Chaos

Predicting noise-induced critical transitions in bistable systems

Cite as: Chaos **29**, 081102 (2019); https://doi.org/10.1063/1.5115348
Submitted: 17 June 2019 . Accepted: 24 July 2019 . Published Online: 20 August 2019

Jinzhong Ma, Yong Xu, Yongge Li, Ruilan Tian, and Jürgen Kurths

Chaos ARTICLE scitation.org/journal/cha

Predicting noise-induced critical transitions in bistable systems

Cite as: Chaos **29**, 081102 (2019); doi: 10.1063/1.5115348
Submitted: 17 June 2019 · Accepted: 24 July 2019 ·
Published Online: 20 August 2019

View Online Export Citation CrossMark

Jinzhong Ma,[1] Yong Xu,[1,2,a)] Yongge Li,[3] Ruilan Tian,[4] and Jürgen Kurths[5,6]

AFFILIATIONS

[1]Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China
[2]MIIT Key Laboratory of Dynamics and Control of Complex Systems, Northwestern Polytechnical University, Xi'an 710072, China
[3]Center for Mathematical Sciences and School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China
[4]Centre for Nonlinear Dynamics Research, Shijiazhuang Tiedao University, Shijiazhuang 050043, China
[5]Potsdam Institute for Climate Impact Research, Potsdam 14412, Germany
[6]Department of Physics, Humboldt University at Berlin, Berlin 12489, Germany

[a)]**Email:** hsux3@nwpu.edu.cn

ABSTRACT

Critical transitions from one dynamical state to another contrasting state are observed in many complex systems. To understand the effects of stochastic events on critical transitions and to predict their occurrence as a control parameter varies are of utmost importance in various applications. In this paper, we carry out a prediction of noise-induced critical transitions using a bistable model as a prototype class of real systems. We find that the largest Lyapunov exponent and the Shannon entropy can act as general early warning indicators to predict noise-induced critical transitions, even for an earlier transition due to strong fluctuations. Furthermore, the concept of the parameter dependent basin of the unsafe regime is introduced via incorporating a suitable probabilistic notion. We find that this is an efficient tool to approximately quantify the range of the control parameter where noise-induced critical transitions may occur. Our method may serve as a paradigm to understand and predict noise-induced critical transitions in multistable systems or complex networks and even may be extended to a broad range of disciplines to address the issues of resilience.

Critical transitions are widespread in natural, social, and technological systems, which can lead to detrimental system breakdowns. Therefore, it is important to develop effective tools to predict noise-induced critical transitions early. Many studies, in recent years, have been devoted to exploring early warning indicators to predict and characterize the onset of critical transitions. However, these early warning indicators are valid only for relatively small perturbations, and they sometimes cannot provide enough time for us to avert an impending noise-induced critical transition. To overcome these problems, in this paper, we verify that the largest Lyapunov exponent and the Shannon entropy can act as general early warning indicators. Furthermore, we introduce the parameter dependent basin of an unsafe regime to quantify the range of the control parameter where noise-induced critical transitions may occur.

I. INTRODUCTION

Complex systems in ecology, economics, sociology, physiology,[1–4] and many other fields often undergo slow changes affected by stochastic events, whose persistent effects may lead to a sudden or irreversible change of the system, such as a transition from one stable state to another one. In typical examples, such a transition is able to cause considerable impacts on human economies and societies[5,6] or even induce some unexpected disasters.[7,8] Therefore, it is crucially important to investigate noise-induced critical transitions so as to predict such a catastrophic event.

Recently, some studies in critical transitions showed that a phenomenon known as "critical slowing-down (CSD)" can be used to detect a sudden transition.[9–11] CSD means that the rate of return to equilibrium following small perturbations slows down as systems approach critical transitions. It can cause an increase in the variance

and autocorrelation of a system prior to a transition.[12–14] Therefore, CSD-related predictions have become a hot topic and are increasingly attracting much attention from various disciplines.[15–17] However, it should be mentioned that the CSD principle holds only for a critical transition sufficiently approaching the deterministic bifurcation, namely, perturbations are relatively small. Many practical systems in fact are usually intrinsically or extrinsically convoluted with varying scales of perturbations.[18,19] Strong fluctuations may induce an earlier critical transition far before the deterministic bifurcation, for which all CSD-based early warning indicators may fail.[20,21] A crucial question is whether these indicators can provide enough time for us to avert an imminent critical transition.[22,23] To counter these problems, in this paper, not only more general early warning indicators will be explored but also the range of the control parameter in which noise-induced critical transitions may occur will be quantified.

Given a noise-induced complex system with a varying control parameter, there are two common features during the process of a transition near a critical value. One is that the instability of the system becomes more prominent,[24–26] and the other is that the height of the potential barrier decreases, namely, the current desirable state is gradually absorbed.[1,27–29] With these in mind, we can carry out a prediction of noise-induced critical transitions via detecting the instability of a system and quantifying the range of the control parameter where the desirable state may be absorbed. In nonlinear stochastic dynamics, the largest Lyapunov exponent can measure the intrinsic instability of the trajectories in a system and detect qualitative changes of the system behavior with varying control parameters.[30–32] In addition, the Shannon entropy is another measure of instability, and it can reflect the change in the dynamics of a system.[33–35] In fact, the likelihood that the height of the potential barrier disappears or a desirable stable state is absorbed depends on its stability against significant perturbations. The traditional linearization-based approach to study stability is too local to adequately assess how stable a state is. A new measure, basin stability, related to the volume of the basin of attraction was recently proposed.[36,37] It is nonlocal, nonlinear, and easily applicable to complex systems. The extension may provide a powerful tool for the prediction of noise-induced critical transitions in complex systems.

In this paper, we analyze the instability of an imminent critical transition through the largest Lyapunov exponent and Shannon entropy, respectively. Furthermore, we extend basin stability to a new concept: the "parameter dependent basin of an unsafe regime" (PDBUR) to predict noise-induced critical transitions. We find that the largest Lyapunov exponent and the Shannon entropy can act as generic early warning indicators of noise-induced critical transitions, and PDBUR can approximately quantify a range of control parameters where noise-induced critical transitions may occur. A combination of them provides sufficient warnings for disaster management groups to hinder noise-induced critical transitions.

The rest of the paper is organized as follows. In Sect. II, the noise-induced critical transitions of a given model are introduced. In Sec. III, the largest Lyapunov exponent and the Shannon entropy are validated as early warning indicators to predict these noise-induced critical transitions. Section IV introduces the concept of PDBUR to approximately quantify the range of the control parameter where noise-induced critical transitions may occur. Finally, discussions are presented in Sec. V.

II. NOISE-INDUCED CRITICAL TRANSITIONS IN BISTABLE SYSTEMS

To describe parameterized phosphorus dynamics in a lake approaching eutrophication under a regime of external perturbations,[38–40] a stochastic bistable model can be defined as

$$dx = f(x,\alpha)dt + \sqrt{\sigma}dB(t), \tag{1}$$

where $f(x,\alpha) = -\frac{\partial}{\partial x}U(x,\alpha) = \alpha - x + \frac{x^8}{x^8+1}$ and $U(x,\alpha)$ denotes the potential function of x. α is a control parameter, $B(t)$ is the Brownian motion, and σ denotes the noise intensity.

Equation (1) can be reduced to the corresponding deterministic system for $\sigma = 0$, and its equilibrium point x_E with changing α can be obtained through

$$f(x_E,\alpha) = 0.$$

The stability of x_E can be determined based on the positive or negative value of the derivative of $f(x,\alpha)$ with respect to x,

$$\left.\frac{df(x,\alpha)}{dx}\right|_{x=x_E} = -1 + \left.\frac{8x^7}{(x^8+1)^2}\right|_{x=x_E}.$$

Then, the geometric structure of the deterministic system vs α is shown in Fig. 1. Three equilibrium points x_{S1}, x_U, and x_{S2}, in the bistable region (shaded part), are exhibited for a given α. x_{S1} in the lower branch means that Eq. (1) remains in the desirable state, while x_{S2} in the upper branch says that Eq. (1) stays in a state space populated by the undesirable state. x_U in the middle branch (dotted line) marks the height of the potential barrier between x_{S1} and x_{S2}. The coordinate values of the two bifurcation points, Fold 1 and Fold 2, are ($\alpha_{Fold1} = 0.660, x_{E\_Fold1} = 0.768$) and ($\alpha_{Fold2} = 0.389, x_{E\_Fold2} = 1.204$), respectively. If Eq. (1) approaches Fold 1, a slight incremental change in α may bring it beyond the bifurcation and induce a transition to the undesirable state x_{S2}.

Figure 2 shows the average state of x in the presence of a noise with changing α. We find that the critical point approaches Fold 1 when σ is small. While the critical transition takes place much earlier with large σ, and the larger the σ the more obvious the phenomenon

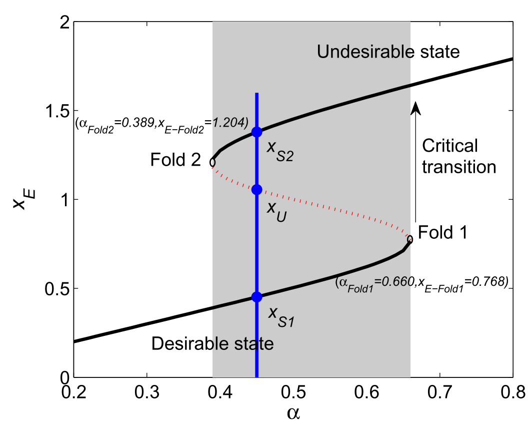

FIG. 1. Geometric structure of Eq. (1) with respect to changing α and $\sigma = 0$.

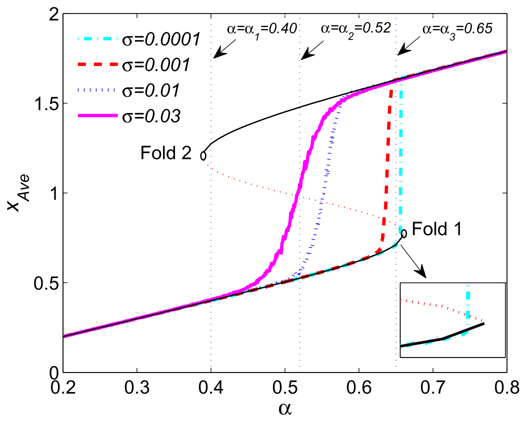

FIG. 2. Noise-induced critical transitions of Eq. (1) vs different σ. x_{Ave} denotes the average state of x in the presence of noise. The inset plot shows that the critical point approaches Fold 1.

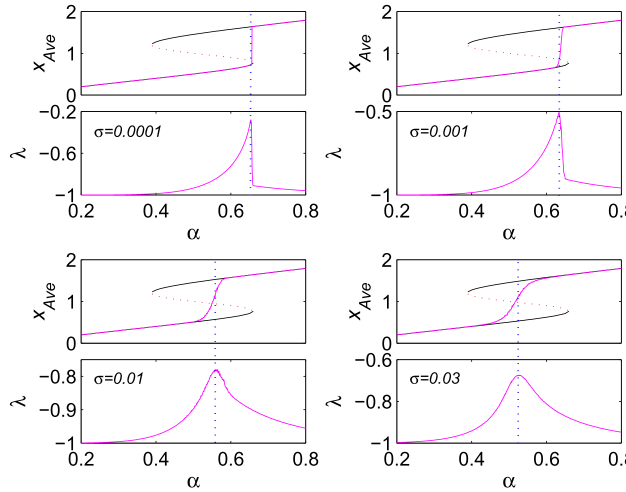

FIG. 3. Largest Lyapunov exponent λ of Eq. (1) for four different σ.

is. It is, thus, evident that stochastic perturbations may induce a transition to the alternative stable state if they are sufficiently large to bring the system over the height of the potential barrier. Here, such large σ values that destroy the bistable structure are not considered.

III. EARLY WARNING INDICATORS OF NOISE-INDUCED CRITICAL TRANSITIONS

In a noise-induced bistable model, the response will become more unstable as it approaches the critical point. Then, a critical transition is easy to occur. As is well known, the largest Lyapunov exponent and the Shannon entropy are appropriate measures for instability of trajectories in a stochastic system. Therefore, we will further examine that the largest Lyapunov exponent and the Shannon entropy may act as general early warning indicators to predict noise-induced critical transitions.

A. Largest Lyapunov exponent

The largest Lyapunov exponent λ is a very powerful tool to investigate complex system dynamics. One of the most advantageous applications is to detect the qualitative changes of the system behavior with changing control parameters.[41,42] Here, it will be used to identify noise-induced critical transitions. In the present paper, the method introduced by Benettin *et al.* will be employed to calculate λ.[43,44]

The λ of Eq. (1) for different σ are shown in Fig. 3. We uncover that an obvious increase of λ can be obtained when the critical transition approaches the deterministic bifurcation. Moreover, an increase in λ is also evident when the critical transition takes place far from the bifurcation point of the corresponding deterministic system due to strong fluctuations. More importantly, the increase in λ appears in advance with the critical transition earlier, as shown by the vertical dashed lines in Fig. 3. It is, thus, clear that the remarkable difference of λ is a general feature to an impending noise-induced critical transition. Through a space diagram in Fig. 4, we can more intuitively witness an earlier increase of λ with increasing σ. Therefore, λ can act as a general early warning indicator to predict noise-induced critical transitions.

B. Shannon entropy

Entropy-based measures are efficient to quantify the dynamical characteristics of complex systems.[45] Entropy analysis with respect to a changing parameter may provide a special signal for an impending noise-induced critical transition.[46]

We use the definition of the entropy for a discrete random variable introduced by Shannon.[33,47] Let X be a discrete random variable with the possible states $X_1, X_2, \ldots, X_n$, whose corresponding probability is P_i, $i = 1, 2, \ldots, n$. Then, the Shannon entropy is defined as

$$H_s = -\sum_{i=1}^{n} P_i \log_2(P_i). \tag{2}$$

To obtain the Shannon entropy of Eq. (1), three parts of α need to be considered, including $\alpha < \alpha_{Fold2}$, $\alpha \in [\alpha_{Fold2}, \alpha_{Fold1}]$, and $\alpha > \alpha_{Fold1}$. For $\alpha \in [\alpha_{Fold2}, \alpha_{Fold1}]$, the expressions to calculate the probability of the two possible states x_{S1} and x_{S2} are

$$P_{11} = \left\langle \frac{1}{T}\{t : x(t) < x_U,\ t \le T\} \right\rangle$$

and

$$P_{12} = \left\langle \frac{1}{T}\{t : x(t) \ge x_U,\ t \le T\} \right\rangle,$$

where T is the time length of a trajectory and $\langle\cdot\rangle$ is the operator to average multiple realizations.

For $\alpha < \alpha_{Fold2}$, we define

$$P_{21} = \left\langle \frac{1}{T}\{t : x(t) < x_{E\_Fold2},\ t \le T\} \right\rangle$$

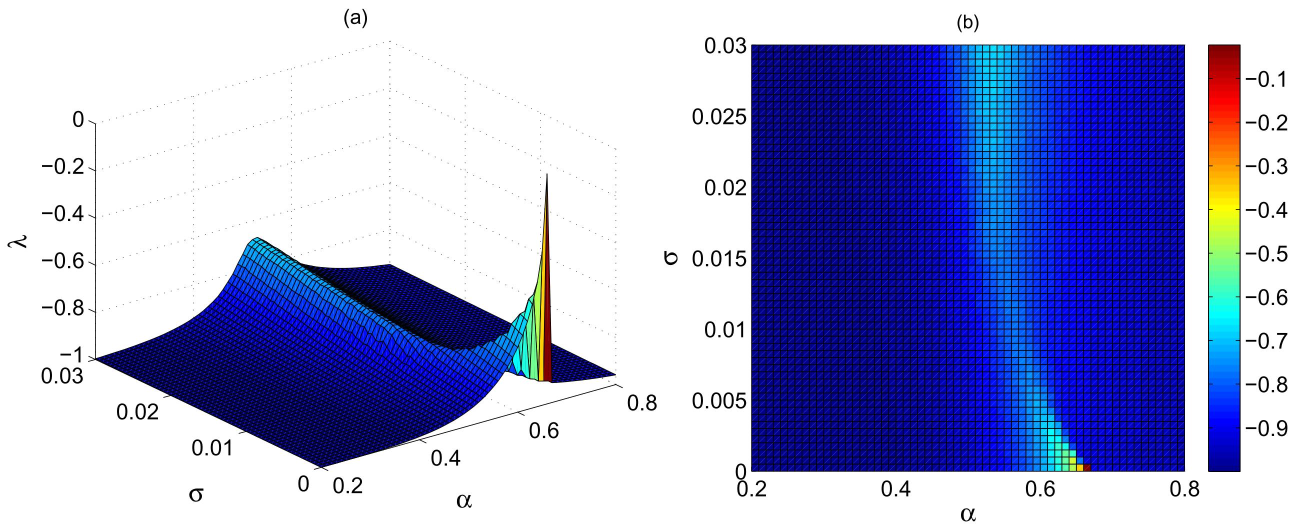

FIG. 4. Largest Lyapunov exponent λ of Eq. (1) vs σ and α. (a) The space diagram of λ. (b) The vertical view of λ.

and

$$P_{22} = \left\langle \frac{1}{T}\{t : x(t) \geq x_{E\_Fold2},\ t \leq T\} \right\rangle.$$

Similarly, the probability in $\alpha > \alpha_{Fold1}$ is given as

$$P_{31} = \left\langle \frac{1}{T}\{t : x(t) < x_{E\_Fold1},\ t \leq T\} \right\rangle$$

and

$$P_{32} = \left\langle \frac{1}{T}\{t : x(t) \geq x_{E\_Fold1},\ t \leq T\} \right\rangle.$$

Based on Eq. (2), the H_s of Eq. (1) with three different parts of α is composed of

$$H_{si} = -P_{i1}\log_2 P_{i1} - P_{i2}\log_2 P_{i2},\ i = 1, 2, 3. \tag{3}$$

Here the initial condition is set as $x(0) = x_{S1}$ corresponding to different α.

Based on Eq. (3), the numerical results of H_s for different σ are shown in Fig. 5. Although an increase in H_s is obtained when the critical transition approaches the deterministic bifurcation, it appears so suddenly that there is not enough time to avert it. Thus, λ is more suitable for predicting the critical transitions than H_s when σ is smaller. To an earlier critical transition under strong fluctuations, a meaningful change of H_s allows us to witness the possibility to warn a noise-induced critical transition. Similar to the result of λ, an increase in H_s appears earlier with the critical transition occurring earlier. From a space diagram of H_s shown in Fig. 6, we find that an earlier increase appears with increasing σ, and it becomes easier to be identified. All our numerical results indicate that both λ and H_s can act as general early warning indicators to predict the noise-induced critical transitions.

IV. PARAMETER DEPENDENT BASIN OF THE UNSAFE REGIME

Beyond establishing more general early warning indicators of impending noise-induced critical transitions, we hope to further quantify the range of the control parameter in which noise-induced critical transitions may occur. In this section, the concept of PDBUR will be introduced by incorporating a suitable probabilistic notion, and it will be an appropriate measure of the influence of Gaussian white noise on critical transitions.

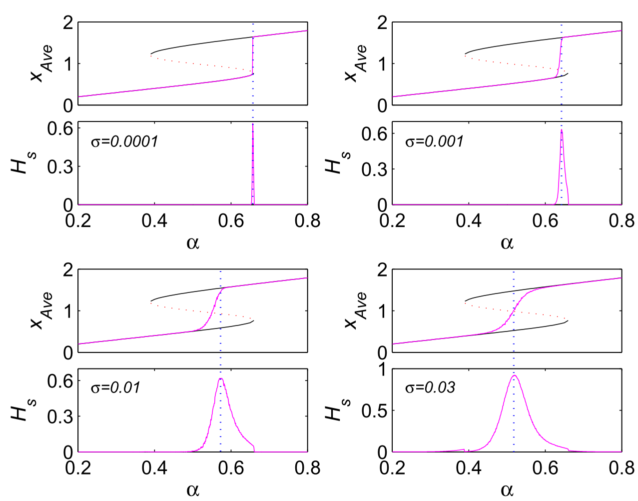

FIG. 5. Shannon entropy H_s of Eq. (1) with four different σ.

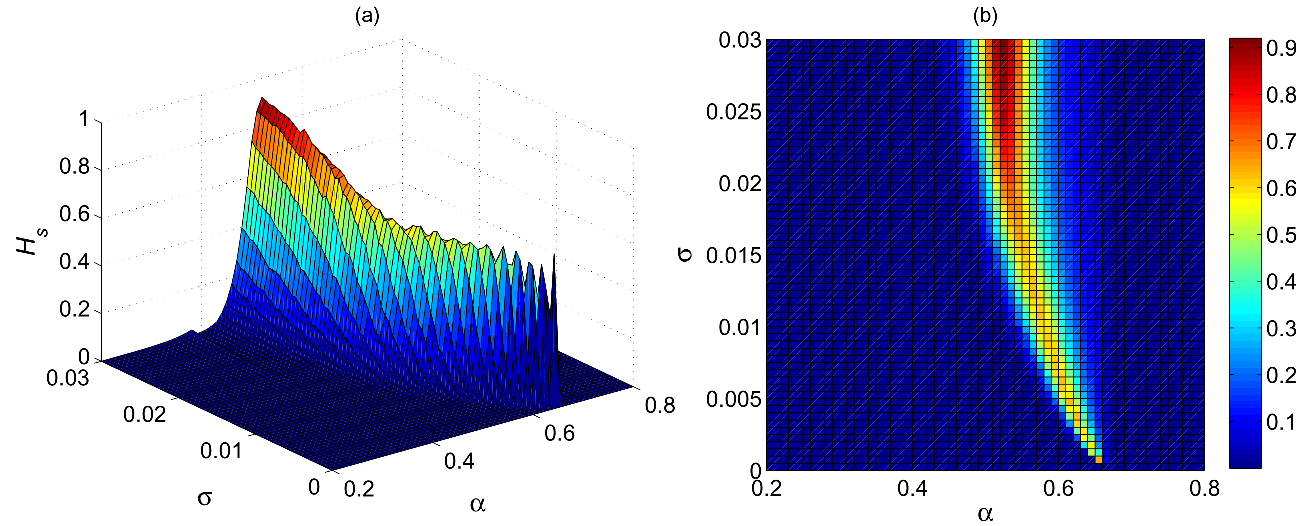

FIG. 6. Shannon entropy H_s of Eq. (1) vs σ and α. (a) The space diagram of H_s. (b) The vertical view of H_s.

A. The absorbed region within $[x_{S1}, x_U]$ under the fixed $\alpha \in [\alpha_{Fold2}, \alpha_{Fold1}]$

Stochastic events may bring a complex system into another contrasting and undesirable stable state, i.e., almost all $x \in [x_{S1}, x_U]$ are absorbed. Therefore, the possibility of a noise-induced critical transition occurring with a fixed $\alpha \in [\alpha_{Fold2}, \alpha_{Fold1}]$ can be estimated via measuring the part of $[x_{S1}, x_U]$ that may be absorbed.

Considering three representative $U(x)$ of Eq. (1) as shown in Fig. 7, we find that the occurrence of a critical transition depends on the perturbations as well as on the barrier height of $U(x)$. In the case of α_1, the potential well of x_{S1} is deeper, a transition occurs only if the perturbations are sufficiently large. The ability to switch between x_{S1} and x_{S2} is almost the same with α_2. Nevertheless, the potential well of x_{S1} becomes shallower in the case of α_3, and the small perturbations may result in a transition to a contrasting dynamical state. In Table I, $[x_{S1}, x_U]$ corresponding to these three different cases of α are presented. Based on the escape probability, we will give a definition of the absorbed region within $[x_{S1}, x_U]$ under the fixed $\alpha \in [\alpha_{Fold2}, \alpha_{Fold1}]$.

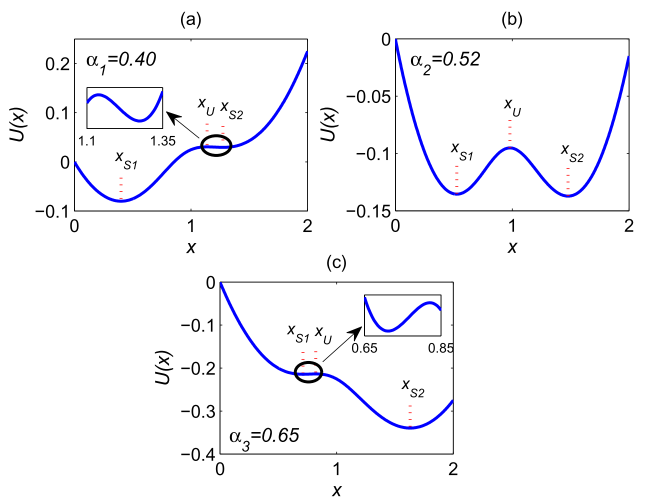

FIG. 7. Potential function $U(x)$ of Eq. (1) for $\sigma = 0$ and three different α.

In fact, if Eq. (1) starts from a point x in the region $[x_{S1}, x_U]$, it will either exit the region from the left side x_{S1} or from the right side x_U. Here, we focus on the escape probability exiting through x_U, denoted as $P(x)$, which can be described by the following equation:[48,49]

$$\mathrm{L}P(x) = f(x,\alpha)\frac{dP(x)}{dx} + \frac{1}{2}\sigma\frac{d^2P(x)}{dx^2} = 0, \tag{4}$$

where L is a generator. For the boundary cases, we suppose that the escape does not occur once the solution path of Eq. (1) reaches x_{S1}. On the contrary, the escape takes place if the solution path reaches x_U. Thus, the boundary conditions of Eq. (4) should be set as $P(x)|_{x=x_{S1}} = 0$ and $P(x)|_{x=x_U} = 1$. A numerical method for solving Eq. (4) can be found in Appendix.

From Fig. 2, we observe that noise-induced critical transitions appear in the cases of $\sigma = 0.01$ and $\sigma = 0.03$ with α_3. To obtain some generic behavior of $P(x)$ in noise-induced critical transitions, $P(x)$

TABLE I. The $[x_{S1}, x_U]$ corresponding to three different cases of α.

α	$[x_{S1}, x_U]$
α_1	[0.401, 1.139]
α_2	[0.526, 0.980]
α_3	[0.712, 0.821]

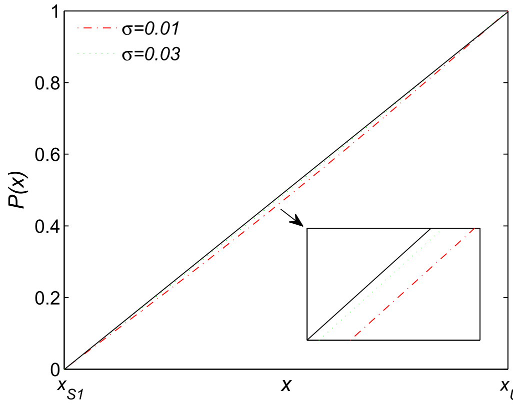

FIG. 8. Escape probability $P(x)$ for α_3 and two different σ. The solid line is $y(x)$. The inset shows that $P(x)$ is very close to $y(x)$.

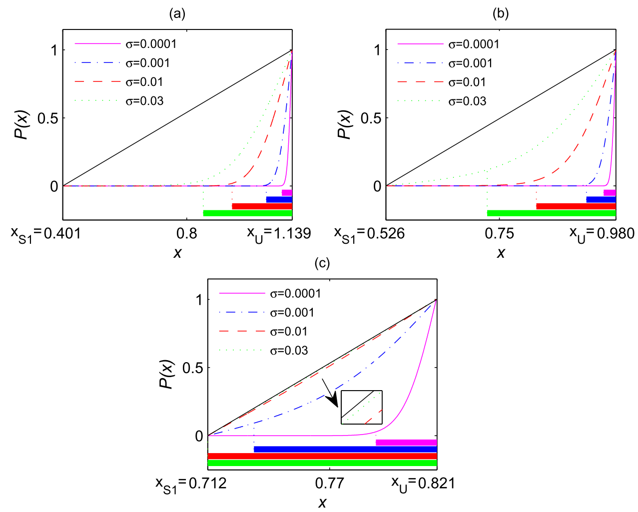

FIG. 9. Escape probability $P(x)$ and the corresponding absorbed region D_{AR} (bar-type) for different σ. (a) $\alpha_1 = 0.40$. (b) $\alpha_2 = 0.52$. (c) $\alpha_3 = 0.65$. The top solid line is $y(x)$. The inset in (c) shows that $P(x)$ is very close to $y(x)$.

under these two cases are simulated in Fig. 8. There is an interesting phenomenon that $P(x)$ is very close to $y(x) = \frac{1}{x_U - x_{S1}}x + \frac{x_{S1}}{x_{S1} - x_U}$, a line connecting $(x_{S1}, 0)$ and $(x_U, 1)$ in the x vs $P(x)$ plane. It means that the tangent slope of $P(x)$ is very close to $\frac{1}{x_U - x_{S1}}$ when the region $[x_{S1}, x_U]$ is almost completely absorbed. That is, we can approximately obtain the possible absorbed region within $[x_{S1}, x_U]$ via detecting the change of the tangent slope of $P(x)$. Therefore, an approximate definition of the absorbed region within $[x_{S1}, x_U]$ is given as:

Definition 1. Assuming that a tagged partition of $[x_{S1}, x_U]$ is a finite sequence

$$x_{S1} = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = x_U,$$

we call the set $[x_k, x_U]$, $x_{S1} \le x_k < x_U$, $k = 0, 1, \ldots, n-1$ satisfying $\frac{P(x_{k+1}) - P(x_k)}{x_{k+1} - x_k} \ge \varepsilon \frac{1}{x_U - x_{S1}}$, $0 < \varepsilon < 1$ as the "absorbed region" within $[x_{S1}, x_U]$ under σ, and it is denoted as D_{AR}.

Within Definition 1, the selection of ε is critical, which may be variable for different systems. If the critical transition occurs quickly, a smaller ε should be considered, such as $\varepsilon < \frac{1}{2}$. On the contrary, a larger ε that approaches 1 should be chosen. The selection of ε makes it possible that we not only obtain a more accurate result, but also have enough time to adapt some management to avert an imminent noise-induced critical transition. Taking $\varepsilon = \frac{1}{2}$ as a case study, in the present paper, D_{AR} under different σ and $\alpha \in [\alpha_{Fold2}, \alpha_{Fold1}]$ will be investigated.

Based on Eq. (4) and Definition 1, $P(x)$ and D_{AR} for different α and σ are shown in Fig. 9. The corresponding interval length of D_{AR} is presented in Table II. Obviously, the larger σ is, the closer $P(x)$ is to $y(x)$, and the larger the range of D_{AR} is. Especially, for $\sigma = 0.01$ and $\sigma = 0.03$ shown in Fig. 8(c), $P(x)$ almost coincides with $y(x)$, and the whole $[x_{S1}, x_U]$ becomes D_{AR}. It indicates that the noise-induced critical transitions have taken place in both cases, and the same conclusion can be observed in Fig. 2. Although the whole $[x_{S1}, x_U]$ is not absorbed, a noise-induced critical transition may also occur. For example, a critical transition in Eq. (1) has occurred under α_3 and $\sigma = 0.001$. A natural question is: for how big $\frac{\mu(D_{AR})}{\mu([x_{S1}, x_U])} \triangleq \mu_P$ is (here μ is a measurement of the interval length), a noise-induced critical transition may occur? Next, we will define PDBUR of noise-induced critical transitions.

B. PDBUR of noise-induced critical transitions

Through the noise-induced critical transition of Eq. (1) with $\sigma = 0.0001$ in Fig. 2, we find that Eq. (1) is about to leave the lower branch of x_E when $\alpha = 0.654$, and it has left it when $\alpha = 0.655$. μ_P in both cases are

$$\mu_P|_{\alpha=0.654} = 0.379$$

and

$$\mu_P|_{\alpha=0.655} = 0.428.$$

Therefore, we approximate PDBUR of α via counting the point where

$$\mu_P \ge \frac{\mu_P|_{\alpha=0.654} + \mu_P|_{\alpha=0.655}}{2} \approx 0.404 \triangleq \mu_{P\_c}.$$

Definition 2. For D_{AR} and $[x_{S1}, x_U]$ under σ and $\alpha \in [\alpha_{Fold2}, \alpha_{Fold1}]$, we call the set of α satisfying $\mu_P \ge \mu_{P\_c}$ as PDBUR, and it is denoted as $UB(\alpha, \sigma)$.

TABLE II. The interval length of D_{AR} under different α and σ.

	σ			
α	0.0001	0.001	0.01	0.03
0.40	[1.106,1.139]	[1.056,1.139]	[0.946,1.139]	[0.853,1.139]
0.52	[0.956,0.980]	[0.922,0.980]	[0.823,0.980]	[0.726,0.980]
0.65	[0.792,0.821]	[0.734,0.821]	[0.712,0.821]	[0.712,0.821]

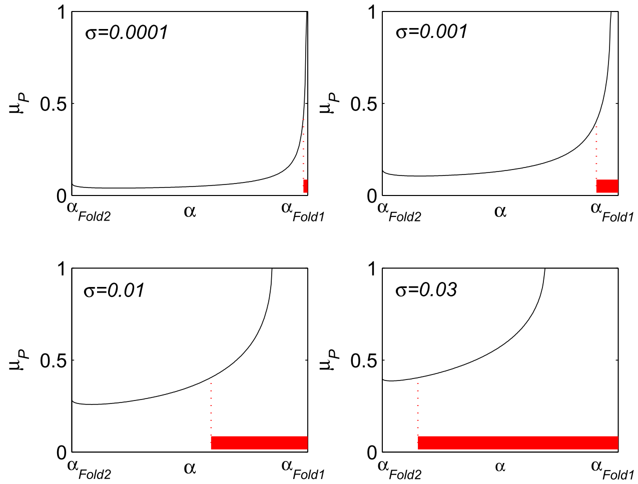

FIG. 10. The measurement μ_P and the PDBUR $UB(\alpha, \sigma)$ with respect to different σ. The bar-type denotes interval length of $UB(\alpha, \sigma)$.

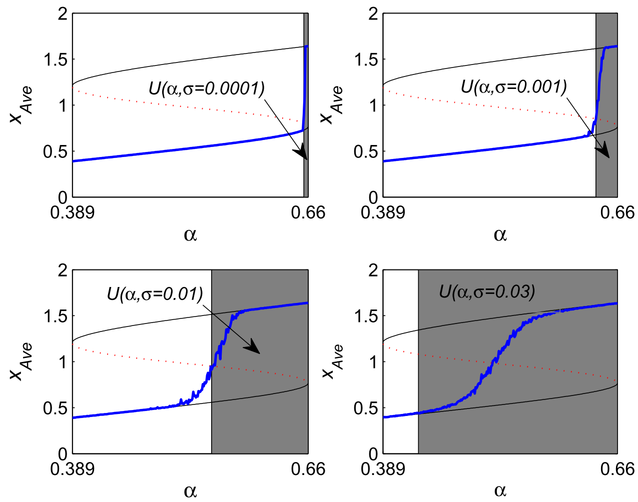

FIG. 11. The PDBUR $UB(\alpha, \sigma)$ (shaded parts) for four different σ.

In Fig. 10, μ_p and $UB(\alpha, \sigma)$ for different σ are shown. We observe that σ plays a crucial role to expand the interval length of $UB(\alpha, \sigma)$: the larger the σ, the larger the range of $UB(\alpha, \sigma)$. When σ is large enough, such as $\sigma = 0.03$, nearly $\mu_P \approx 85\%$ of the bistable region $[\alpha_{Fold2}, \alpha_{Fold1}]$ becomes $UB(\alpha, \sigma)$. This phenomenon is also associated with an earlier noise-induced critical transition taking place under strong fluctuations.

To make the results more intuitive, $UB(\alpha, \sigma)$ and x_{Ave} under four different σ are simultaneously shown in Fig. 11. Obviously, $UB(\alpha, \sigma)$ can be quantified when a critical transition impending. Because the critical transition occurs more suddenly with $\sigma = 0.03$, $UB(\alpha, \sigma)$ should be considered once Eq. (1) leaves the desirable state. Our results also reflect this phenomenon. Therefore, a satisfactory result has been found based on Definitions 1 and 2 by taking $\varepsilon = \frac{1}{2}$. More accurate results may be realized in terms of the other ε.

Figure 12 indicates the μ_p with respect to α and σ. Once α and σ enter $UB(\alpha, \sigma)$, there is a strong possibility that a noise-induced critical transition occurs. Now, some management should be adapted to avert it. As the well-known concept of basin stability, the introduced

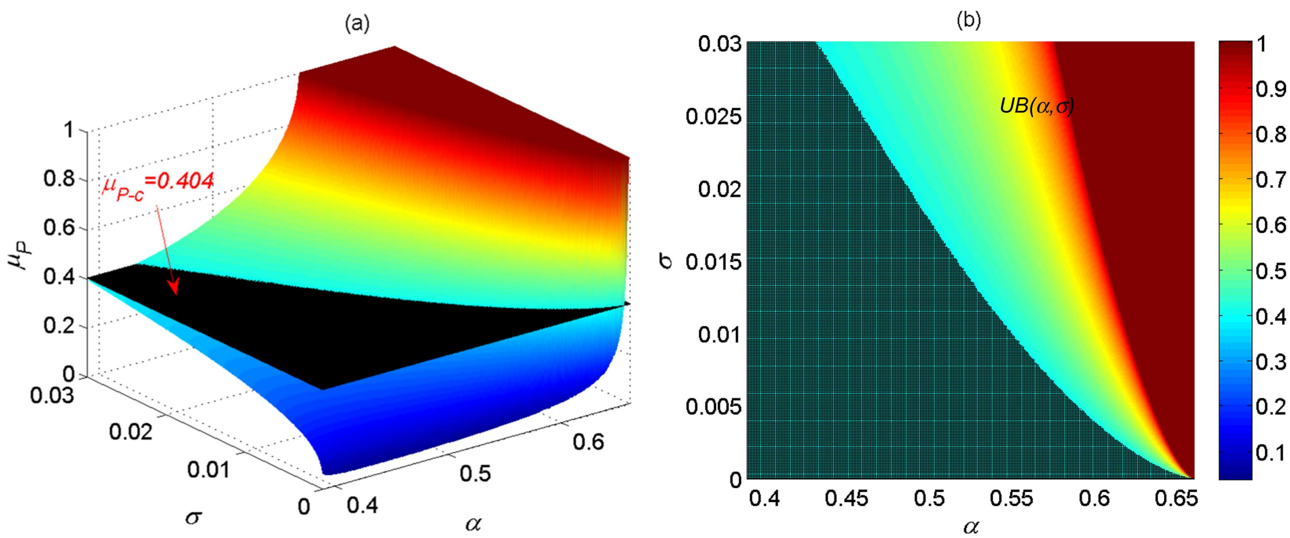

FIG. 12. The measurement μ_P vs σ and α. (a) The space diagram of μ_P. The inserted plane represents $\mu_{P\_c} = 0.404$. (b) The vertical view of μ_P and $UB(\alpha, \sigma)$ can be visualized on the upper right.

$UB(\alpha,\sigma)$ is the important geometric structure that can be used to predict the noise-induced critical transitions in complex systems.

V. DISCUSSIONS

In this paper, we have focused on a Gaussian white noise-induced eutrophication model as a concrete example to show the prediction of noise-induced critical transitions. To understand the role of stochastic events with respect to the emergence of a critical transition, the existing results on stochastic dynamic theory should be used for its prediction. With this in mind, we carry out a measure of the instability to an impending critical transition. Our results show that the largest Lyapunov exponent and the Shannon entropy can serve as generic early warning indicators to predict noise-induced critical transitions. Considering some practical difficulties in averting an imminent noise-induced critical transition, for example, the largest Lyapunov exponent or the Shannon entropy sometimes is too late to warn a transition. Therefore, PDBUR is further introduced to approximately quantify the range of the control parameter where noise-induced critical transitions may occur from a nonlocal perspective.

Furthermore, some problems should be addressed. For example, what will happen for these catastrophic transitions in multistable or high-dimensional systems, or the effects of different kinds of random noises like Lévy noise still deserves more concentrations. In addition, noises different from the Gaussian one and the coupling of multistable or high-dimensional systems may induce rich dynamical behaviors with unexpected critical transitions. Whether the definitions and methods proposed in this paper can be extended to warn more complex critical transitions should be explored in future.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grant No. 11772255, the Fundamental Research Funds for the Central Universities, the Shaanxi Science Fund for Distinguished Young Scholars, and Innovation Foundation for Doctor Dissertation of Northwestern Polytechnical University. Y. Li thanks the China Postdoctoral Science Foundation funded project.

APPENDIX

Let $x_{S1} = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = x_U$ be a tagged partition of $[x_{S1}, x_U]$ and Δx be the step length. Based on the second-order difference, Eq. (4) can be rewritten as

$$\frac{\sigma}{2}\frac{P(x_{j-1}) - 2P(x_j) + P(x_{j+1})}{\Delta x^2} + f(x_j,\alpha)\frac{P(x_{j+1}) - P(x_{j-1})}{2\Delta x} = 0, \quad j = 1, \ldots, n-1.$$

Then, we get

$$\operatorname{diag}\left(\frac{\sigma}{2}\right) AP(x) + a + \operatorname{diag}(f(x,\alpha))BP(x) + b = \left(\operatorname{diag}\left(\frac{\sigma}{2}\right) A + \operatorname{diag}(f(x,\alpha))B\right) P(x) + (a+b) = 0,$$

where

$$\operatorname{diag}\left(\frac{\sigma}{2}\right) = \frac{1}{\Delta x^2}\begin{pmatrix} \frac{\sigma}{2} & & & & & \\ & \frac{\sigma}{2} & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \frac{\sigma}{2} & \\ & & & & & \frac{\sigma}{2} \end{pmatrix}, \quad a = \frac{1}{\Delta x^2}\begin{pmatrix} \frac{\sigma}{2}P(x_0) \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \frac{\sigma}{2}P(x_n) \end{pmatrix},$$

$$\operatorname{diag}(f(x,\alpha)) = \frac{1}{2\Delta x}\begin{pmatrix} f(x_1,\alpha) & & & & & \\ & f(x_2,\alpha) & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & f(x_{n-2},\alpha) & \\ & & & & & f(x_{n-1},\alpha) \end{pmatrix},$$

$$b = \frac{1}{2\Delta x}\begin{pmatrix} -f(x_1,\alpha)P(x_0) \\ 0 \\ \vdots \\ \vdots \\ 0 \\ f(x_{n-1},\alpha)P(x_n) \end{pmatrix}, \quad A = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}, \quad P(x) = \begin{pmatrix} P(x_1) \\ P(x_2) \\ \vdots \\ \vdots \\ P(x_{n-2}) \\ P(x_{n-1}) \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}.$$

Finally, $P(x)$ can be obtained as

$$\begin{aligned} P(x) &= \left(\mathrm{diag}\left(\frac{\sigma}{2}\right)A + \mathrm{diag}(f(x,\alpha))B\right)^{-1}(-(a+b)) \\ &= -\left(\mathrm{diag}\left(\frac{\sigma}{2}\right)A + \mathrm{diag}(f(x,\alpha))B\right)^{-1}(a+b). \end{aligned}$$

REFERENCES

[1] M. Scheffer, S. Carpenter, J. A. Foley, C. Folke, and B. Walker, "Catastrophic shifts in ecosystems," Nature **413**, 591–596 (2001).
[2] R. M. May, S. A. Levin, and G. Sugihara, "Ecology for bankers," Nature **451**, 893–895 (2008).
[3] W. A. Brock, *Tipping Points, Abrupt Opinion Changes, and Punctuated Policy Change* (Department of Economics, UW Madison, 2004).
[4] J. G. Venegas *et al.*, "Self-organized patchiness in asthma as a prelude to catastrophic shifts," Nature **434**, 777–782 (2005).
[5] F. Berkes, C. Folke, and J. Colding, *Linking Social and Ecological Systems: Management Practices and Social Mechanisms for Building Resilience* (Cambridge University Press, Cambridge, 1998).
[6] M. Scheffer, *Critical Transitions in Nature and Society* (Princeton University Press, Princeton, NJ, 2009).
[7] A. D. Barnosky *et al.*, "Has the earth's sixthmass extinction already arrived?," Nature **471**, 51–57 (2011).
[8] A. D. Barnosky *et al.*, "Approaching a state shift in earth's biosphere," Nature **486**, 52–58 (2012).
[9] C. Wissel, "A universal law of the characteristic return time near thresholds," Oecologia **65**, 101–107 (1984).
[10] K. Wiesenfeld and B. McNamara, "Small-signal amplification in bifurcating dynamical systems," Phys. Rev. A **33**, 629–642 (1986).
[11] M. Scheffer *et al.*, "Early-warning signals for critical transitions," Nature **461**, 53–59 (2009).
[12] S. R. Carpenter and W. A. Brock, "Rising variance: A leading indicator of ecological transition," Ecol. Lett. **9**, 311–318 (2006).
[13] V. Dakos *et al.*, "Slowing down as an early warning signal for abrupt climate change," Proc. Natl. Acad. Sci. U.S.A. **105**, 14308–14312 (2008).
[14] V. Dakos *et al.*, "Methods for detecting early warnings of critical transitions in time series illustrated using simulated ecological data," PLoS ONE **7**, e41010 (2012).
[15] E. A. Gopalakrishnan, Y. Sharma, T. John, P. S. Dutta, and R. I. Sujith, "Early warning signals for critical transitions in a thermoacoustic system," Sci. Rep. **6**, 35310 (2016).
[16] X. Zhang, C. Kuehn, and S. Hallerberg, "Predictability of critical transitions," Phys. Rev. E **92**, 052905 (2015).
[17] C. F. Clements, M. A. McCarthy, and J. L. Blanchard, "Early warning signals of recovery in complex systems," Nat. Commun. **10**, 1681 (2019).
[18] Z. Wang, Y. Xu, and H. Yang, "Lévy noise induced stochastic resonance in an FHN model," Sci. China Tech. Sci. **59**, 371–375 (2016).
[19] Y. Xu, H. Li, H. Wang, W. Jia, X. Yue, and J. Kurths, "The estimates of the mean first exit time of a bi-stable system excited by Poisson white noise," J. Appl. Mech. **84**, 091004 (2017).
[20] R. Liu, P. Chen, K. Aihara, and L. Chen, "Identifying early-warning signals of critical transitions with strong noise by dynamical network markers," Sci. Rep. **5**, 17501 (2015).
[21] J. Ma, Y. Xu, J. Kurths, H. Wang, and W. Xu, "Detecting early-warning signals in periodically forced systems with noise," Chaos **28**, 113601 (2018).
[22] R. Biggs, S. R. Carpenter, and W. A. Brock, "Turning back from the brink detecting an impending regime shift in time to avert it," Proc. Natl. Acad. Sci. U.S.A. **106**, 826–831 (2009).
[23] P. D. Ditlevsen and S. J. Johnsen, "Tipping points: Early warning and wishful thinking," Geophys. Res. Lett. **37**, L19703 (2010).
[24] L. Moreau and E. Sontag, "Balancing at the border of instability," Phys. Rev. E **68**, 020901 (2003).
[25] R. V. Solé and S. Valverde, "Information transfer and phase transitions in a model of internet traffic," Physica A **289**, 595–605 (2001).
[26] F. Nazarimehr, S. Jafari, S. M. R. H. Golpayegani, M. Perc, and J. C. Sprott, "Predicting tipping points of dynamical systems during a period-doubling route to chaos," Chaos **28**, 073102 (2018).
[27] V. N. Livina, F. Kwasniok, and T. M. Lenton, "Potential analysis reveals changing number of climate states during the last 60 kyr," Clim. Past. **6**, 77–82 (2010).

[28] Y. Xu, J. Li, J. Feng, H. Zhang, W. Xu, and J. Duan, "Lévy noise-induced stochastic resonance in a bistable system," Eur. Phys. J. B **86**, 198 (2013).
[29] Y. Li, Y. Xu, and J. Kurths, "First-passage-time distribution in a moving parabolic potential with spatial roughness," Phys. Rev. E **99**, 052203 (2019).
[30] S. P. Kuznetsov and D. I. Trubetskov, "Chaos and hyperchaos in a backward-wave oscillator," Radiophys. Quant. Eletron. **47**, 341–355 (2004).
[31] S. P. Kuznetsov, "Example of a physical system with a hyperbolic attractor of the Smale-Williams type," Phys. Rev. Lett. **95**, 144101 (2005).
[32] C. Matsuoka and K. Hiraide, "Computation of entropy and Lyapunov exponent by a shift transform," Chaos **25**, 103110 (2015).
[33] C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication* (University of Illinois Press, Urbana, 1998).
[34] R. Yan and R. X. Gao, "Approximate entropy as a diagnostic tool for machine health monitoring," Mech. Syst. Signal Pr. **21**, 824–839 (2007).
[35] G. Poveda and H. D. Salas, "Statistical scaling, Shannon entropy, and generalized space-time q-entropy of rainfall fields in tropical South America," Chaos **25**, 075409 (2015).
[36] P. J. Menck, J. Heitzig, N. Marwan, and J. Kurths, "How basin stability complements the linear-stability paradigm," Nat. Phys. **9**, 89–92 (2013).
[37] Y. Zheng, L. Serdukova, J. Duan, and J. Kurths, "Transitions in a genetic transcriptional regulatory system under Lévy motion," Sci. Rep. **6**, 29274 (2016).
[38] S. R. Carpenter, D. Ludwig, and W. A. Brock, "Management of eutrophication for lakes subject to potentially irreversible change," Ecol. Appl. **9**, 751–771 (1999).
[39] R. Wang *et al.*, "Flickering gives early warning signals of a critical transition to a eutrophic lake state," Nature **492**, 419–422 (2012).
[40] S. R. Carpenter, "Eutrophication of aquatic ecosystems: Bistability and soil phosphorus," Proc. Natl. Acad. Sci. U. S. A. **102**, 10002–10005 (2005).
[41] A. E. Hramov, A. A. Koronovskii, and M. K. Kurovskaya, "Zero Lyapunov exponent in the vicinity of the saddle-node bifurcation point in the presence of noise," Phys. Rev. E **78**, 036212 (2008).
[42] F. Nazarimehr, S. Jafari, S. M. R. H. Golpayegani, and J. C. Sprott, "Can Lyapunov exponent predict critical transitions in biological systems?," Nonlinear Dyn. **88**, 1493–1500 (2017).
[43] G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn, "Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, Part I: Theory," Meccanica **15**, 9–20 (1980).
[44] G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn, "Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, Part II: Numerical application," Meccanica **15**, 21–30 (1980).
[45] J. Venkatramani, S. Sarkar, and S. Gupta, "Investigations on precursor measures for aeroelastic flutter," J. Sound Vib. **419**, 318–336 (2018).
[46] F. Nazarimehr, K. Rajagopal, A. J. M. Khalaf, A. Alsaedi, V. T. Pham, and T. Hayat, "Investigation of dynamical properties in a chaotic flow with one unstable equilibrium: Circuit design and entropy analysis," Chaos Solitons Fractals **115**, 7–13 (2018).
[47] J. Wu, Y. Xu, H. Wang, and J. Kurths, "Information-based measures for logical stochastic resonance in a synthetic gene network under lévy flight superdiffusion," Chaos **27**, 063105 (2017).
[48] T. Gao, J. Duan, X. Li, and R. Song, "Mean exit time and escape probability for dynamical systems driven by lévy noises," SIAM J. Sci. Comput. **36**, A887–A906 (2014).
[49] L. Serdukova, Y. Zheng, J. Duan, and J. Kurths, "Stochastic basins of attraction for metastable states," Chaos **26**, 073117 (2016).