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Characteristics and synchronization of time-delay systems driven by a common noise

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Abstract. We investigate the characteristics of time-delay systems in the presence of Gaussian noise. We show that the delay time embedded in the time series of time-delay system with constant delay cannot be estimated in the presence noise for appropriate values of noise intensity thereby forbidding any possibility of phase space reconstruction. We also demonstrate the existence of complete synchronization between two independent identical time-delay systems driven by a common noise without explicitly establishing any external coupling between them.

1 Introduction

Synchronization of chaotic systems driven by common signals has been an area of extensive research since the pioneering works of Fujisaka and Yamada [1,2] and of Pecora and Carroll [3]. Since the identification of complete (identical) chaotic synchronization, different kinds of chaotic synchronizations have been identified and demonstrated both theoretically and experimentally (cf. [4–7]). Recently, synchronization in coupled time-delay systems with or without time-delay coupling has become an active area of research by exploiting the infinite dimensional nature of the underlying systems, which exhibits a large number of positive Lyapunov exponents as a function of the delay time, for potential applications [8–16].

An interesting application of chaos synchronization is secure communication. Several approaches for chaos synchronization and control of chaos with application to secure communication have been the focus of many recent investigations [17–20]. However, it has been shown that messages masked even by hyperchaotic signals of time-delay systems can be extracted by using nonlinear dynamic forecasting techniques as the local dynamics does not reflect more complicated dynamics significantly [21–23], which is considered to be a serious drawback of time-delay systems with a constant delay. Nevertheless, it has been shown that delay time modulation (time-dependent delay) wipes off any imprints of the delay time carved in the time series of a time-delay system and that the reconstructed phase trajectory of the system is not collapsed into a simple manifold [24]. Synchronization, encryption, and communication in coupled time-delay systems in the presence of chaotic/stochastic delay time modulation has also been demonstrated [25,26]. We have also shown that even a simple sinusoidal modulation of the delay time can remove the imprints of the delay time in the time series of any time-delay systems [27].

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Further, the effect of noise on synchronization has also been studied extensively [28–31]. In particular, noise-induced and noise-enhanced synchronizations has been investigated in coupled chaotic oscillators without delay [30,31]. It has been shown that noise can enhance phase synchronization and this phenomenon has been experimentally verified in electrochemical oscillators [30]. In addition, the mechanism of noise-induced synchronization has been revealed and it has been shown that a common noise can induce phase synchronization in nonidentical chaotic systems [31].

In this connection, in the present manuscript, we will show that additive Gaussian noise in the system is capable of masking the information about the delay time encarved in the time series of dynamical systems with a constant delay. Therefore, using the available nonlinear dynamic forecasting techniques that have been used to estimate the delay time from a time series, one cannot retrieve the information about the delay time from the system trajectory for appropriate values of noise intensities. Hence, the corresponding phase space cannot be collapsed into a simple manifold to enable the possibility of any phase space reconstruction. As noise is an inherent part of the dynamics in real world experiments, our results in this manuscript illustrate that even in dynamical systems with a constant delay it is difficult to retrieve the information about the delay time. Further, we will also show that a common noise can drive two independent time-delay systems to evolve in complete synchrony.

In particular, we have used two measures, namely, the length of polygon line and the filling factors, which have been used in the literature to estimate the delay time encarved in the time series, and show that these measures cannot reveal the delay time for suitable values of noise intensity. Further, we will also show that two independent time-delay systems with constant delays driven by a common Gaussian noise can be synchronized above certain threshold values of the noise intensity without any explicit coupling between them.

The plan of the paper is as follows. In Sec. 2, we briefly point out the dynamical details of the model system we have consider for our study, while in Sec. 3 we discuss the effect of noise on the estimates of the length of polygon line and the filling factor that have been used in the literature to estimate the delay time embedded in the time series of a dynamical system. In Sec. 4, we will demonstrate the emergence of complete synchronization between two identical time-delay systems that are driven by a common noise without establishing any explicit coupling between them. Finally in Sec. 5, we summarize our results.

2 Model of a piecewise linear time-delay system

We consider the following first-order delay differential equation introduced by Lu and He [32] and discussed in detail in [33,34],

$$\dot{x}(t) = -ax(t) + bf(x(t-\tau)), \tag{1}$$

where a and b are parameters, τ is the time delay and f is an odd piecewise linear function defined as

$$f(x) = \begin{cases} 0, & x \leq -4/3 \\ -1.5x - 2, & -4/3 < x \leq -0.8 \\ x, & -0.8 < x \leq 0.8 \\ -1.5x + 2, & 0.8 < x \leq 4/3 \\ 0, & x > 4/3 \end{cases} \tag{2}$$

We have shown [34] that systems of the form (1) exhibit hyperchaotic behavior for suitable parameter values. For our present study, we find that for the choice of the parameters $a = 1.0$, $b = 1.2$ and $\tau = 20$ with the initial condition $x(t) = 0.9$, $t \in (-\tau, 0)$, Eq. (1) exhibits hyperchaos. The corresponding pseudoattractor (obtained by embedding) is shown in the Fig. 1(a). The hyperchaotic nature of Eq. (1) is confirmed by the existence of multiple positive Lyapunov exponents. The first ten maximal Lyapunov exponents for the same parameters but as a function

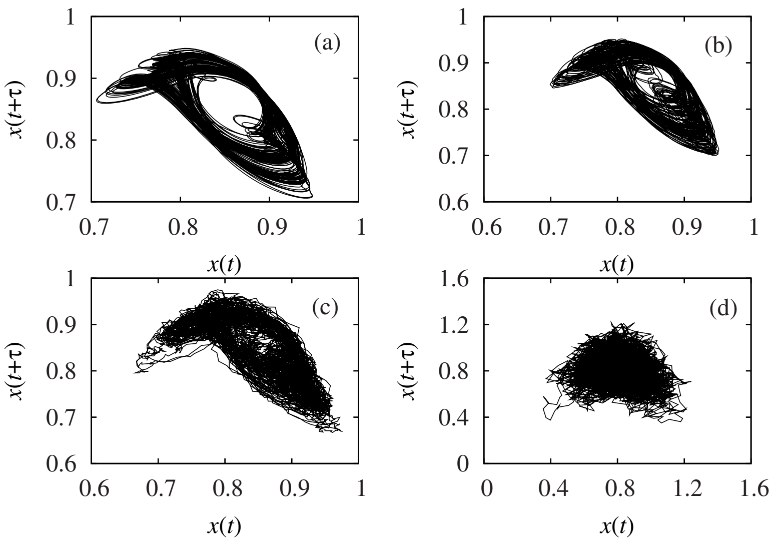

Fig. 1. The hyperchaotic attractors of the scalar time-delay system (1) for the parameter values $a = 1.0, b = 1.2, \tau = 20$ for different values of Gaussian noise intensity D_0. (a) $D_0 = 0$, (b) $D_0 = 0.0001$, (c) $D_0 = 0.001$ and (d) $D_0 = 0.01$.

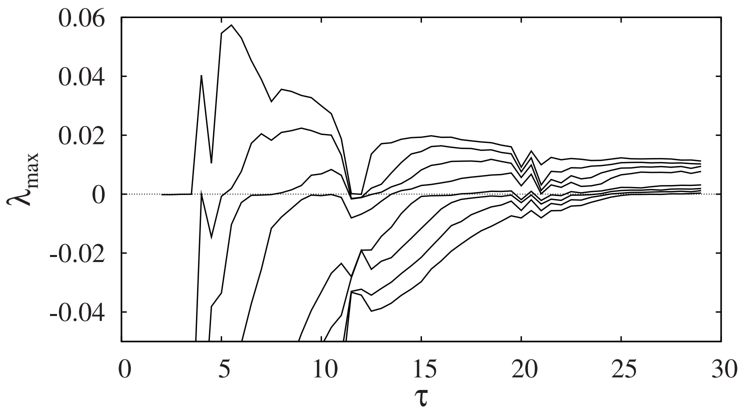

Fig. 2. The first ten maximal Lyapunov exponents λ_{max} of the scalar time-delay equation (1) for the parameter values $a = 1.0$, $b = 1.2$, $\tau \in (2, 29)$.

of the time-delay τ are shown in Fig. 2. In the following, we will demonstrate that by adding Gaussian noise, $\sqrt{2}D_0\xi(t)$, where D_0 is the noise intensity and $\xi(t)$ is a zero mean Gaussian noise, it is impossible to estimate the delay time encarved in the time series of the piecewise time-delay system with constant delay for the above values of the parameters and for appropriate values of the noise intensity.

3 Effect of Gaussian noise

Now, we consider the following two measures, namely, the length of polygon line [36] and the filling factor [35], that have been used in the literature to estimate the delay time embedded in a time series of a dynamical system. We will show that for increasing noise intensity D_0 these estimates lose their ability to extract the information about the delay time, thereby indicating that even in systems with constant delay one cannot retrieve any information in the presence of noise.

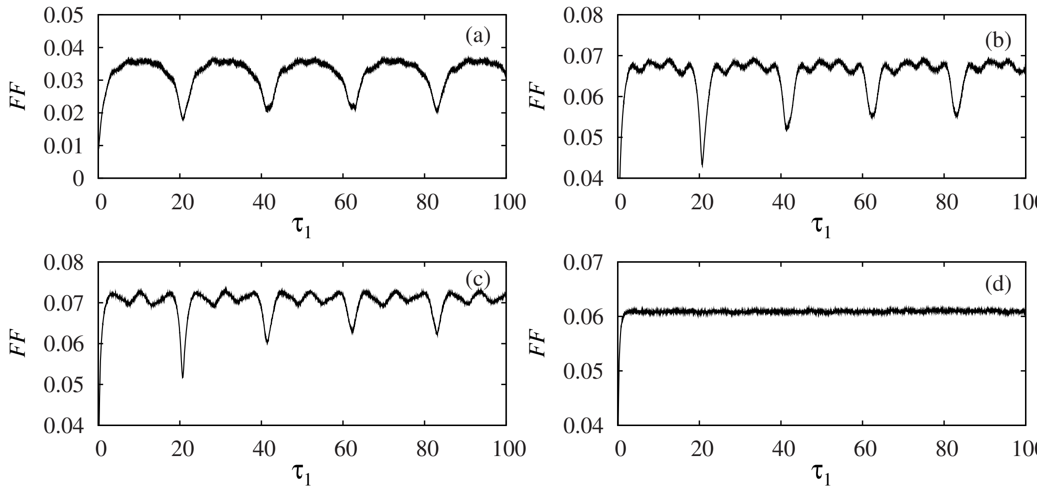

Fig. 3. Estimates of the measure filling factor FF for the trajectories of the corresponding attractors in Fig. 1 for different values of Gaussian noise intensity D_0. (a) $D_0 = 0$, (b) $D_0 = 0.0001$, (c) $D_0 = 0.001$ and (d) $D_0 = 0.01$.

3.1 Filling factor (FF)

First, we will compute the filling factor [35] for the hyperchaotic trajectory $x(t)$ of the time-delay system (1) by projecting it onto the pseudospace $(x, x_{\tau_1}, \dot{x})$ with P^{3N} equally sized hypercubes, where the delayed time series $x_{\tau_1} = x(t - \tau_1)$ is constructed from $x(t)$ for various values of τ_1. The filling factor is the number of hypercubes which are visited by the projected trajectory, normalized to the total number of hypercubes, P^{3N}. Figure 3(a) shows the filling factor estimated from the time series (the corresponding phase portrait is shown in Fig. 1(a)) of the piecewise linear time-delay system with constant delay, $\tau = 20$, in the absence of any noise, that is $D_0 = 0$. In this case, one can identify the existence of an underlying time-delay-induced instability [35] which induces local minima in the filling factor at $\tau_1 \approx n\tau,\ n = 1, 2, 3 \ldots$ From the locations of these minima, one can identify the value of the time-delay parameter τ of the system (1) under consideration. Figures 3(b) and (c) show the filling factor estimated from the time series of the attractors shown in Fig. 1(b) and (c) for the values of the noise intensity $D_0 = 0.0001$ and $D_0 = 0.001$, respectively, which still betrays the delay time by its local minima. However, on further increasing D_0 these local minima disappear slowly and for the value of noise intensity $D_0 = 0.01$, estimates of the filling factor, corresponding to the attractor in Fig. 1(d), do not show any local minima as in Fig. 3(d). From these figures, one can conclude that the imprints of the delay time embedded in the projected trajectory is completely erased due to the presence of sufficiently strong Gaussian noise.

3.2 Length of polygon line (LPL)

Next, to calculate the length of polygon line [36], the trajectory in $(x, x_{\tau_1}, \dot{x})$ space is restricted to a two-dimensional surface. The restriction in dimension is effected by intersecting the projected trajectory with a surface $k(x, x_{\tau_1}, \dot{x}) = 0$. Consequently, the number of times the trajectory traverses the surface and the corresponding intersection points can be calculated. One then orders the points with respect to the values of x_{τ_1}, and a simple measure for the alignment of the points is the length L of the polygon line connecting all the ordered points. Figure 4(a) shows the length of polygon line LPL estimated from the flow of the attractor shown in Fig. 1(a), which shows the local minima corresponding to the delay time of the piecewise linear time-delay system. The length of the polygon line estimated for the values of noise intensity $D_0 = 0.0001$ and $D_0 = 0.001$ from the trajectories of the attractors in Figs. 1(b) and (c) are depicted in

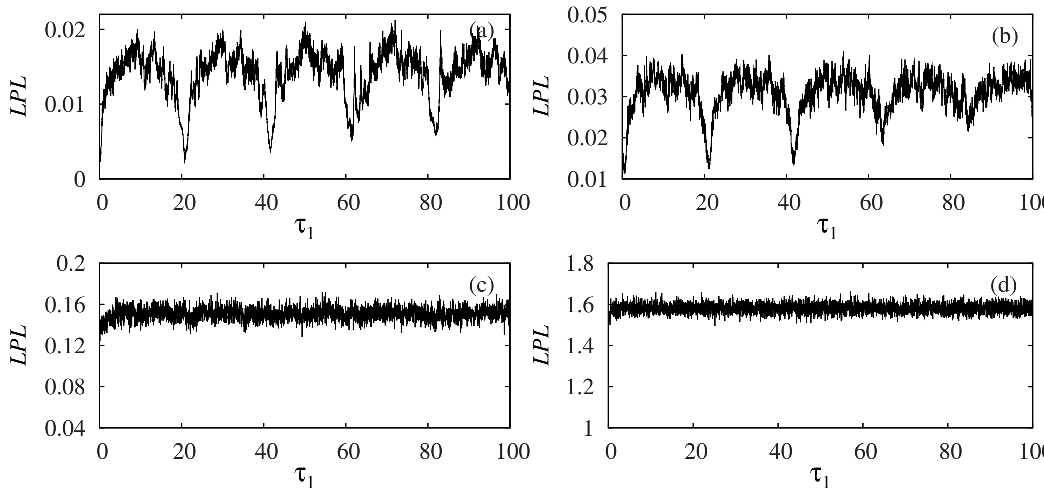

Fig. 4. Estimates of the measure length of polygon line LPL for the trajectories of the corresponding attractors in Fig. 1.

Figs. 4(b) and (c), respectively. These figures indeed indicate the time-delay by local minima at integral multiples of τ. Figure 4(d) shows the estimates of LPL from the trajectory of the attractor shown in Fig. 1(d) for $D_0 = 0.01$, which does not reveal any imprints of the delay time of the piecewise linear time-delay system with constant time-delay.

Hence, it is evident that even in dynamical systems with constant delay, the nonlinear dynamic forecasting techniques is not capable of extracting the delay time in the presence of Gaussian noise of appropriate intensity. This indicates that the phase space cannot be collapsed into a simple manifold to enable phase space reconstruction even in time-delay systems with a single constant delay. In the following section, we will demonstrate the existence of synchronization between two independent time-delay systems without any explicit coupling but driven by a common noise.

4 Synchronization by a common noise

In this section, we will demonstrate the emergence of synchronization between two piecewise linear time-delay systems without any coupling between them but instead driven by a common Gaussian noise in analogy with the investigation of noise-induced synchronization [31]. To be specific, we consider the following identical time-delay systems

$$\dot{x_1}(t) = -ax_1(t) + bf(x_1(t-\tau)) + \sqrt{2}D_0\xi(t), \tag{3}$$

$$\dot{x_2}(t) = -ax_2(t) + bf(x_2(t-\tau)) + \sqrt{2}D_0\xi(t), \tag{4}$$

where a, b are positive constants and τ is the delay time, whose values are fixed as the same as in the Sec. 2, D_0 is the noise intensity, $\xi(t)$ is Gaussian white noise and $f(x)$ is of the same form as in Eq. (2).

On increasing the noise intensity, D_0, we found that complete synchronization is established between two piecewise linear time-delay systems driven by a common Gaussian noise and without any explicit coupling between them. To quantify the degree of synchronization, we define the synchronization error as follows

$$\eta = \left\langle \sqrt{(x_1(t) - x_2(t))^2 + (x_1(t-\tau) - x_2(t-\tau))^2} \right\rangle, \tag{5}$$

where $\langle\cdot\rangle$ denotes the time average and we have averaged the error for $N = 200,000$ time steps in our simulations. The synchronization error, η, is illustrated in Fig. 5 as a function of noise

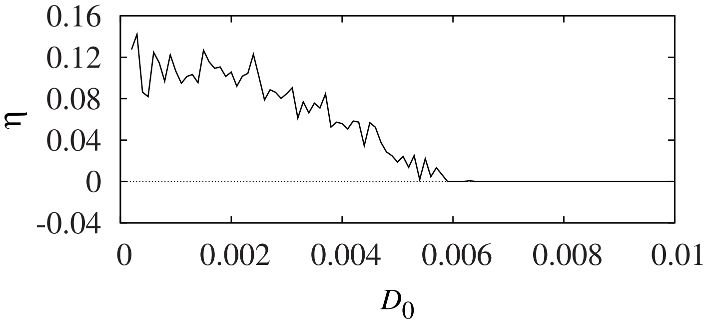

Fig. 5. Synchronization error, η, as a function of noise intensity in the range of $D_0 \in (0, 0.01)$.

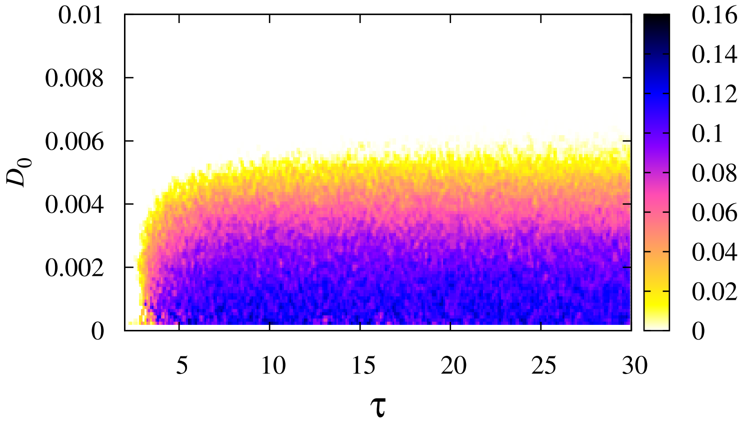

Fig. 6. (Color online) Synchronization error, η, as a function of both the time-delay and the noise intensity in the range of $\tau \in (2, 30)$ and $D_0 \in (0, 0.01)$.

intensity D_0 for the same values of all the parameters as discussed in the Sec. 2. It is clear from this figure that the degree of synchronization error decreases on increasing the noise intensity D_0 and for $D_0 > 0.006$, η approaches zero indicating that the two time-delay systems driven by a common Gaussian noise oscillates in complete synchrony with each other. We have also estimated the synchronization error as functions of both the noise intensity D_0 and the time-delay τ to investigate whether a common noise is capable of inducing synchronization in a wide range of delay times. The synchronization error is depicted in Fig. 6 in the range of $\tau \in (2, 30)$ and $D_0 \in (0, 0.01)$. This figure illustrates that synchronization is achieved even for feeble values of the noise intensity for small values of the delay time, $\tau \in (2, 3)$. The critical value of D_0 required for synchronization increases with the delay time in the range $\tau \in (3, 7)$ and for $\tau > 7$, the two time-delay systems are synchronized for $D_0 > 0.006$ as shown in Fig. 5. This confirms that complete synchronization can also be induced in time-delay systems that are driven by a common noise in a wide range of time-delay without any explicit couplings between them.

5 Summary and conclusion

To conclude, we have shown that nonlinear dynamic forecasting techniques, such as the length of polygon line and the filling factor, are not capable of retrieving the delay time encarved in the time series even in delay dynamical systems with constant delay in the presence of noise. In particular, we have shown that for values of the noise intensity $D_0 \geq 0.01$, the aforesaid estimates failed to indicate the delay times by their local minima. In other words, the noise in the dynamical systems are capable of masking the delay time embedded in the dynamics, and the phase space cannot be collapsed into a simple manifold to enable phase space reconstruction even in systems with a single constant delay. We have also shown that a common Gaussian noise can drive two independent identical time delay systems such that they evolve in complete synchrony with each other for suitable values of the noise intensity in a wide range of time delays. With respect to the applications to secure communication and, in particular, to the task of masking of delays, we can interpret our findings as another instance where noise may play a constructive role.

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References

1. H. Fujisaka, T. Yamada, Prog. Theor. Phys. **69**, 32 (1983)
2. H. Fujisaka, T. Yamada, Prog. Theor. Phys. **70**, 1240 (1983)
3. L.M. Pecora, T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990)
4. See for example, Chaos **7** (1997), special focus issue on chaotic synchronization, edited by L. Pecora
5. A.S. Pikovsky, M.G. Rosenblum, J. Kurths, *Synchronization - A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001)
6. S. Boccaletti, J. Kurths, G. Osipov, D.L. Valladares, C.S. Zhou, Phys. Rep. **366**, 1 (2002)
7. See for example, Int. J. Bifurcation Chaos **10** (2000), special focus issue on phase synchronization, edited by J. Kurths
8. E.M. Shahverdiev, S. Sivaprakasam, K.A. Shore, Phys. Rev. E **66**, 037202 (2002)
9. S. Boccaletti, L.M. Pecora, A. Pelaez, Phys. Rev. E **63**, 066219 (2001)
10. K.M. Cuomo, A.V. Oppenheim, Phys. Rev. Lett. **71**, 65 (1993)
11. S. Hayes, C. Grebogy, E. Ott, A. Mark, Phys. Rev. Lett. **73**, 1781 (1994)
12. J. Garcia-Ojalvo, R. Roy, Phys. Rev. Lett. **86**, 5204 (2001)
13. G.D. VanWiggeren, R. Roy, Phys. Rev. Lett. **88**, 097903 (2002)
14. D.V. Senthilkumar, M. Lakshmanan, J. Kurths, Phys. Rev. E **74**, 035205(R) (2006)
15. D.V. Senthilkumar, M. Lakshmanan, J. Kurths, Chaos **18**, 023118 (2008)
16. D.V. Senthilkumar, J. Kurths, M. Lakshmanan, Phys. Rev. E **79**, 066208 (2009)
17. L. Kocarev, U. Parlitz, Phys. Rev. Lett. **74**, 5028 (1995)
18. B.B. Zhou, R. Roy, Phys. Rev. E **75**, 026205 (2007)
19. X. Wang, X. Wu, Y. He, G. Aniwar, Int. J. Mod. Phys. B **22**, 3709 (2008)
20. A. Argyris, D. Syvridis, L. Larger, V. Annovazzi-Lodi, P. Colet, I. Fischer, J. Garcia-Ojalvo, Claudio R. Mirasso, L. Pesquera, K.A. Shore, Nature **438**, 343 (2005)
21. G. Perez, H. Cerdeira, Phys. Rev. Lett. **74**, 1970 (1995)
22. K.M. Short, A.T. Parker, Phys. Rev. E **58**, 1159 (1998)
23. H. Voss, J. Kurths, Phys. Lett. A **234**, 336 (1997)
24. W.-H. Kye, M. Choi, S. Rim, M.S. Kurdoglyan, C.-M. Kim, Y.-J. Park, Phys. Rev. E **69**, 055202(R) (2004)
25. W.-H. Kye, M. Choi, M.S. Kurdoglyan, C.-M. Kim, Y.-J. Park, Phys. Rev. E **70**, 046211 (2004)
26. W.-H. Kye, M. Choi, C.-M. Kim, Y.-J. Park, Phys. Rev. E **71**, 045202(R) (2005)
27. D.V. Senthilkumar, M. Lakshmanan, Chaos **17**, 013112 (2007)
28. V.S. Anishchenko, V.V. Astachov, A.B. Neiman, T.E. Vadivasova, L. Schimansky-Geier, *Nonlinear Dynamics of Chaotic and Stochastic Systems* (Springer, Berlin, 2002)
29. L. Schimansky-Geier, D. Abbott, A. Neiman, C. van den Broeck (eds.), *Noise in Complex Systems and Stochastic Dynamics* (Proceedings of SPIE, Vol. 5114, Bellingham, WA, 2003)
30. C. Zhou, J. Kurths, I.Z. Kiss and J.L. Hudson, Phys. Rev. Lett. **89**, 014101 (2002)
31. C. Zhou, J. Kurths, Phys. Rev. Lett. **88**, 230602 (2002)
32. H. Lu, Z. He, IEEE Trans. Circuits Syst. I **43**, 700 (1996)
33. P. Thangavel, K. Murali, M. Lakshmanan, Int. J. Bifurcation Chaos **8**, 2481 (1998)
34. D.V. Senthilkumar, M. Lakshmanan, Int. J. Bifurcation Chaos **15**, 2895 (2005)
35. M.J. Bunner, Th. Meyer, A. Kittel, J. Parisi, Phys. Rev. E **56**, 5083 (1997)
36. M.J. Bunner, M. Popp, Th. Meyer, A. Kittel, J. Parisi, Phys. Rev. E **54**, 3082(R) (1996)