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Transient chaos in the Lorenz-type map with periodic forcing

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We consider a case study of perturbing a system with a boundary crisis of a chaotic attractor by periodic forcing. In the static case, the system exhibits persistent chaos below the critical value of the control parameter but transient chaos above the critical value. We discuss what happens to the system and particularly to the transient chaotic dynamics if the control parameter periodically oscillates. We find a non-exponential decaying behavior of the survival probability function, study the impact of the forcing frequency and amplitude on the escape rate, analyze the phase-space image of the observed dynamics, and investigate the influence of initial conditions. *Published by AIP Publishing.*
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Transient chaos is a dynamical phenomenon where a system exhibits irregular behavior for a finite but possibly long time and after that arrives at some stable state. To statistically quantify such finite-time chaotic dynamics, one should measure the survival probability function describing how fast an average trajectory leaves the chaotic region. A typical form of this function is an exponential decaying one characterized by some escape rate—the function's slope in a logarithmic scale. Here, we study the case where the simple system, a Lorenz-type map, with transient chaotic dynamics is subject to periodic forcing by its control parameter. We find that the lifetime of trajectories moving in a chaotic manner can be changed—increased or decreased—due to such periodic forcing. We analyze the scaling law for the survival probability function, the role of the initial conditions and of the forcing parameters in the escaping process.

I. INTRODUCTION

Many real systems are subject to external forcing of different kinds; even a simple periodic fluctuation of some parameter in a dynamical system can give rise to drastic changes in the phase space.[1] In particular, chaotic dynamics can be shaped by forcing in a way that can lead to the emergence of new regimes with new features of chaotic attractors,[2] to a suppression and creation of chaos,[3] or to converting transient chaos into a sustained one by properly chosen small perturbations.[4]

In general, when a control parameter in a system is slowly varying, the resulting phenomena can be studied in terms of the theory of dynamic bifurcations.[5] It was shown in Refs. 6–13 that new effects, e.g., persistent or delayed subcritical activity in the supercritical region, appear in such systems with fixed points, limit cycles, or chaotic attractors. The chaotic case was studied in Ref. 14 for the Lorenz-type map[15,16] derived from the analysis of the Lorenz attractor.[17] The main outcome of Ref. 14 was that with a slowly changing control parameter one observes a delayed exit of phase trajectories from the chaotic area in the supercritical region, the survival probability decays non-exponentially, and by varying the rate of the control parameter changing one can control the average delay time.

In this paper, we study the impact of periodic forcing on transient chaotic dynamics displayed by a simple system that exhibits a boundary crisis of the chaotic attractor.

II. TRANSIENT CHAOS UNDER PERIODIC FORCING

We consider a Lorenz-type map of the form

$$x_{n+1} = x_n + F_H(x_n) - \gamma, \tag{1}$$

where $F_H(x) = x(x-0.1)(1-x) - 0.3H(x-0.45)$ is a cubic nonlinearity with a discontinuity at $x_d = 0.45$ controlled by the Heaviside function $H(x)$ [$H(x)=0$ if $x<0$, and $H(x)=1$ if $x \geq 0$]. Such map appears, e.g., when studying a discrete-time model of neural activity[18–21] where it is part of the two-dimensional map[22] and relates to the membrane potential dynamics. The model is rich for various dynamic regimes observed in biological neurons and allows us to study collective activity of neuronal networks (see, e.g., Refs. 23 and 24).

In the current paper, the region of interest is $S = (0.2 < x < 0.6; 0.01 < \gamma < 0.03)$; here, the map (1) undergoes a boundary crisis of the chaotic attractor at $\gamma = \gamma_{cr} \simeq 0.0198$ as γ increases.[14] For $\gamma < \gamma_{cr}$, there are two attractors, a stable fixed point $O_s = x_s^*(\gamma)$ and a chaotic attractor $\Omega(\gamma)$, located in the invariant region $I(\gamma) = [b(\gamma) < x < c(\gamma)]$ (explicit expressions are given in the Appendix). Their basins are separated by an unstable fixed point $O_u = x_u^*(\gamma)$ and the following inequality holds: $x_u^*(\gamma) < x_s^*(\gamma) < b(\gamma) < c(\gamma)$. For $\gamma > \gamma_{cr}$, the attractor Ω does no longer exist, and the only remaining one is the stable fixed point O_s. However, above the critical value $\gamma = \gamma_{cr}$, there is a nonattracting chaotic set $\Xi(\gamma)$, where trajectories move rather irregularly for some finite lifetimes τ whose values depend on initial conditions.

The set Ξ with transient chaos is characterized by the distribution of τ and the survival probability $P(n)$.[25] The

lifetime of a certain trajectory is equal to the amount of time when it moves in Ξ before escaping to O_s. Two trajectories starting at close-by points can have lifetimes differing by orders of magnitude. The survival probability determines the probability of finding a point with the lifetime longer than n. The survival probability $P(n)$ can be calculated as $N(n)/N_0$, where $N(n)$ is a number of trajectories survived until the instant n and N_0 is an initial number of trajectories.

Now we ask what happens when the control parameter γ induces a periodic forcing $\gamma \sim A\sin(\omega n)$ into the map (1) thus converting it into

$$\begin{aligned} x_{n+1} &= x_n + F_H(x_n) - \gamma_n, \\ \gamma_n &= \gamma_0 + A\sin(\omega n). \end{aligned} \tag{2}$$

An example of what impact this periodic forcing can have is shown in Fig. 1 where the two trajectories start from the same initial conditions $(x_0, \gamma_0) = (0.25, 0.001)$. In the static case (1), one observes a chaotic motion for about $\tau \approx 250$. However, when the periodic driving is introduced, the lifetime of chaotic behavior becomes more than 1000.

III. LIFETIME, SURVIVAL PROBABILITY, AND ESCAPE RATE

We are interested in the impact of the frequency of the periodic forcing on the chaotic lifetimes. We start with $N_0 = 10^5$ points uniformly distributed in the interval $b(\gamma_0) < x_0 < c(\gamma_0)$ where $\gamma_0 = \gamma_{cr} + \delta$. δ is a control parameter that gives a "central" point around which oscillations occur, here $\delta = 0.001$. We let the trajectories move according to Eq. (2) until they escape the region of transient chaos which can be indicated by $x_{\tau+1} < b(\gamma_{\tau+1})$, thus giving the corresponding lifetime τ. Note that although in the static case the lower border of chaotic motion is determined by the unstable fixed points $O_u = x_u^*(\gamma)$ in the region $\gamma > \gamma_{cr}$, in the dynamic case due to the continuous oscillation of γ_n some trajectories can move outside the static border and then come back to the chaotic area. Hence, it is more appropriate to consider escaped only those trajectories that intersect the border defined by $x = b(\gamma)$.

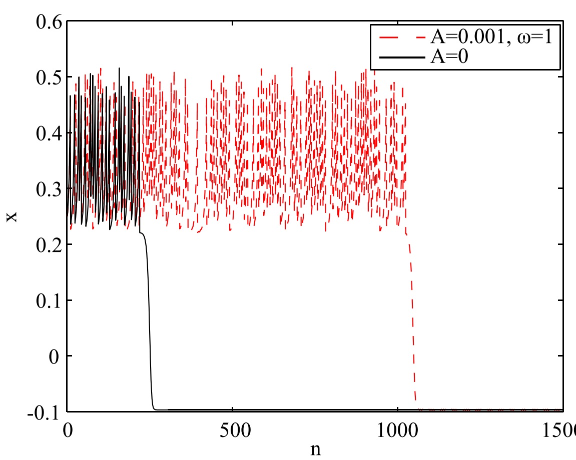

FIG. 1. Dynamics of x in the static case (1) for $x_0 = 0.25$, $\gamma = 0.001$ (solid), and in the dynamic case (2) for the same initial conditions $(x_0, \gamma_0) = (0.25, 0.001)$ and $A = 0.001$, $\omega = 1$ (dashed).

The described procedure results in probability distributions of the lifetime shown in Fig. 2 for different forcing frequencies. These are decaying in average functions indicating that most likely values of the lifetime are rather moderate. However, there are quite a few trajectories with lifetimes close to 1000 contributing to the long tails. Note that for the smallest frequency $\omega = 0.01$ there is an extra little peak at about $\tau \simeq 600$ in contrast to the other distributions. One more distinctive feature emerges in the case $\omega = 0.1$ where the probability function is not so smooth as for the other ω's but appears to be rather discontinuous.

To understand these effects, we compute how the survival probability decays $P(n)$ in time for the cases shown in Fig. 2. Note that for typical chaotic transients, the scaling law for $P(n)$ is exponential, i.e., $P(n) = e^{-\kappa n}$, where κ is the escape rate characterizing how fast in average trajectories leave the transient chaotic area. Figure 3 shows that for the system under periodic forcing this is clearly not the case for some frequencies. For the minimum frequency $\omega = 0.001$, one observes an exponential behavior with a slightly increasing slope. Such behavior can be understood based on the fact that this case corresponds to a very slow motion through the sets $\Xi(\gamma)$ of transient chaos with different γ's and hence different escape rates κ's. The motion takes place towards $\gamma > \gamma_0 > \gamma_{cr}$, and all the trajectories leave the chaotic region before γ begins to decrease according to Eq. (2). A greater frequency $\omega = 0.01$ leads to a faster motion towards the region $\gamma > \gamma_0$. Hence, the surviving probability first decays faster than in the previous case. However, after γ_n turns around, trajectories move through γ-layers with smaller escape rates. So, the decreasing behavior of $P(n)$ slows down and accelerates again after γ_n begins to increase, hence resulting in a total escape from the chaotic area. Similar oscillating dynamics of $P(n)$ is seen for $\omega = 0.1$ due to the same reasons. However, most trajectories have time to oscillate around $\gamma = \gamma_0$ many times before ceasing to move in a chaotic manner. The high-frequency oscillation $\omega = 1$ gives rise to an exponential decaying process for $P(n)$ similar to those observed in static cases because here the trajectories visit repeatedly the γ-layers at both sides of the central point $\gamma = \gamma_0$ resulting in a rather smooth averaging of the escapes rates $\kappa(\gamma)$. Note that for higher values of ω the situation is very similar up to the natural upper bound $\omega_{max} = \pi$ corresponding to a degenerate case of switching between two different γ-values (the period is equal to 2).

We can analytically derive the scaling laws for $P(n)$ based on the escaping properties of static transient chaotic sets $\Xi(\gamma)$ which create a skeleton for the set formed by Eq. (2). According to Ref. 14, the escape rate for $\gamma > \gamma_{cr}$ in Eq. (1) depends on γ as $\kappa(\gamma) = \alpha(\gamma - \gamma_{cr})$, where $\alpha \simeq 7.48$. Taking into account the exponential form of the survival probability function for each $\Xi(\gamma)$, we can write for the escaping process in Eq. (2) the following equation:

$$P(n) = \exp\left(-\sum_{k=0}^{n} \kappa_k\right). \tag{3}$$

Note that we assume $A \leq \delta$ in order to ensure that trajectories do not enter the region of persistent chaos in $\gamma < \gamma_{cr}$. Using $\gamma_k = \gamma_{cr} + \delta + A\sin(\omega k)$, one obtains

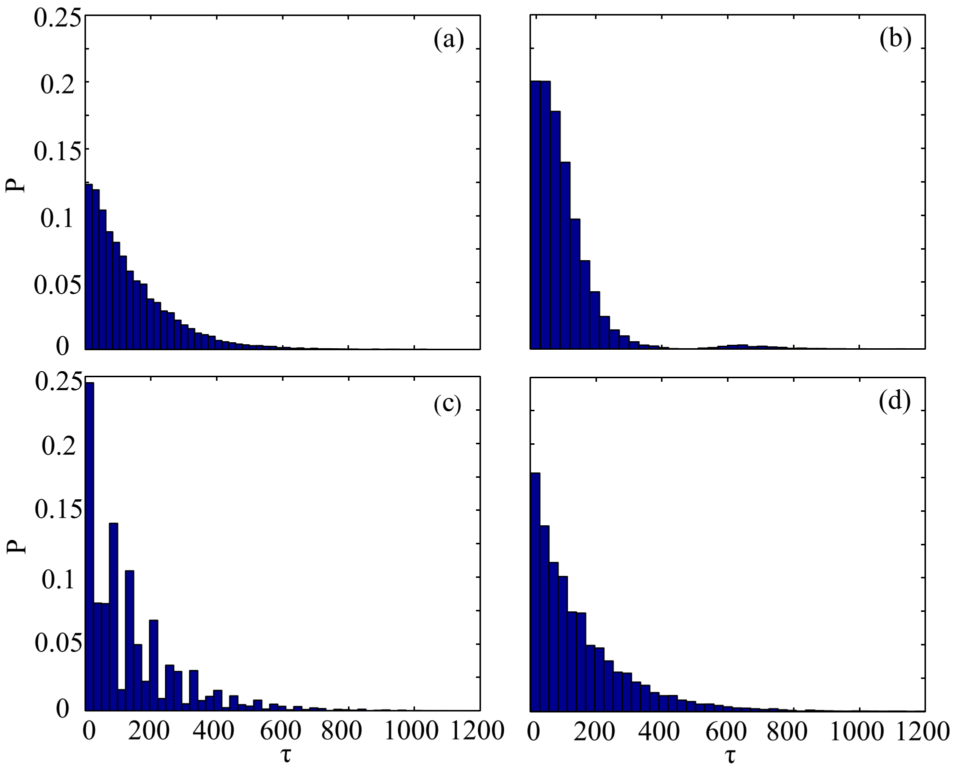

FIG. 2. Probability distributions of lifetimes τ for $A = 0.001$, $\delta = 0.001$, and different frequencies: (a) $\omega = 0.001$, (b) $\omega = 0.01$, (c) $\omega = 0.1$, (d) $\omega = 1$.

$$P(n) = \exp\left(-\alpha\sum_{k=0}^{n}[\delta + A\sin(\omega k)]\right) \simeq \exp\left(-\alpha\left[\delta n + \frac{A}{\omega}(1-\cos(\omega n))\right]\right). \quad (4)$$

In a low-frequency case, the survival probability can be approximated as $P(n) \simeq \exp(-\alpha\delta n)$ hence confirming the strong similarity to the exponential form shown in Fig. 3 for $\omega = 0.001$. In a high-frequency case, Eq. (4) gives $P(n) \simeq \exp(-\alpha[\delta n + A/\omega])$. Our calculations show that these expressions are in good agreement with numerical results of Fig. 3, but they best hold when most of the trajectories still move in the transient chaotic area.

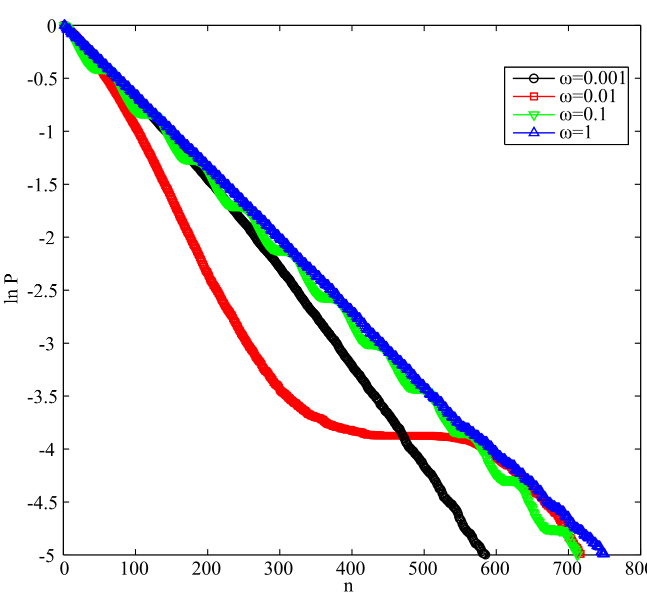

FIG. 3. Dynamics of the survival probability shown as $\ln P(n)$ for $A = 0.001$, $\delta = 0.001$, and different frequencies $\omega = 0.001, 0.01, 0.1, 1$.

Note that these findings are rather different from those obtained in Ref. 14 for the case of the γ-input increasing at a constant rate $\gamma_n = \gamma_0 + \varepsilon n$. There, the surviving probability decreases in a non-exponential manner as $P(n) = \exp(-\alpha\varepsilon n^2/2)$ which is much slower than here at the beginning of the escaping process. However, as time progresses, the escaping rate increases in the former case, while it remains on average constant in the latter case for most frequencies (see Fig. 3). Hence, depending on the relation between the rate of change ε and the parameters (A,ω,δ), the escaping process for the periodically driven input can terminate earlier or later than that for the constantly increasing input. For example, let us consider the low-frequency case ($\omega = 0.001$) shown in Fig. 3 and compare it with the results of Ref. 14. We define the termination of the escaping process as the moment n_f when less than 1% of trajectories still remains moving in the transient chaotic area; for convenience we assume $\ln P(n_f) = -5$. The moment $n^* = 2\delta/\varepsilon$ when the surviving probabilities for both processes become equal to each other should satisfy $n^* < n_f$ in order to ensure that the case of a constantly increasing input leads to an earlier termination than in the case of a periodically driven input. This condition implies that for $\varepsilon > 3 \times 10^{-6}$, most trajectories leave the transient chaotic set earlier in the former case than in the latter.

Although the survival probability decays non-exponentially for some range of forcing frequencies, one can characterize this process by the escape rate κ. While for the static case κ equals the absolute value of the slope of $P(n)$, in the dynamic case this quantity gives the slope of the linear trend in $P(n)$. Figure 4(a) shows the escape rate as a function of the amplitude A for different frequencies. Note that κ is

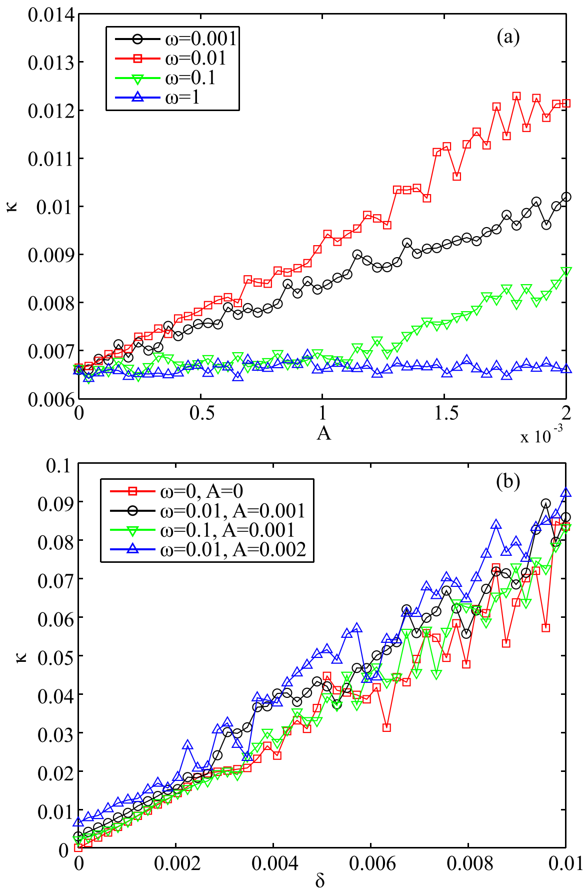

FIG. 4. (a) Escape rate versus amplitude for different frequencies $\omega = 0.001, 0.01, 0.1, 1$. (b) Escape rate versus δ for different frequencies and amplitudes of forcing. In all cases, $\gamma_0 = 0.001$, $\delta = 0.001$, $x_0 \in [b(\gamma), c(\gamma)]$.

calculated using the first 500 iterations after the trajectories start from $\gamma = \gamma_0$ and x_0 is randomly taken from a uniform distribution in the interval $x_0 \in [b(\gamma), c(\gamma)]$. All the curves start from a value $\kappa \approx 6.5 \times 10^{-3}$ at $A = 0$ corresponding to the escape rate of the (static) transient chaotic area at $\gamma = \gamma_0$. For small frequencies $\omega = 0.001, 0.01$, it is clearly seen that with increasing amplitude A, the escape rate on average grows at a more or less constant rate. This can be understood by the fact that for larger A the trajectories visit areas in $\gamma > \gamma_0$ where in the static case the farther from $\gamma = \gamma_{cr}$, the larger the escape rate. Note that the curve corresponding to $\omega = 0.01$ is over that of $\omega = 0.001$ which is consistent with the curves in Fig. 3, where $P(n)$ for the first case initially decreases faster than for the second one.

For the greater frequency value $\omega = 0.1$, the escape rate almost does not change for $A < 0.001$ and begins to slowly increase for $A > 0.001$; in the case $\omega = 1$, $\kappa(A)$ is on average constant within the region of interest. It is interesting to observe that although for $A > 0.001$ the trajectories spend some time in the region of persistent chaos $\gamma < \gamma_{cr}$ (because $\delta = 0.001$), they cannot decrease their escape rate, since the farther they come into the area of (static) persistent chaos, the deeper they move into the region of (static) transient chaos with larger escape rates.

While the point $\gamma_0 = \gamma_{cr} + \delta$ moves deeper towards the region $\gamma > \gamma_{cr}$, the escape rate increases for different frequencies and amplitudes because for larger δ's the trajectories are influenced by (static) γ-layers with greater escape rates. Figure 4(b) illustrates such behavior for the static case ($\omega = 0$, $A = 0$) and three dynamic cases which show a similar dependence $\kappa(\delta)$.

To understand the dynamic mechanisms behind the escaping dynamics described, we study the structure of the transient chaotic set formed by Eq. (2). Initially, we take a large number $N_0 = 10^5$ of points uniformly distributed in the interval $b(\gamma_0) < x_0 < c(\gamma_0)$ where $\gamma_0 = \gamma_{cr} + \delta$ and look for the trajectories with maximum lifetimes as γ_n oscillates according to (2). Such trajectories that continue moving chaotically for a long time are close to an invariant structure of the transient chaotic set. Figure 5 shows a typical result obtained for $\omega = 1$, $A = 0.001$, and $\gamma_0 = \gamma_{cr} + 0.001$. The left (right) vertical line denotes the critical (central) point $\gamma = \gamma_{cr}$ ($\gamma = \gamma_0$), respectively. One observes several gaps through which the trajectories leave the transient chaotic set. Namely, the widest gap near $x = 0.45$ is a region such that once a trajectory falls into it, at the next moment it escapes from the transient chaotic set. The narrower gaps below the former one are its preimages; the lower and the narrower a gap, the higher the order of the preimage.

IV. CONCLUSION

In general, periodic forcing leads to an acceleration of the escaping process in a statistical sense. However, as it is seen in Fig. 1, one can obtain a prolongation of chaotic behavior by an appropriate choice of the oscillation parameters. This can be achieved by inspecting the diagram similar to that shown in Fig. 6. Here the lifetime τ of transient chaotic dynamics is depicted by color coding depending on the

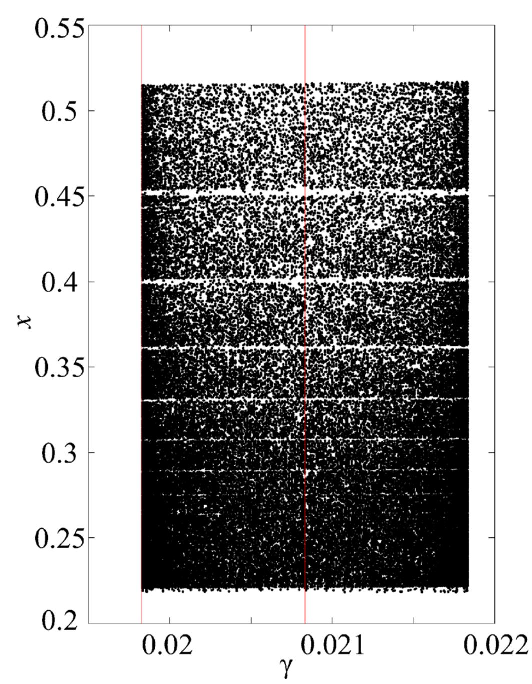

FIG. 5. Structure of the transient chaotic set formed by the dynamics of Eq. (2) for $A = 0.001$, $\delta = 0.001$, $\omega = 1$.

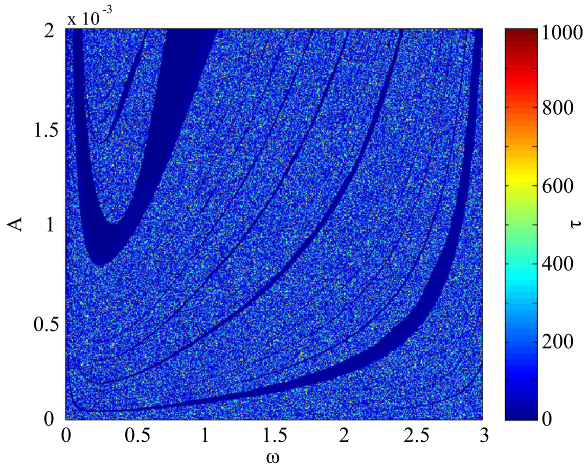

FIG. 6. Lifetimes τ for trajectories started from the same initial condition $x_0 = 0.25$, $\gamma_0 = \gamma_{cr} + 0.001$ depending on the forcing frequency ω and the amplitude A.

frequency ω and the amplitude A of γ_n-oscillation for the fixed initial condition $(x_0, \gamma_0) = (0.25, \gamma_{cr} + 0.001)$. The compact blue islands correspond to the pairs (ω, A) leading to very short chaotic transients. These values result in a fast entrance of trajectories into the gaps of escaping shown in Fig. 5. The lifetime in the rest of the plane is rather sensitive to ω and A, and there are many points with $\tau > 1000$. Similar diagrams arise for trajectories starting from different initial conditions. So by varying the parameters of oscillations (ω and A), one can control whether shortening or prolongation of the lifetime in comparison to the static case.

It should be noted that the problem of controlling the lifetime of transient chaotic behavior raised here is important in different contexts.[25,26] Transient chaos plays an essential role when studying problems of biological systems,[27–31] climate processes,[32,33] laser dynamics,[34,35] power grids,[36] chemical reactions,[37] etc. Depending on the application, either prolongation or shortening can be required. For example, in neuroscience, the disappearance of chaotic dynamics can be resulted from some pathological processes,[28] so it is vital to maintain the chaotic behavior. On the contrary, in ecology, if one intends to obtain a stable population size, one needs to prevent shorten transient chaotic behavior in order to reach an equilibrium state.

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APPENDIX: EXPLICIT EXPRESSIONS

The expressions for the borders of the invariant region containing the chaotic attractor are the following:

$$b(\gamma) = 0.236625 - \gamma, \quad c(\gamma) = 0.536625 - \gamma.$$

The coordinates $x_s^*(\gamma)$ and $x_u^*(\gamma)$ of the fixed points O_s and O_u, correspondingly, can be calculated from the following expressions:

$$x_{s,u}^*(\gamma) = \frac{1+a}{3} - \frac{2(1-a+a^2)^{1/2}}{3}\cos\left(\frac{\phi}{3} \mp \frac{\pi}{3}\right),$$

$$\cos\phi = -\frac{27}{2(1-a+a^2)^{3/2}}\left[\gamma - \frac{(1+a)(2a^2-5a+2)}{27}\right],$$

where $a = 0.1$.

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