac{2\psi+\sum_{m\in\mathcal{A}}g_m^2}{2\lambda_2(L)}\right)\right.\\ &\qquad\times\lambda_i^2(L)-(a\check{\phi}+k)\lambda_i^2(L)\\ &\qquad\left.+\left(\psi+\frac{1}{2}\sum_{m\in\mathcal{A}}g_m^2\right)\lambda_i(L)\right]\upsilon_i^2\\ &\quad+2\sum_{i\in\mathcal{V}}\sum_{j>i}p_i^T\left[\left(a\check{\phi}+k-\frac{2\psi+\sum_{m\in\mathcal{A}}g_m^2}{2\lambda_2(L)}\right)L^2\right.\\ &\qquad\left.-((a\check{\phi}+k)L-(\psi+\frac{1}{2}\sum_{m\in\mathcal{A}}g_m^2)I_N)L\right]p_j\upsilon_i\upsilon_j\\ &=\sum_{i=2}^{N}p_i^Tp_i\left[-\frac{1}{\lambda_2(L)}\lambda_i(L)+1\right]\\ &\qquad\times\left(\psi+\frac{1}{2}\sum_{m\in\mathcal{A}}g_m^2\right)\lambda_i(L)\upsilon_i^2\\ &\le 0.\end{aligned}\tag{36}$$

Integrating both sides of (5) yields

$$\mathbb{E}\left\{\sum_{i\in\mathcal{V}}\int_0^\infty d\epsilon_i(t)\right\}=\mathbb{E}\sum_{i\in\mathcal{V}}\int_0^\infty\alpha\left[\sum_{j\in\mathcal{N}_i}e_{ij}\right]^T\left[\sum_{j\in\mathcal{N}_i}e_{ij}\right]dt. \tag{37}$$

$\mathcal{M}_c$ can be calculated as follows:

$$\mathcal{M}_c = \mathbb{E}\Big\{\frac{\alpha}{N}\int_0^{\infty} x^T(L^2 \otimes I_n)x dt\Big\}. \tag{38}$$

We obtain the following inequality from Corollary 2:

$$\mathbb{E}\mathscr{L}V \le \mathbb{E}\Big\{x^T(t)\Big[\Big(\Big(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2\Big)I_N - a\check{\phi}L - kL\Big)L \otimes I_n\Big]x(t)\Big\}. \tag{39}$$

Thus, we find

$$\begin{aligned}\mathcal{M}_c &= \mathbb{E}\Big\{\frac{\alpha}{N}\int_0^{\infty} x^T(t)(L^2 \otimes I_n)x(t)dt\Big\}\\
&\le \mathbb{E}\Big\{\frac{\alpha\lambda_2(L)}{N[(a\check{\phi}+k)\lambda_2(L)-\mathscr{A}]}\int_0^{\infty} x^T(t)\\
&\quad\times[((a\check{\phi}+k)L - \mathscr{A}I_N)L\otimes I_n]x(t)dt\Big\}\\
&\le -\mathbb{E}\Big\{\frac{\alpha\lambda_2(L)}{N[(a\check{\phi}+k)\lambda_2(L)-\mathscr{A}]}\times\Big[\int_0^{\infty}\mathscr{L}V dt\Big]\Big\}\\
&= \mathbb{E}\Big\{\frac{\alpha\lambda_2(L)}{N[(a\check{\phi}+k)\lambda_2(L)-\mathscr{A}]}[V_0 - V_\infty]\Big\}\\
&= \mathbb{E}\Big\{\frac{\alpha\lambda_2(L)}{N[(a\check{\phi}+k)\lambda_2(L)-\mathscr{A}]}\Big[q_0\\
&\quad+\sum_{i\in\mathcal{V}}\frac{\phi_i}{2\alpha}(2a\epsilon_{i,\infty}-\epsilon_{i,\infty}^2)\Big]\Big\}\\
&\le \mathbb{E}\Big\{\frac{\alpha\lambda_2(L)}{N[(a\check{\phi}+k)\lambda_2(L)-\mathscr{A}]}\\
&\quad\times\Big[q_0 + \frac{aN\hat{\phi}}{\alpha}\mathcal{M}_c - \frac{N\check{\phi}}{2\alpha}\mathcal{M}_c^2\Big]\Big\}\end{aligned} \tag{40}$$

where $\mathscr{A} = \psi + (1/2)\sum_{m\in\mathcal{A}} g_m^2$, $V_0 = V(0)$, $V_\infty = \lim_{t\to\infty} V(t)$, and $q_0 = (1/4)\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\|e_{ij}(0)\|^2$. Let $\tilde{\phi} = \frac{\hat{\phi}}{\check{\phi}}$. By solving the last inequality (40), we have

$$\mathcal{M}_c \le \bar{\mathcal{M}}_c = \mathbb{E}\left\{\mathscr{X} + \sqrt{\mathscr{X}^2 + \frac{2q_0\alpha}{\check{\phi}N}}\right\} \tag{41}$$

where

$$\mathscr{X} = \frac{2\psi + \sum_{m\in\mathcal{A}} g_m^2 - 2k\lambda_2(L)}{2\lambda_2(L)\check{\phi}} + a(\tilde{\phi} - 1). \tag{42}$$

According to the inequality $\sqrt{a^2+b^2} \le a+b$, where a and $b \in \mathbb{R}$ are $a, b \ge 0$

$$\begin{aligned}\bar{\mathcal{M}}_c \le \hat{\mathcal{M}}_c &= \mathbb{E}\Big\{\mathscr{X} + \Big|\mathscr{X}\Big| + \sqrt{\frac{2q_0\alpha}{\check{\phi}N}}\Big\}\\
&= \begin{cases}\mathbb{E}\Big\{2\mathscr{X} + \sqrt{\frac{2q_0\alpha}{\check{\phi}N}}\Big\}, & \text{if } \mathscr{X} \ge 0,\\ \mathbb{E}\sqrt{\frac{2q_0\alpha}{\check{\phi}N}}, & \text{otherwise}\end{cases}\end{aligned} \tag{43}$$

in the following, we will present an upper bound for $\mathcal{M}_s$. Denote $\mathcal{U} = (u_{ij})$ with $u_{ij} = -(1/N)$ if $i \ne j$ and $u_{ii} = 1 - (1/N)(\forall i = 1, 2, \ldots, N)$ and $\mathscr{W} = (1/m-1)\mathcal{U}^T\mathcal{U}$. Thus, $\mathcal{M}_s$ can be written as follows:

$$\mathcal{M}_s = \mathbb{E}\int_0^{\infty} x^T(t)(\mathscr{W}\otimes I_n)x(t)dt. \tag{44}$$

The following inequality holds according to [21]:

$$\mathscr{W} \le \frac{1}{(N-1)\lambda_2^2(L)}L^2.$$

From (38) and (44), it can be checked that

$$\begin{aligned}\mathcal{M}_s &\le \mathbb{E}\Big\{\frac{1}{(N-1)\lambda_2^2(L)}\int_0^{\infty} x^T(t)(L^2\otimes I_n)x(t)dt\Big\}\\
&\le \mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\alpha}\mathcal{M}_c\Big\}\\
&\le \mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\alpha}\Big[\mathscr{X} + \sqrt{\mathscr{X}^2 + \frac{2q_0\alpha}{\check{\phi}N}}\Big]\Big\}.\end{aligned} \tag{45}$$

By using $\sqrt{a^2+b^2} \le a+b$, one has

$$\bar{\mathcal{M}}_s \le \mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\alpha}\Big[\mathscr{X} + |\mathscr{X}| + \sqrt{\frac{2q_0\alpha}{\check{\phi}N}}\Big]\Big\}. \tag{46}$$

Hence, one has from (46)

$$\hat{\mathcal{M}}_s = \begin{cases}\mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\alpha}\Big[2\mathscr{X} + \sqrt{\frac{2q_0\alpha}{\check{\phi}N}}\Big]\Big\}, & \text{if } \mathscr{X} \ge 0\\ \mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\alpha}\sqrt{\frac{2q_0\alpha}{\check{\phi}N}}\Big\}, & \text{otherwise}\end{cases} \tag{47}$$

this completes the proof. ■

If $\phi_i = \phi, i \in \mathcal{V}$, we have the following results.

Corollary 3: If all assumptions and conditions of Corollary 2 are satisfied, then when $\epsilon_i(0) = 0, (\forall i \in \mathcal{V})$, an upper bound of the mean control gain $\mathcal{M}_c$ is

$$\mathcal{M}_c \le \hat{\mathcal{M}}_c = \begin{cases}\mathbb{E}\Big\{2\mathscr{Y} + \sqrt{\frac{2q_0\alpha}{\phi N}}\Big\}, & \text{if } \mathscr{Y} \ge 0\\ \mathbb{E}\Big\{\sqrt{\frac{2q_0\alpha}{\phi N}}\Big\}, & \text{otherwise}\end{cases} \tag{48}$$

where $q_0 = (1/4)\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\|e_{ij}(0)\|^2$, $\mathscr{Y} = (2\psi + \sum_{m\in\mathcal{A}} g_m^2 - 2k\lambda_2(L)/2\lambda_2(L)\phi)$. An upper bound of $\mathcal{M}_s$ is

$$\mathcal{M}_s \le \hat{\mathcal{M}}_s = \begin{cases}\mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\alpha}\Big[2\mathscr{Y} + \sqrt{\frac{2q_0\alpha}{\phi N}}\Big]\Big\}, & \text{if } \mathscr{Y} \ge 0\\ \mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\alpha}\sqrt{\frac{2q_0\alpha}{\phi N}}\Big\}, & \text{otherwise.}\end{cases} \tag{49}$$

Proof: The proof is straightforward to obtain from Theorem 4 and hence omitted here. ■

Remark 9: From Theorem 4, it is observed that the noise intensities greatly influence upper bounds of $\mathcal{M}_c$ and $\mathcal{M}_s$. Therefore, the smaller the intensities of stochastic disturbances, the better the synchronization performance is.

IV. Optimization Methods for Solving Pinning Synchronization of Stochastic Dynamical Networks

In this section, based on EAs and SDP, we will propose two optimization algorithms for solving the pinning synchronization of stochastic dynamical network in (8) and (9), respectively.

A. SDP for Distributed Pinning Synchronization

We convert the criteria in Theorem 1 or 2 into an SDP problem when the number of driver nodes l is fixed [17] and the driver nodes are selected according to certain rules. Following Theorems 1 and 2 to solve the stochastic dynamical network in (8) and (9), the optimization problem is formulated as follows:

$$\min k$$
$$\text{subject to}$$
$$\begin{cases} [(\psi + \frac{1}{2}\sum_{m\in\mathcal{A}} g_m^2) I_N - aL\mathscr{H} - kL]L \le 0 \\ \text{Assumption 2 should be satisfied} \end{cases} \tag{50}$$

where $\mathscr{H} = \hat{I}_N$ for Theorem 1 and $\mathscr{H} = \mathcal{I}$ for Theorem 2

$$\hat{I}_N(i,j) = \begin{cases} 1, & \text{if } i = j, \delta_{\mathcal{R}_1}(i) = 1 \\ 0, & \text{otherwise} \end{cases}$$

should be provided before optimization and $\mathcal{I} = \text{diag}\{\phi_1, \ldots, \phi_N\}$.

B. Constraint Optimization Evolutionary Algorithms

EAs have been used to identify controlling regions in complex networks by treating the problem into single objective optimization problems [10], [25] and constraint optimization problems [7]. However, in these works, we have to design control gains for driver nodes, which inevitably result in huge complexity.

1) jDE: A self-adaptive differential evolution algorithm (jDE) was proposed by [32], in which the scaling factor F and the crossover probability C_r are encoded into the individual $\vec{x}_{i,s} = \langle x_{i,s}, F_{i,s}, Cr_{i,s}\rangle, i = 1, \ldots, SP$, ($SP$ is the population size), s is the generation number and adjusted by two new arguments τ_1 and τ_2. The newly generated $F_{i,s+1}$ and $Cr_{i,s+1}$ are used before the mutation is implemented. Because of simplicity and effectiveness of jDE, it has been widely used in single objective, constraint, and multiobjective optimization problems.

2) IDyHF: An improved dynamic hybrid framework (IDyHF) was proposed in [7], which is based on jDE [32] and DyHF [33]. IDyHF is a COEA and it has been demonstrated that IDyHF improves the performance of DyHF by replacing the search engine in the global scheme of DyHF using jDE. Such a simple replacement makes the IDyHF more powerful to deal with constraint optimization problems, since jDE can efficiently adjust the control parameters in differential evolution and thus adapt to the search situations [7], [32].

The global search model in IDyHF is to refine the overall performance of the population, which adopts the following steps.

Step 1: Each target vector $\vec{x}_{i,s}$ $(i = 1, 2, \ldots, SP)$ is utilized to generate a trial vector $\vec{u}_{i,s}$ through the mutation and crossover operations of the jDE. The control parameters of jDE is updated according to the self-adaptive scheme [32].

Step 2: Compute the two objectives, i.e., $\Sigma_1(x)$ and the $\Sigma_2(x)$, for the trial vector $\vec{u}_{i,s}$.

Step 3: Based on the multiobjective optimization methods, if $\vec{u}_{i,s}$ dominates $\vec{x}_{i,s}$, the trial vector $\vec{u}_{i,s}$ will replace the target vector $\vec{x}_{i,s}$, otherwise no replacement occurs.

For more details, please refer to [7] and references therein.

C. Mixed Optimization Method for Distributed Pinning Synchronization

The mixed optimization method proposed here is an evolutionary algorithm based on a convex optimization method (EACOM), which includes a COEA and a convex optimization method. Here, we embed convex optimization method into the COEA to solve the pinning distributed synchronization of the dynamical network in (8) or (9) with multiple stochastic disturbances. The proposed EACOM takes the advantage of finding feasible solutions for the matrix computation and dealing with the selection of driver nodes efficiently. The main steps of EACOM are listed as follows:

Step 1: Initialize a population $\mathscr{P}$ with SP individuals and $(\#\mathcal{Q} + 1)$ dimension size. One individual in the population is initialized by the method in [17], which means that the driver nodes are selected according to degree information in a descending way. The first $\#\mathcal{Q}$ dimensions represent the locations of driver nodes, i.e., $\hat{I}_N$. The $(\#\mathcal{Q} + 1)$th dimension stands for the value of the coupling strength k. The first part of the encoding scheme follows the one in [25].

Step 2: Calculate the fitness values of $\mathscr{P}$ and their violation values Σ according to (50), where Σ can be obtained from linear matrix inequality box in MATLAB and Yalmip. In this step, convex optimization methods and COEAs are included for constraint handling together.

Step 3: Sort the individuals in $\mathscr{P}$ according to Pareto dominance. If two solutions are both infeasible, the solution having a less violation value dominates the other one. If both solutions are feasible, the solution having a less fitness value dominates the other one. If one solution is feasible and the other is not, the feasible solution always dominates the other one.

Step 4: Using IDyHF to perform global and local searches, thus the individuals are updated adaptively.

Step 5: Check whether the termination condition is satisfied or not. If so, the best solution is recorded. If not, go to Step 2.

Remark 10: It is worth mentioning that in Step 1, the population $\mathscr{P}$ is initialized by using the information of degree satisfying the SDP in (50). Due to the elitism of IDyHF, EACOM will always perform not worse than the method in [17], which will also be verified in the following section. Moreover, EAs with an elitism method (i.e., the best individual survives with probability one) like EACOM are able to find the global optimum with probability 1 if the number of generations tends to infinity. Such kind of statement has been proved using the concept of nonhomogeneous Markov chains [34].

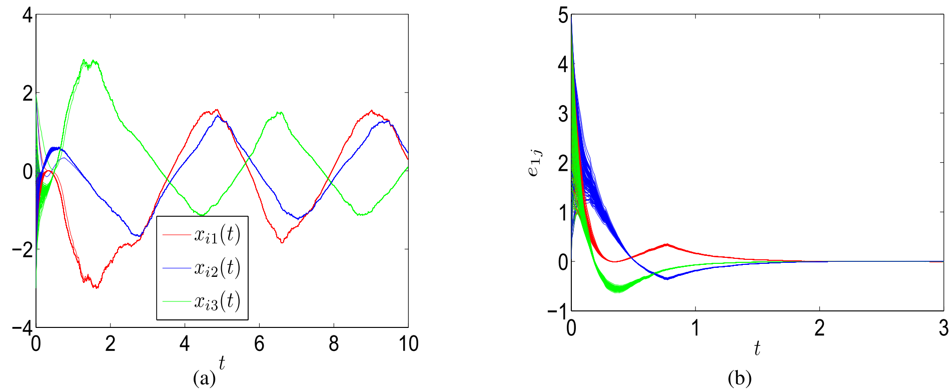

Fig. 1. Synchronization errors and state trajectories of pinning distributed the nonlinear stochastic dynamical network in (8) under $\mathcal{R}_1$. (a) State trajectories; (b) synchronization errors.

TABLE I

COMPARISON WITH THE METHOD IN [17] AND THE MEAN RESULTS OF EACOM IN 20 RUNS OF THIS PAPER

	$\#\mathcal{Q}=2$	$\#\mathcal{Q}=4$	$\#\mathcal{Q}=8$	$\#\mathcal{Q}=10$	$\#\mathcal{Q}=16$	$\#\mathcal{Q}=24$
[17]	3.12	3.09	2.98	2.84	2.51	0.01
EACOM	2.90	2.74	2.53	2.20	1.23	0.0091

Although we cannot ensure that EACOM can find an optimum in every run, the statistical tests can be carried out to show its effectiveness by running the algorithm for $\bar{M} \geq 20$ times [10], [33], [35].

V. NUMERICAL EXAMPLES

In this section, two simulation examples are presented to demonstrate the effectiveness of the proposed results and methods for pinning distributed synchronization of stochastic dynamical networks with multiple disturbances.

A. Model Description

A Hopfield neural network is considered on each node, which shows chaotic behavior [36]

$$dx_i = [-Cx_i + Ah(x_i)]dt + \sum_{m\in\mathcal{A}} \sigma_m(x_i,t)d\omega_m(t) \tag{51}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t)]^T$; the matrices A and C are picked as follows:

$$A = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The nonlinear function $h(x_i) = [h(x_{i1}), h(x_{i2}), h(x_{i3})]^T$ is chosen as $h(x_i) = ((|x_i+1| - |x_i-1|)/2)$. $\mathcal{A}$ is taken as $\mathcal{A} = \{1,2\}$. The noise intensity functions are chosen as $\sigma_1(x_i,t) = 0.1x_i(t)$ and $\sigma_2(x_i,t) = 0.11x_i(t)$. Hence, the noise intensities in Assumption 1 are $g_1 = 0.1$ and $g_2 = 0.11$. The simulation time is set as $T = 10$. The step size of our algorithm is chosen as 0.005. The graph considered here is a scale-free network [37], in which the degree distribution follows a power law. The growth starts from three nodes and no edges. At each step, a new node with three edges is added to the existing graph until N achieves a predefined value. Repeating this method, we will generate a scale-free network, which the graph $\mathcal{G}$ to be connected is satisfied. The parameter setting for IDyHF follows the method in [7]. The maximum number of fitness evaluation is $f_{e,\max} = D * \varsigma$, where D is the dimension size of the problem and $\varsigma = 1000$ is an adjustable parameter for balancing the tradeoff between the computation resources and accuracies. Results are obtained for a total of 20 trials to show the reliability of EACOM [10], [33], [35]. The initial range for IDyHF is $(0, N+1)$ for the first $\#\mathcal{Q}$ dimension and $(0, \nu)$ for the last dimension, where $\nu = 5$ for $N = 25$ and $\nu = 10$ for $N = 100$. To make Assumption 2 in (50) satisfied, we need to consider the following inequality constraint:

$$\begin{bmatrix} 2(C+\psi) - \Phi * W^2 - 2\Delta & -A \\ -A^T & \Phi \end{bmatrix} < 0. \tag{52}$$

B. Example 1

In this example, pinning distributed synchronization of the complex network in (8) is investigated using EACOM. The parameters are set as $\alpha = 0.5, N = 100, \#\mathcal{Q} = 3$. It can be checked that $W = 1$.

$\hat{I}_N$ is chosen according the descending degree of each node in the network [17]. Using linear matrix inequality toolbox and Yalmip [38] to solve (50), one can find as a result $k = 6.5$. However, if we use EACOM, the mean value of k achieved in 20 runs is 3.74 and the minimum value of k achieved in 20 runs is 3.68. The feasible solution for $k = 3.68$ is listed as follows:

$$\begin{aligned} &\Delta = 0.0150, \quad \psi = 5.5943, \quad a = 50689, \\ &\Phi = \mathrm{diag}\{6.6897, 7.3914, 5.7121\} \end{aligned} \tag{53}$$

which shows that the proposed EACOM outperforms the one in [17], since EACOM can efficiently select $\hat{I}_N$ to reduce possible conservativeness. The pinning distributed synchronization of the complex network in (8) with multiple stochastic disturbances is ensured in mean square under fixed pinning set $\mathcal{R}_1$. The synchronization errors and state trajectories under $\mathcal{R}_1$ are shown in Fig. 1(a) and (b), which verify the effectiveness of the proposed method.

To further illustrate our method, a network with $N = 25$ is considered. The k achieved by the SDP [17] and EACOM is shown in Table I, in which the mean results of EACOM of 20 runs are given. Note that all the final solutions of EACOM are feasible here. It is shown that the proposed method can efficiently reduce the conservativeness induced by the naive selection of $\hat{I}_N$, which manifests the reliability of the proposed method.

C. Example 2

In this example, the distributed pinning synchronization of the complex network in (9) is studied using SDP. The parameters are set as $\alpha = 0.5, N = 100, \#\mathcal{Q} = 3, \phi_1 = \phi_2 = \phi_3 = 0.5, \phi_i = 0, i = 4, \ldots, N$.

Using linear matrix inequality toolbox and Yalmip [38] to solve (50), k found is 3.7995. One feasible solution is listed as follows:

$$\begin{aligned}&\Delta = 4.1765 \times 10^{-6}, \quad \psi = 5.5641, \quad a = 203.2593,\\ &\Phi = \mathrm{diag}\{6.6515, 7.3604, 5.6678\}.\end{aligned} \tag{54}$$

Thus, the pinning distributed synchronization of the complex network in (9) with multiple stochastic disturbances is guaranteed in mean square under switching pinning set $\mathcal{R}_2(t)$.

VI. Conclusion

The problem of pinning distributed synchronization has been investigated for a class of nonlinear dynamical networks with multiple stochastic disturbances, based on fixed pinning and switching pinning schemes. Using the Lyapunov stability theory and graph theory, criteria are presented for pinning mean square distributed synchronization of stochastic dynamical networks. For the fixed pinning scheme, an EA-based convex optimization method has been developed to select driver nodes and find feasible solutions for matrix computation. For a Bernoulli switching pinning strategy, upper bounds of mean control gain and convergence rate have been derived to show the relationship between the synchronization performance and systems' parameters. Simulation examples are presented to illustrate the advantage and effectiveness of the proposed method. In the future, we can apply our results to industrial processes [39], [40].

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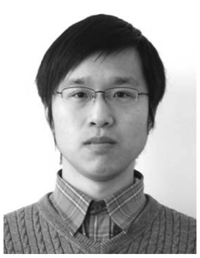

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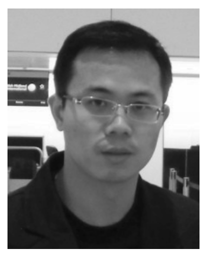

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