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Exponential lag function projective synchronization of memristor-based multidirectional associative memory neural networks via hybrid control

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This paper is concerned with the exponential lag function projective synchronization of memristive multidirectional associative memory neural networks (MMAMNNs). First, we propose a new model of MMAMNNs with mixed time-varying delays. In the proposed approach, the mixed delays include time-varying discrete delays and distributed time delays. Second, we design two kinds of hybrid controllers. Traditional control methods lack the capability of reflecting variable synaptic weights. In this paper, the controllers are carefully designed to confirm the process of different types of synchronization in the

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MMAMNNs. Third, sufficient criteria guaranteeing the synchronization of system are derived based on the derive-response concept. Finally, the effectiveness of the proposed mechanism is validated with numerical experiments.

Keywords: Multidirectional associative memory; memristor; hybrid control; lag function projective; exponential synchronization.

1. Introduction

Being an highly important property of the brain-systems and enormous social phenomenon, the associative memory has attracted considerable attention. Owing to the complexity of human brains, many-to-many associative memory is a basic dynamical behavior of the human cognition. Comparing to the one-to-many association and many-to-one association, many-to-many associative memory can describe the evolutionary process of the storage pattern in the human brains more effectively. Thus, it can be widely applied in artificial intelligence, pattern recognition, and so on.

Since M. Hagiwara first proposed the multidirectional associative memory neural networks (MAMNNs) in 1990,[1] owing to its ability to simulate many-to-many association, the synchronization and the stable analysis of various MAMNNs have become a research hot spot.[2–4] With the concept of memristor,[5] memristive neural networks (MNNs) has provoked crucial attention.[6–12] Inspired by this, according to replace the conventional resistor with the MAMNNs, the memristive multidirectional associative memory neural networks (MMAMNNs) can process more information capacity, which would expressively enhance the applications of MNNs for data clustering,[13] optimization and massively parallel computing,[14] real-time encoding and compression,[15] and machine learning.[16,17]

Synchronization as a fundamental dynamic behavior of NNs,[18] has widely potential application in biological and chemical systems,[19] power grid networks,[20,21] and so on. In 1988, Kosko proposed the bidirectional associative memory neural networks (BAMNNs).[22] This kind of neural networks is similar with MAMNNs in terms of the structure. Moreover, as a special class of MMAMNNs, the research on synchronization of memristive bidirectional associative memory neural networks (MBAMNNs) has attached considerable attention.[23–27] Nevertheless, the synchronization of MMAMNNs are rarely reported in literatures. Additionally, time delays can significantly influence the dynamic behaviors of a chaotic system,[28–36] so the appropriate control strategies are required to make systems stable. Therefore, the synchronization and stability of MMAMNNs with mixed time-varying delays desire much more research attention. This is the first motivation of this paper.

Inspired by the synchronization of diverse dynamic chaotic systems, Mainieri and Rehacek[37] first proposed the projective synchronization. This is a type of synchronization which affected by the scaling factor. Because of its widely feature, the projective synchronization can achieve faster communication according to extend binary digital to M-nary digital.[38] Furthermore, the projective synchronization can be treated as a general form of anti-synchronization and complete synchronization. In recent years, some researchers have studied projective synchronization of MNNs with or without delays.[39–41] However, there are a lot of useful information ignored in the study of the MMAMNNs, such as the characters of the memristor, the structure of the MMAMNNs, etc. How to utilize these useful information to do further research on MMAMNNs is still a challenging problem. Thus, this is the second motivation of this paper.

Furthermore, among the various types of synchronization, owing to the inevitable signal transmitted delay between the drive and response systems, the lag synchronization[42] can be a rational scheme from the point of practical applications and the features of channel. However, the lag synchronization problem of the MMAMNNs is seldom illustrated in literatures. When investigating the MMAMNNs, previous studies have demonstrated that only deterministic parameters have been considered, and the lag synchronization conditions are derived based only on the fixed-value parameters. The situation of the parameters mismatched in some MNNs is often occurred, and its characteristic can be reasonably expressed by some mathematical methods.[43,44] Hence, under this circumstances, it is valuable to study the lag synchronization of MMAMNNs with parameters mismatched. At present, although there are some studies on the stability of MAMNNs,[2–4] the exponential lag function projective synchronization of MMAMNNs with parameters mismatched has not been reported to the best of our knowledge.

In this paper, we concentrate on the exponential lag function projective synchronization control of MMAMNNs with mixed time-varying delays. With the aid of the set-valued mapping, the mathematical model of memristor, differential inclusions, linear feedback controller, adaptive linear feedback controller, and the definition of exponential lag function projective synchronization, two new sufficient criteria are derived to guarantee the exponential lag function projective synchronization of MMAMNNs with and without distributed time-varying delay. The main contributions of this paper can be summarized as follows:

(i) We first investigate the exponential lag function projective synchronization of MMAMNNs with mixed time-varying delays based on the proposed definition.
(ii) On the basis of analyzing the characters of memristor mathematical model, we discuss the parameters mismatched issue between the drive-response systems. Based on the achievements, two kinds of hybrid linear feedback controller and adaptive linear feedback controller are designed to fit the features of the memristor, rather than treating them as some constants.
(iii) We obtain two sufficient criteria to ensure the exponential lag function projective synchronization of the proposed model. Meanwhile, the scaling factor is a variable instead of a constant, thus, our results can easily extend to the other types of synchronization which depend on the scaling factor and lag factor.

The rest of this paper is organized as follows. In Sec. 2, we introduced the mathematical model of MMAMNNs, the definition of exponential lag function projective synchronization, two assumptions and two lemmas. In Sec. 3, the main results of this paper, including two theorems and two corollaries, are presented. Two numerical examples are presented in Sec. 4 while Sec. 5 concludes this paper with some insights provided.

Notations: For $r > 0$, $\boldsymbol{C}([-r, 0], \mathbb{R}^{n_k})$ denotes the Banach space of all continuous functions mapping $[-r, 0]$ into $\mathbb{R}^{n_k}$ with q-norm or ∞-norm by the following forms, respectively. $\mathbb{R}^{n_k}$ denotes the n_k-dimensional Euclidean space. The superscript T represents matrix or vector transposition. We define the norm of the vector as $\|x_{ki}\|$ indicates the 2-norm of a vector x_{ki}, i.e. $\|x_{ki}\| = (\sum_{i=1}^{n_k} x_{ki}^2)^{\frac{1}{2}}$. $co[a, b]$ denotes the closure of the convex hull generated by real numbers or matrices a and b.

2. Neural Networks Model and Preliminaries

In Sec. 2.1, the neural networks model will be presented step by step. In Secs. 2.2 and 2.3, the error systems and useful lemmas are provided.

2.1. *Neural networks model*

In order to better illustrate the MMAMNNs, first we describe a general class of MAMNNs[2] as follows:

$$\frac{dx_{ki}(t)}{dt} = -d_{ki}x_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} a_{pjki} f_{pj}(x_{ki}(t - \tau_{pjki})) + I_{ki}, \quad i \in I, \ \ k \in K\,. \tag{1}$$

Based on the physical properties of the memristor and system (1), the proposed MMAMNNs with mixed time-varying delays is described by the following differential equations:

$$\begin{aligned} \frac{dx_{ki}(t)}{dt} &= -d_{ki}x_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} a_{pjki}(x_{ki}(t)) f_{pj}(x_{ki}(t)) \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} b_{pjki}(x_{ki}(t - \tau_{pjki}(t))) g_{pj}(x_{pj}(t - \tau_{pjki}(t))) \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} c_{pjki}(x_{ki}(t)) \int_{t-\mu_{pjki}(t)}^{t} h_{pj}(x_{ki}(s))ds + I_{ki}\,, \\ &\qquad t \geq 0, \ \ i \in I, \ \ k \in K\,, \end{aligned} \tag{2}$$

where $m \geq 3$ denotes the number of fields in system (2), thus we have $k \in K \triangleq \{1, 2, \ldots, m\}$ and $p \in P \triangleq \{1, 2, \ldots, m\}$. And $x_{ki}(t)$ is the state of the ith neuron in the field k at time t, n_p, n_k represent to the number of neurons in the field p and k, then we define $i \in I \triangleq \{1, 2, \ldots, n_k\}$ and $j \in J \triangleq \{1, 2, \ldots, n_p\}$.

Then d_{ki} is a positive constant which stands for an amplification function; $f_{pj}(\cdot)$, $g_{pj}(\cdot)$ and $h_{pj}(\cdot)$ are the feedback functions. The $\tau_{pjki}(t)$ denotes the discrete time-varying delay which satisfy $0 \leq \tau_{pjki}(t) \leq \tau$ and $\dot{\tau}_{pjki}(t) \leq \tau_0 < 1$. And $0 \leq \mu_{pjki}(t) \leq \mu$, $\dot{\mu}_{pjki}(t) \leq \mu_0 < 1$ represents the distributed time-varying delay. The I_{ki} is the external input on the ith unit of field k.

Due to digital computer applications demanding only two memory conditions,[45] a memristor requires to demonstrate only two sufficient distinct equilibrium states. And a basic form of the memristor includes a junction, which can be switched from a low to a high resistive state and vice versa.[46] The switch starts at a threshold voltage. According to the analysis in Ref. 45, Fig. 1 illustrates the simplification current characteristic of a memristor, and we apply the following mathematical model of the memristance.[47–49] We define the $a_{pjki}(\cdot)$, $b_{pjki}(\cdot)$, $c_{pjki}(\cdot)$ are the memristive connection weights as followed:

$$a_{pjki}(x_{ki}(t)) = \frac{W^f_{pjki}}{C_{ki}} \times \mathrm{sgn}_{pjki}\,,$$

$$b_{pjki}(x_{ki}(t - \tau_{pjki}(t))) = \frac{W^g_{pjki}}{C_{ki}} \times \mathrm{sgn}_{pjki}\,,$$

$$c_{pjki}(x_{ki}(t)) = \frac{W^h_{pjki}}{C_{ki}} \times \mathrm{sgn}_{pjki}\,,$$

where $\mathrm{sgn}_{pjki} = \mathrm{sgn}_{kipj} = 1$ for $pj \neq ki$ and $\mathrm{sgn}_{pjki} = \mathrm{sgn}_{kipj} = -1$ for $pj = ki$. Then the C_{ki} is the capacitor, W^f_{pjki}, W^g_{pjki} and W^h_{pjki} are the memductances of memristors R^f_{pjki}, R^g_{pjki} and R^h_{pjki}, respectively. In addition, R^f_{pjki} presents the memristor between the feedback function $f_{pj}(x_{ki}(t))$ and $x_{ki}(t)$; R^g_{pjki}

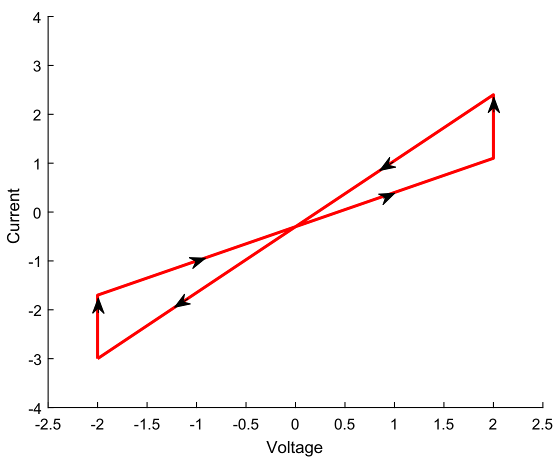

Fig. 1. Typical current–voltage characteristic of a memristor.

presents the memristor between the feedback function $g_{pj}(x_{pj}(t-\tau_{pjki}(t)))$ and $x_{pj}(t-\tau_{pjki}(t))$; R^h_{pjki} presents the memristor between the feedback function $\int_{t-\mu_{pjki}(t)}^{t} h_{pj}(x_{ki}(s))ds$ and $x_{ki}(t)$. The complex-valued memductance functions $a_{pjki}(x_{ki}(t))$, $b_{pjki}(x_{ki}(t-\tau_{pjki}(t)))$, and $c_{pjki}(x_{ki}(t))$ are given by

$$a_{pjki}(x_{ki}(t)) = \begin{cases} a^*_{pjki}, & x_{ki}(t) \le \overline{T}_{ki}\,, \\ a^{**}_{pjki}, & x_{ki}(t) > \overline{T}_{ki}\,, \end{cases}$$

$$c_{pjki}(x_{ki}(t)) = \begin{cases} c^*_{pjki}, & x_{ki}(t) \le \overline{T}_{ki}\,, \\ c^{**}_{pjki}, & x_{ki}(t) > \overline{T}_{ki}\,, \end{cases}$$

$$b_{pjki}(x_{ki}(t-\tau_{pjki}(t))) = \begin{cases} b^*_{pjki}, & x_{ki}(t-\tau_{pjki}(t)) \le \overline{T}_{ki}\,, \\ b^{**}_{pjki}, & x_{ki}(t-\tau_{pjki}(t)) > \overline{T}_{ki}\,, \end{cases}$$

where a^*_{pjki}, a^{**}_{pjki}, b^*_{pjki}, b^{**}_{pjki}, c^*_{pjki}, and c^{**}_{pjki} are constant numbers.

Assume that $q \ge 1$ is a positive integer and $\mathbb{R}^{n_k}$ be the space of n_k-dimensional real column vectors, for any $x_k = (x_{k_1}, x_{k_2}, \ldots, x_{k_{n_k}})^T \in \mathbb{R}^{n_k}$. $\|x_k\|$ denotes a vector norm defined by

$$\|x_k\|_q = \left[\sum_{i=1}^{n_k} |x_{ki}|^q\right]^{\frac{1}{q}}, \quad \text{or} \quad \|x_k\|_\infty = \max_{i\in I} |x_{ki}|\,.$$

The initial conditions associated with system (2) are given by

$$x_{ki}(s) = \varphi_{ki}(s), \quad s \in [-r, 0]\,,$$

where $r = \max\{\tau, \mu\}$, $\tau = \max_{i\in I, j\in J} \tau_{pjki}(t)$, $\mu = \max_{i\in I, j\in J} \mu_{pjki}(t)$. Then $\varphi_k(s) = (\varphi_{k_1}(s), \varphi_{k_2}(s), \ldots, \varphi_{k_{n_k}}(s))^T \in \boldsymbol{C}([-r,0], \mathbb{R}^{n_k})$.

$$\|\varphi_k\|_q = \sup_{s\in[-r,0]} \left[\sum_{i=1}^{n_k} |\varphi_{ki}|^q\right]^{\frac{1}{q}}, \quad \text{or} \quad \|\varphi_k\|_\infty = \sup_{s\in[-r,0]} |\varphi_{ki}|\,.$$

In this paper, solutions of all systems are treated as the following are intended in Fillpov's sense.[50] For a continuous function $l(t)$: $\mathbb{R} \to \mathbb{R}$, $D^+l(t)$ is named the upper right Dini derivative and defined as $D^+l(t) = \overline{\lim}_{k\to 0^+} \frac{1}{k}(l(t+k) - l(t))$.

Now, we do the following assumptions for system (2):

(A1) The activation functions $f_{pj}(\cdot)$, $g_{pj}(\cdot)$ and $h_{pj}(\cdot)$ are Lipschitz conditions. That is for any $p \in P, j \in J$, there exist real numbers L^f_{pj}, L^g_{pj}, and L^h_{pj}, such that

$$|f_{pj}(u) - f_{pj}(v)| \le L^f_{pj}|u - v|\,,$$

$$|g_{pj}(u) - g_{pj}(v)| \le L^g_{pj}|u - v|\,,$$

$$|h_{pj}(u) - h_{pj}(v)| \le L^h_{pj}|u - v|\,,$$

for all $u, v \in R$ and $u \ne v$.

(A2) The activation functions $f_{pj}(\cdot)$, $g_{pj}(\cdot)$ and $h_{pj}(\cdot)$ are bounded. That is for any $p \in P, j \in J$, there exist real numbers M_{pj}^f, M_{pj}^g, and M_{pj}^h, such that

$$|f_{pj}(u)| \le M_{pj}^f, \quad |g_{pj}(u)| \le M_{pj}^g, \quad |h_{pj}(u)| \le M_{pj}^h \,,$$

for all $u \in R$.

In this paper, we consider the system (2) as the drive system and the response system is as follows:

$$\begin{aligned} \frac{dy_{ki}(t)}{dt} &= -d_{ki}y_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} a_{pjki}(y_{ki}(t))f_{pj}(y_{ki}(t)) \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))g_{pj}(y_{pj}(t-\tau_{pjki}(t))) \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} c_{pjki}(y_{ki}(t)) \int_{t-\mu_{pjki}(t)}^{t} h_{pj}(y_{ki}(s))ds + U_{ki}(t) + I_{ki}, \\ &\qquad t \ge \sigma,\ i \in I,\ k \in K\,, \end{aligned} \tag{3}$$

where

$$a_{pjki}(y_{ki}(t)) = \begin{cases} a^*_{pjki}, & y_{ki}(t) \le \underline{T}_{ki}\,, \\ a^{**}_{pjki}, & y_{ki}(t) > \underline{T}_{ki}\,, \end{cases} \qquad c_{pjki}(y_{ki}(t)) = \begin{cases} c^*_{pjki}, & y_{ki}(t) \le \underline{T}_{ki}\,, \\ c^{**}_{pjki}, & y_{ki}(t) > \underline{T}_{ki}\,, \end{cases}$$

$$b_{pjki}(y_{ki}(t-\tau_{pjki}(t))) = \begin{cases} b^*_{pjki}, & y_{ki}(t-\tau_{pjki}(t)) \le \underline{T}_{ki}\,, \\ b^{**}_{pjki}, & y_{ki}(t-\tau_{pjki}(t)) > \underline{T}_{ki}\,, \end{cases}$$

for a.e. $t \ge \sigma, j \in J, p \in P$, which initial conditions $y_k(s) = (\phi_{k_1}(s), \phi_{k_2}(s), \ldots, \phi_{k_{n_k}}(s))^T = \phi_{k\sigma}(s) \in \boldsymbol{C}([-r,0], \mathbb{R}^{n_k})$. Then $\phi_{k\sigma}(s) = \phi_k(s+\sigma)$ for all $s \in [-r,0]$. And $U_{ki}(t)$ is the appropriate control input that will be designed.

Through the theories of differential inclusions and set-valued map,[50–52] systems (2) and (3) can be transformed as follows:

$$\begin{aligned} \frac{dx_{ki}(t)}{dt} &\in -d_{ki}x_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} co[a_{pjki}(x_{ki}(t))]f_{pj}(x_{ki}(t)) \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} co[b_{pjki}(x_{ki}(t-\tau_{pjki}(t)))]g_{pj}(x_{pj}(t-\tau_{pjki}(t))) \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} co[c_{pjki}(x_{ki}(t))] \int_{t-\mu_{pjki}(t)}^{t} h_{pj}(x_{ki}(s))ds + I_{ki}, \quad t \ge 0\,, \end{aligned} \tag{4}$$

and

$$\frac{dy_{ki}(t)}{dt} \in -d_{ki}y_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} co[a_{pjki}(y_{ki}(t))]f_{pj}(y_{ki}(t))$$
$$+ \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} co[b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))]g_{pj}(y_{pj}(t-\tau_{pjki}(t)))$$
$$+ \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} co[c_{pjki}(y_{ki}(t))]\int_{t-\mu_{pjki}(t)}^{t} h_{pj}(y_{ki}(s))ds + U_{ki}(t) + I_{ki}, \quad t \geq \sigma\,, \tag{5}$$

where

$$co[a_{pjki}(x_{ki}(t))] = \begin{cases} a^{*}_{pjki}, & x_{ki}(t) < \overline{T}_{ki}\,, \\ co[a^{*}_{pjki}, a^{**}_{pjki}], & x_{ki}(t) = \overline{T}_{ki}, \\ a^{**}_{pjki}, & x_{ki}(t) > \overline{T}_{ki}\,, \end{cases}$$
$$co[a_{pjki}(y_{ki}(t))] = \begin{cases} a^{*}_{pjki}, & y_{ki}(t) < \underline{T}_{ki}\,, \\ co[a^{*}_{pjki}, a^{**}_{pjki}], & y_{ki}(t) = \underline{T}_{ki}\,, \\ a^{**}_{pjki}, & y_{ki}(t) > \underline{T}_{ki}\,, \end{cases} \tag{6}$$

$$co[b_{pjki}(x_{ki}(t-\tau_{pjki}(t)))] = \begin{cases} b^{*}_{pjki}, & x_{ki}(t-\tau_{pjki}(t)) < \overline{T}_{ki}\,, \\ co[b^{*}_{pjki}, b^{**}_{pjki}], & x_{ki}(t-\tau_{pjki}(t)) = \overline{T}_{ki}\,, \\ b^{**}_{pjki}, & x_{ki}(t-\tau_{pjki}(t)) > \overline{T}_{ki}\,, \end{cases}$$
$$co[b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))] = \begin{cases} b^{*}_{pjki}, & y_{ki}(t-\tau_{pjki}(t)) < \underline{T}_{ki}\,, \\ co[b^{*}_{pjki}, b^{**}_{pjki}], & y_{ki}(t-\tau_{pjki}(t)) = \underline{T}_{ki}\,, \\ b^{**}_{pjki}, & y_{ki}(t-\tau_{pjki}(t)) > \underline{T}_{ki}\,, \end{cases} \tag{7}$$

$$co[c_{pjki}(x_{ki}(t))] = \begin{cases} c^{*}_{pjki}, & x_{ki}(t) < \overline{T}_{ki}\,, \\ co[c^{*}_{pjki}, c^{**}_{pjki}], & x_{ki}(t) = \overline{T}_{ki}\,, \\ c^{**}_{pjki}, & x_{ki}(t) > \overline{T}_{ki}\,, \end{cases}$$
$$co[c_{pjki}(y_{ki}(t))] = \begin{cases} c^{*}_{pjki}, & y_{ki}(t) < \underline{T}_{ki}\,, \\ co[c^{*}_{pjki}, c^{**}_{pjki}], & y_{ki}(t) = \underline{T}_{ki}\,, \\ c^{**}_{pjki}, & y_{ki}(t) > \underline{T}_{ki}\,. \end{cases} \tag{8}$$

It is clearly that the set-valued map

$$\begin{aligned}\frac{dx_{ki}(t)}{dt} &\mapsto -d_{ki}x_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} co[a_{pjki}(x_{ki}(t))]f_{pj}(x_{ki}(t)) \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} co[b_{pjki}(x_{ki}(t-\tau_{pjki}(t)))]g_{pj}(x_{pj}(t-\tau_{pjki}(t))) \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} co[c_{pjki}(x_{ki}(t))]\int_{t-\mu_{pjki}(t)}^{t} h_{pj}(x_{ki}(s))ds + I_{ki}, \quad t\geq 0\,, \qquad (9)\end{aligned}$$

which has non-empty compact convex values. Thus, it is upper-semi continuous.

2.1.1. *Error systems*

Then, we define the synchronization error $e_{ki}(t)$ as follows: $e_{ki}(t) = y_{ki}(t) - \alpha(t)x_{ki}(t-\sigma)$, and the scaling factor $\alpha(t)$ is bounded which satisfied $|\alpha(t)| \leq \Delta$. By the theories of differential inclusion and set-valued map. Combining with the systems (4) and (5), we get the following synchronization error system:

$$\frac{de_{ki}(t)}{dt} = \frac{dy_{ki}(t)}{dt} - \alpha(t)\frac{dx_{ki}(t-\sigma)}{dt} - \frac{d\alpha(t)}{dt}x_{ki}(t-\sigma), \quad t\geq\sigma\,. \qquad (10)$$

Remark 1. In the literature, the existing results concerning the error system[53–55] require the scaling factor $\alpha(t)$ as a constant. As mentioned in Sec. 1, in some important applications, the scaling factor $\alpha(t)$ may be generated as a variable to change with time. Thus, in such case, these results will lose efficacy. The error system (10) is more rational in practical applications.

Thus, we have

$$\begin{aligned}\frac{de_{ki}(t)}{dt} &\in -\{d_{ki}y_{ki}(t) - d_{ki}x_{ki}(t-\sigma)\alpha(t)x_{ki}(t-\sigma)\} \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\{co[a_{pjki}(y_{ki}(t))]f_{pj}(y_{pj}(t)) \\ &\quad - co[a_{pjki}(x_{ki}(t-\sigma)))]\alpha(t)f_{pj}(x_{pj}(t-\sigma))\} \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\{co[b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))]g_{pj}(y_{pj}(t-\tau_{pjki}(t))) \\ &\quad - co[b_{pjki}(x_{ki}(t-\tau_{pjki}(t-\sigma)))]\alpha(t)g_{pj}(x_{pj}(t-\tau_{pjki}(t-\sigma)))\}\end{aligned}$$

$$+\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\left\{co[c_{pjki}(y_{ki}(t))]\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(y_{ki}(s))ds\right.$$

$$\left.-co[c_{pjki}(x_{ki}(t-\sigma))]\alpha(t)\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(y_{ki}(s-\sigma))ds\right\}$$

$$+(1-\alpha(t))I_{ki}-\frac{d\alpha(t)}{dt}x_{ki}(t-\sigma)+U_{ki}(t),\quad t\geq\sigma\,. \tag{11}$$

Definition 1. A function $x_k = (x_{k_1}, x_{k_2}, \ldots, x_{k_{n_k}})^T$ with initial conditions $x_k(t) = \varphi_k(t) \in \boldsymbol{C}([-r, 0], \mathbb{R}^{n_k})$ is the solution of system (2), which is evidently continuous.[56] Furthermore, based on the conditions (A1) and (A2), this local solution $x_k(t)$ can be extended to the interval $[0, +\infty]$ under the Filippov's sense.[50,52]

Definition 2. For $\forall t \geq 0$, systems (2) and (3) are said to achieve exponentially lag function projective synchronization, if there exist $M \geq 1$ and $\varepsilon > 0$ such that

$$\|y_{ki}(t)-\alpha(t)x_{ki}(t-\sigma)\|\leq M\exp\{-\varepsilon(s-\sigma)\}\|\phi_{ki}(t)-\alpha(t)\varphi_{ki}(t-\sigma)\|\,, \tag{12}$$

where ε is defined as the degree of exponential synchronization.

Remark 2. Enlightened by Refs. 57 and 58, we proposed the definition of the exponential lag function projective synchronization. Based on the definition, the type of synchronization can extend to exponential complete synchronization and anti-synchronization with or without lag easily. Comparing with the existing results, the results are more flexible and functional.

2.2. *Some useful lemmas*

Lemma 1. (see Ref. 50) *Suppose that* $M(k)\colon \mathbb{R}^n \mapsto \mathbb{R}$ *is* C*-regular, and that* $k(t)\colon [0, +\infty) \mapsto \mathbb{R}$ *is absolutely continuously on any compact interval of* $[0, +\infty)$*. Then,* $V(k(t))\colon [0, +\infty) \mapsto \mathbb{R}$ *are differential for a.e.* $t \in [0, +\infty)$*, then we get*

$$\frac{dV(k(t))}{dt}=\theta(t)\frac{dk(t)}{dt},\quad \forall\theta(t)\in\partial V(k(t))\,.$$

Lemma 2. *Under Assumption* (A1), *the following estimations are true*:

(i)
$$|co[a_{pjki}(y_{ki}(t))]f_{pj}(y_{pj}(t))-A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))|\leq A_{pjki}L_{pj}^{f}|e_{pj}(t)|+L_a\,,$$

$$|-co[a_{pjki}(x_{ki}(t-\sigma))]\alpha(t)f_{pj}(x_{pj}(t-\sigma))+A_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma))|\leq L_a\Delta\,,$$

(ii)
$$|co[b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))]g_{pj}(y_{pj}(t-\tau_{pjki}(t)))-B_{pjki}g_{pj}(\alpha(t-\tau_{pjki}(t)))x_{pj}(t-\tau_{pjki}(t)-\sigma))|\leq B_{pjki}L_{pj}^{g}|e_{pj}(t-\tau_{pjki}(t))|+L_b\,,$$

$$\begin{aligned}&|-co[b_{pjki}(x_{ki}(t-\tau_{pjki}(t)-\sigma))]\alpha(t)g_{pj}(x_{pj}(t-\tau_{pjki}(t)-\sigma))\\&\quad+B_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\tau_{pjki}(t)-\sigma))|\le L_b\Delta\,,\end{aligned}$$

(iii)
$$\begin{aligned}&\left|co[c_{pjki}(y_{ki}(t))]\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(y_{pj}(s))ds\right.\\&\quad\left.-C_{pjki}\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(\alpha(s)x_{pj}(s-\sigma))ds\right|\\&\le C_{pjki}L_{pj}^{h}\int_{t-\mu_{pjki}(t)}^{t}|e_{pj}(s)|ds+L_c\,,\\&\left|-co[c_{pjki}(x_{ki}(t-\sigma))]\alpha(t)\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(x_{pj}(s-\sigma))ds\right.\\&\quad\left.+C_{pjki}\alpha(t)\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(x_{pj}(s-\sigma))ds\right|\le L_c\Delta\,,\end{aligned}$$

where

$$\begin{aligned}A_{pjki}&=\max\{|a^{*}_{pjki}|,|a^{**}_{pjki}|\}\,,\\B_{pjki}&=\max\{|b^{*}_{pjki}|,|b^{**}_{pjki}|\}\,,\\C_{pjki}&=\max\{|c^{*}_{pjki}|,|c^{**}_{pjki}|\}\,,\end{aligned}$$

and

$$\begin{aligned}L_a&=\max(|a^{*}_{pjki}-A_{pjki}|M^{f}_{pj},|a^{**}_{pjki}-A_{pjki}|M^{f}_{pj})\,,\\L_b&=\max(|b^{*}_{pjki}-B_{pjki}|M^{g}_{pj},|b^{**}_{pjki}-B_{pjki}|M^{g}_{pj})\,,\\L_c&=\max(|c^{*}_{pjki}-C_{pjki}|\mu M^{h}_{pj},|c^{**}_{pjki}-C_{pjki}|\mu M^{h}_{pj})\,,\end{aligned}$$

for $t\ge\sigma$, $i\in I,j\in J,k\in K,p\in P$.

Proof. (i) For $y_{ki}(t)\le\underline{T}_{ki}, x_{ki}(t-\sigma)\le\overline{T}_{ki}$, then

$$\begin{aligned}&|co[a_{pjki}(y_{ki}(t))]f_{pj}(y_{pj}(t))-A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))|\\&\quad=|a^{*}_{pjki}f_{pj}(y_{pj}(t))-A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))|\\&\quad=|a^{*}_{pjki}f_{pj}(y_{pj}(t))-a^{*}_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))\\&\qquad+a^{*}_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))-A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))|\\&\quad=|a^{*}_{pjki}(f_{pj}(y_{pj}(t))-f_{pj}(\alpha(t)x_{pj}(t-\sigma)))\\&\qquad+(a^{*}_{pjki}-A_{pjki})f_{pj}(\alpha(t)x_{pj}(t-\sigma))|\\&\quad\le|a^{*}_{pjki}(f_{pj}(y_{pj}(t))-f_{pj}(\alpha(t)x_{pj}(t-\sigma)))|\\&\qquad+|(a^{*}_{pjki}-A_{pjki})f_{pj}(\alpha(t)x_{pj}(t-\sigma))|\,.\end{aligned}\tag{13}$$

Due to Assumptions (A1) and (A2), we have

$$
\begin{aligned}
&|co[a_{pjki}(y_{ki}(t))]f_{pj}(y_{pj}(t)) - A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))| \\
&\quad \le A_{pjki}L^f_{pjki}|f_{pj}(y_{pj}(t)) - f_{pj}(\alpha(t)x_{pj}(t-\sigma))| + |a^*_{pjki} - A_{pjki}|M^f_{pjki} \\
&\quad = A_{pjki}L^f_{pjki}|e_{pj}(t)| + |a^*_{pjki} - A_{pjki}|M^f_{pjki}\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&|-co[a_{pjki}(x_{ki}(t-\sigma))]\alpha(t)f_{pj}(x_{pj}(t-\sigma)) + A_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma))| \\
&\quad = |-a^*_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma)) + A_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma))| \\
&\quad = |(A_{pjki} - a^*_{pjki})\alpha(t)f_{pj}(x_{pj}(t-\sigma))| \\
&\quad \le |A_{pjki} - a^*_{pjki}|\Delta M^f_{pjki}\,.
\end{aligned}
\tag{14}
$$

(ii) For $y_{ki}(t) > \underline{T}_{ki}$, if $x_{ki}(t-\sigma) > \overline{T}_{ki}$, we get

$$
\begin{aligned}
&|co[a_{pjki}(y_{ki}(t))]f_{pj}(y_{pj}(t)) - A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))| \\
&\quad = |a^{**}_{pjki}f_{pj}(y_{pj}(t)) - A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))| \\
&\quad = |a^{**}_{pjki}f_{pj}(y_{pj}(t)) - a^*_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma)) \\
&\qquad + a^{**}_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma)) - A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))| \\
&\quad = |a^{**}_{pjki}(f_{pj}(y_{pj}(t)) - f_{pj}(\alpha(t)x_{pj}(t-\sigma))) \\
&\qquad + (a^{**}_{pjki} - A_{pjki})f_{pj}(\alpha(t)x_{pj}(t-\sigma))| \\
&\quad \le A_{pjki}L^f_{pjki}|e_{pj}(t)| + |a^{**}_{pjki} - A_{pjki}|M^f_{pjki}\,.
\end{aligned}
\tag{15}
$$

And

$$
\begin{aligned}
&|-co[a_{pjki}(x_{ki}(t-\sigma))]\alpha(t)f_{pj}(x_{pj}(t-\sigma)) + A_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma))| \\
&\quad = |-a^{**}_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma)) + A_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma))| \\
&\quad = |(A_{pjki} - a^{**}_{pjki})\alpha(t)f_{pj}(x_{pj}(t-\sigma))| \\
&\quad \le |A_{pjki} - a^{**}_{pjki}|\Delta M^f_{pjki}\,.
\end{aligned}
\tag{16}
$$

If $x_{ki}(t-\sigma) \le \overline{T}_{ki}$, we have

$$
\begin{aligned}
&|-co[a_{pjki}(x_{ki}(t-\sigma))]\alpha(t)f_{pj}(x_{pj}(t-\sigma)) + A_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma))| \\
&\quad = |-a^*_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma)) + A_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma))| \\
&\quad = |(A_{pjki} - a^*_{pjki})\alpha(t)f_{pj}(x_{pj}(t-\sigma))| \\
&\quad \le |A_{pjki} - a^*_{pjki}|\Delta M^f_{pjki}\,.
\end{aligned}
\tag{17}
$$

Similarly, we can get the conclusion of (ii) and (iii) of Lemma 2. The proof is completed. □

3. Main Results

In this section, two kinds of hybrid controllers will be proposed corresponding to the two kinds of MMAMNNs models, that is, with and without distributed time-varying delay.

3.1. *Exponentially lag function projective synchronization via hybrid linear control*

In this section, we will derive some criteria to guarantee the exponentially lag function projective synchronization of systems (2) and (3).

Treat the synchronization error $e_{ki}(t)$ as $e_{ki}(t) = y_{ki}(t) - \alpha(t)x_{ki}(t-\sigma)$ for $i \in I, k \in K$. Then we design the controller $U_{ki}(t)$ in response system (3) as follows:

$$U_{ki}(t) = U_{ki_1}(t) + U_{ki_2}(t) + U_{ki_3}(t) + U_{ki_4}(t) + U_{ki_5}(t)\,, \tag{18}$$

where

$$\begin{aligned}
U_{ki_1} &= \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[A_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma)) - (1+\Delta)L_a \\
&\quad - A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))]\,,\\
U_{ki_2} &= \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[B_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\tau_{pjki}(t)-\sigma)) \\
&\quad - B_{pjki}f_{pj}(\alpha(t-\tau_{pjki}(t))x_{pj}(t-\tau_{pjki}(t)-\sigma)) - (1+\Delta)L_b])\,,\\
U_{ki_3} &= \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\Bigg[C_{pjki}\alpha(t)\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(x_{pj}(s-\sigma))ds \\
&\quad - C_{pjki}\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(\alpha(s)x_{pj}(s-\sigma))ds - (1+\Delta)L_c\Bigg]\,,\\
U_{ki_4} &= (\alpha(t)-1)I_{ki} + \dot{\alpha}(t)x_{ki}(t-\sigma)\,,\\
U_{ki_5} &= -\eta_{ki}e_{ki}(t)\,,
\end{aligned}$$

which η_{ki} is a positive constant determined in later.

Remark 3. Consider the error system (10) and the mathematical model of memristance (6)–(8), the switching rules of the drive system are employed in designing the controller (18). Thus, the present design strategy is more favorable than those neglecting the state transition of the drive-response systems. The design procedure is partly stimulated by Ref. 57, where the lag synchronization for MNNs was addressed.

For convenience, we denote

$$\lambda_{ki} = d_{ki} - \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} A_{pjki} L_{pj}^{f} \,,$$

$$\beta_{ki} = \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} A_{pjki} L_{pj}^{g} \,, \tag{19}$$

$$\xi_{ki} = \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} \frac{C_{pjki} L_{pj}^{h}}{\varepsilon} \,.$$

Based on the above discussions, we give the following assumption for system parameters and control strength.

(A3) $\lambda_{ki} + \eta_{ki} - \beta_{ki} - \xi_{ki} > 0$, for any $i \in I, k \in K$.

For each $i \in I, j \in J, k \in K, p \in P$, consider the following function:

$$G_{ki}(\varepsilon_{ki}) = \lambda_{ki} + \eta_{ki} - \varepsilon_{ki} - \beta_{ki} e^{\varepsilon_{ki}\tau} - \xi_{ki} e^{\varepsilon_{ki}\mu}, \quad \varepsilon_{ki} > 0 \,. \tag{20}$$

It is clearly to see that $\dot{G}_{ki}(\varepsilon_{ki}) < 0$, $G(0) > 0$. Since $G_{ki}(\varepsilon_{ki})$ is continuous and $G_{ki}(\varepsilon_{ki}) \to -\infty$ as $\varepsilon_{ki} \to +\infty$. Thus, there exists a positive number ε_{ki}^{*} such that $G_{ki}(\varepsilon_{ki}^{*}) > 0$. Let $\varepsilon = \min_{i\in I, k\in K} \varepsilon_{ki}^{*}$, then we have

$$\lambda_{ki} + \eta_{ki} - \varepsilon_{ki} - \beta_{ki} e^{\varepsilon\tau} - \xi_{ki} e^{\varepsilon\mu} > 0 \,, \tag{21}$$

for any $i \in I, k \in K$.

Based on the controller (18), the following results can be obtained.

Theorem 1. *Suppose that Assumptions* (A1)–(A3) *hold, the MMAMNNs systems* (2) *and* (3) *are exponentially lag function projective synchronized under the hybrid linear controller* (18).

Proof. Calculating the Clarke's generalized gradient of absolute function $|e_{ki}(t)|$ by Lemma 1. It is clearly that $\theta_{ki} e_{ki}(t) = |e_{ki}(t)|$, so

$$\begin{aligned} \frac{d|e_{ki}(t)|}{dt} &= \theta_{ki}(t) \frac{de_{ki}(t)}{dt} \\ &= \theta_{ki}(t) \Bigg\{ [d_{ki} y_{ki}(t) - d_{ki} x_{ki}(t-\sigma) \alpha(t) x_{ki}(t-\sigma)] \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} [co[a_{pjki}(y_{ki}(t))] f_{pj}(y_{pj}(t)) - A_{pjki} f_{pj}(\alpha(t) x_{pj}(t-\sigma)) \\ &\quad - (1+\Delta) L_a] + \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} [-co[a_{pjki}(x_{ki}(t-\sigma))] \alpha(t) f_{pj}(x_{pj}(t-\sigma)) \\ &\quad + A_{pjki} \alpha(t) f_{pj}(x_{pj}(t-\sigma))] \end{aligned}$$

$$+\sum_{\substack{p=1,\\p\neq k}}^{m}\sum_{j=1}^{n_p}[co[b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))]g_{pj}(y_{pj}(t-\tau_{pjki}(t)))$$

$$-B_{pjki}g_{pj}(\alpha(t-\tau_{pjki}(t))x_{pj}(t-\tau_{pjki}(t)-\sigma))]$$

$$+\sum_{\substack{p=1,\\p\neq k}}^{m}\sum_{j=1}^{n_p}[-co[b_{pjki}(x_{ki}(t-\tau_{pjki}(t)-\sigma))]\alpha(t)$$

$$\times g_{pj}(x_{pj}(t-\tau_{pjki}(t)-\sigma))+B_{pjki}\alpha(t)f_{pj}$$

$$\times(x_{pj}(t-\tau_{pjki}(t)-\sigma))-(1+\Delta)L_b]$$

$$+\sum_{\substack{p=1,\\p\neq k}}^{m}\sum_{j=1}^{n_p}\left[co[c_{pjki}(y_{ki}(t))]\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(y_{pj}(s))ds\right.$$

$$\left.-C_{pjki}\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(\alpha(s)x_{pj}(s-\sigma))ds-(1+\Delta)L_c\right]$$

$$+\sum_{\substack{p=1,\\p\neq k}}^{m}\sum_{j=1}^{n_p}\left[-co[c_{pjki}(x_{ki}(t-\sigma))]\alpha(t)\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(x_{pj}(s-\sigma))ds\right.$$

$$\left.\left.+C_{pjki}\alpha(t)\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(x_{pj}(s-\sigma))ds\right]-\eta_{ki}e_{ki}(t)\right\},\quad t\geq\sigma\,. \tag{22}$$

According to Lemma 2, we can derive

$$\frac{d|e_{ki}(t)|}{dt}=\theta_{ki}(t)\frac{de_{ki}(t)}{dt}\leq-(d_{ki}+\eta_{ki})|e_{ki}(t)|$$
$$+\sum_{\substack{p=1,\\p\neq k}}^{m}\sum_{j=1}^{n_p}A_{pjki}L_{pj}^{f}|e_{pj}(t)|+\sum_{\substack{p=1,\\p\neq k}}^{m}\sum_{j=1}^{n_p}B_{pjki}L_{pj}^{g}|e_{pj}(t-\tau_{pjki}(t))|$$
$$+\sum_{\substack{p=1,\\p\neq k}}^{m}\sum_{j=1}^{n_p}C_{pjki}L_{pj}^{h}\int_{t-\mu_{pjki}(t)}^{t}|e_{pj}(s)|ds,\quad t\geq\sigma\,. \tag{23}$$

Denote

$$\omega=\max_{i\in I,k\in K}\{|\phi_{ki}(s)-\alpha(s)\varphi_{ki}(s-\sigma)|\}\,.$$

Then we design the Lyapunov function such that

$$V_{ki}(t)=e^{\varepsilon(t-\sigma)}|e_{ki}(t)|\,. \tag{24}$$

Case 1. For $t \in [\sigma - r, \sigma)$.

Due to $t \geq \sigma - r$, and $i \in I, k \in K$. Let

$$P_{ki}(t) = V_{ki}(t) - h\omega\,, \tag{25}$$

where $h > 1$ is a constant.

From the definitions of $V_{ki}(t)$ and ω. It is easy to check that

$$P_{ki}(t) < 0\,, \tag{26}$$

for all $t \in [\sigma - r, \sigma)$.

Case 2. For $t \in [\sigma, +\infty)$.

In the following, we will testify that

$$P_{ki}(t) < 0\,, \tag{27}$$

for all $t \in [\sigma, +\infty)$ and $i \in I, k \in K$.

Otherwise, there exist $l \in I, d \in K$, and $t^* \in [\sigma, +\infty)$ such that

$$P_{dl}(t^*) = 0, \quad P_{mn}(t^*) \leq 0, \quad n \in I\backslash\{l\}, \quad m \in K\backslash\{d\}, \quad \frac{dP_{dl}(t^*)}{dt^*} \geq 0\,, \tag{28}$$

and for any $i \in I, k \in K$

$$P_{ki}(t) < 0\,, \tag{29}$$

for $t \in [\sigma, t^*]$.

Combining with Eqs. (26) and (29), we obtain

$$P_{ki}(t) < 0\,, \tag{30}$$

for $t \in [\sigma - r, t^*)$.

Calculating the time derivation of $P_{dl}(t^*)$. In line with the solution of system (22), by the chain rule in Lemma 2 and using Eqs. (28)–(30), we have

$$\begin{aligned} \frac{dP_{dl}(t^*)}{dt^*} &= \varepsilon e^{\varepsilon(t^*-\sigma)}|e_{dl}(t^*)| + e^{\varepsilon(t^*-\sigma)}\frac{d|e_{dl}(t^*)|}{dt^*} \\ &\leq \varepsilon e^{\varepsilon(t^*-\sigma)}|e_{dl}(t^*)| + e^{\varepsilon(t^*-\sigma)}\Bigg[-(d_{dl}+\eta_{dl})|e_{dl}(t^*)| \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} A_{pjdl}L^f_{pj}|e_{pj}(t^*)| + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjdl}L^g_{pj}|e_{pj}(t^* - \tau_{pjki}(t^*))| \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} C_{pjdl}L^h_{pj}\int_{t^*-\mu_{pjki}(t^*)}^{t^*}|e_{pj}(s)|ds\Bigg]\,. \end{aligned} \tag{31}$$

During the definition of $V_{ki}(t)$, we get the following inequation:

$$\begin{aligned}
&\varepsilon e^{\varepsilon(t^*-\sigma)}|e_{dl}(t^*)| + e^{\varepsilon(t^*-\sigma)}\frac{d|e_{dl}(t^*)|}{dt^*}\\
&\quad\leq -(d_{dl}+\eta_{dl}-\varepsilon)V_{dl}(t^*) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} A_{pjdl}L_{pj}^{f}V_{pj}(t^*)\\
&\qquad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjdl}L_{pj}^{g}e^{\varepsilon\tau_{pjki}(t^*)}V_{pj}(t^*-\tau_{pjki}(t^*))\\
&\qquad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} C_{pjdl}L_{pj}^{h}\int_{-\mu_{pjki}(t^*)}^{0} V_{pj}(s+t^*)e^{-\varepsilon s}ds\\
&\quad\leq -(d_{dl}+\eta_{dl}-\varepsilon)V_{dl}(t^*) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} A_{pjdl}L_{pj}^{f}V_{pj}(t^*)\\
&\qquad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjdl}L_{pj}^{g}e^{\varepsilon\tau}V_{pj}(t^*-\tau_{pjki}(t^*))\\
&\qquad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} C_{pjdl}L_{pj}^{h}\int_{-\mu}^{0} V_{pj}(s+t^*)e^{-\varepsilon s}ds\\
&\quad\leq -(d_{dl}+\eta_{dl}-\varepsilon)h\omega + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} A_{pjdl}L_{pj}^{f}h\omega\\
&\qquad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjdl}L_{pj}^{g}e^{\varepsilon\tau}h\omega + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} C_{pjdl}L_{pj}^{h}\frac{e^{\varepsilon\mu}h\omega}{\varepsilon}\\
&\quad< (\varepsilon-\lambda_{dl}-\eta_{dl}+\beta_{dl}e^{\varepsilon\tau}+\xi_{dl}e^{\varepsilon\mu})h\omega\\
&\quad< 0\,,
\end{aligned} \tag{32}$$

which leads to a contradiction with Eq. (28). Hence, the inequation Eq. (27) holds.

Let $h \to 1$, then from Eq. (27) and the definition of $V_{ki}(t)$, we have

$$|e_{ki}(t)| < \omega e^{-\varepsilon(t-\sigma)}\,, \tag{33}$$

for any $t \geq \sigma$ and $i \in I, k \in K$, which demonstrates that

$$\|y_{ki}(t)-\alpha(t)x_{ki}(t-\sigma)\| \leq \omega e^{-\varepsilon(t-\sigma)} = \|\phi_{ki}(s)-\alpha(s)\varphi_{ki}(s-\sigma)\|e^{-\varepsilon(s-\sigma)}\,. \tag{34}$$

Hence, according to the Definition 1, systems (2) and (3) can achieve exponentially function lag synchronization under the hybrid linear controller (18). The proof of Theorem 1 is completed. □

Remark 4. In this paper, the essence of projective function lag synchronization is that the delayed system converges to the stable. At present, the research on projective function lay exponentially synchronization of the MMAMNNs with mixed time-varying delays is few. A lot of models about projective function lag synchronization of NNs are special cases of our proposed model.[60–62] Here, we give a corollary as the special case.

If $\sigma = 0$ in system (8), that is to say the error system becomes the form such as $e_{ki}(t) = y_{ki}(t) - \alpha(t)x_{ki}(t)$, then we have the following corollary.

Corollary 1. *Based on the Assumptions* (A1) *and* (A2), *we define the synchronization error* $e_{ki}(t)$ *as follows*: $e_{ki}(t) = y_{ki}(t) - \alpha(t)x_{ki}(t)$. *By the theories of differential inclusion and associate with the systems* (2) *and* (3), *we have the following synchronization error system*

$$\frac{de_{ki}(t)}{dt} = \frac{dy_{ki}(t)}{dt} - \alpha(t)\frac{dx_{ki}(t)}{dt} - \frac{d\alpha(t)}{dt}x_{ki}(t), \quad t \geq 0\,. \tag{35}$$

Based on the synchronization error system (35), we design the hybrid linear controller as follows:

$$U_{ki}(t) = U_{ki_1}(t) + U_{ki_2}(t) + U_{ki_3}(t) + U_{ki_4}(t) + U_{ki_5}(t)\,, \tag{36}$$

where

$$U_{ki_1} = \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[A_{pjki}\alpha(t)f_{pj}(x_{pj}(t)) - (1+\Delta)L_a - A_{pjki}f_{pj}(\alpha(t)x_{pj}(t))],$$

$$U_{ki_2} = \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[B_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\tau_{pjki}(t)))$$
$$- B_{pjki}f_{pj}(\alpha(t-\tau_{pjki}(t))x_{pj}(t-\tau_{pjki}(t)) - (1+\Delta)L_b]\,,$$

$$U_{ki_3} = \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\left[C_{pjki}\alpha(t)\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(x_{pj}(s))ds\right.$$
$$\left. - C_{pjki}\int_{t-\mu_{pjki}(t)}^{t}h_{pj}(\alpha(s)x_{pj}(sa))ds - (1+\Delta)L_c\right],$$

$$U_{ki_4} = (\alpha(t)-1)I_{ki} + \dot{\alpha}(t)x_{ki}(t)\,,$$

$$U_{ki_5} = -\eta_{ki}e_{ki}(t)\,,$$

and η_{ki} satisfies

$$\lambda_{ki} + \eta_{ki} - \varepsilon_{ki} - \beta_{ki} e^{\varepsilon\tau} - \xi_{ki} e^{\varepsilon\mu} > 0\,, \tag{37}$$

for any $i \in I, k \in K$.

Proof. This proof can be derived directly by taking $\sigma = 0$ in Theorem 1. Thus, it is omitted here. □

Remark 5. There is a drawback of the linear feedback controller, that is the strength of linear feedback must be maximal. In some ways, it will cause a kind of waste in practice. Comparing with linear control, the control gains of adaptive control increase based on the adaptive law.[59] Therefore, the adaptive control is more practical and that is the reason we choose adaptive control in the following.

3.2. *Exponentially lag function projective synchronization via hybrid adaptive linear control*

In this section, we consider the exponentially lag function projective synchronization of MMAMNNs with discrete time-varying delay via a hybrid adaptive controller. By constructing a novel Lyapunov function and designing an appropriate adaptive controller, a useful criterion for lag function projective synchronization is proposed. First, we define the drive-response systems as follows:

$$\begin{aligned}\frac{dx_{ki}(t)}{dt} &= -d_{ki}x_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} a_{pjki}(x_{ki}(t))f_{pj}(x_{ki}(t))\\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} b_{pjki}(x_{ki}(t-\tau_{pjki}(t)))g_{pj}(x_{pj}(t-\tau_{pjki}(t))) + I_{ki},\\ &\qquad t \geq 0,\ \ i \in I,\ \ k \in K\,,\end{aligned} \tag{38}$$

and

$$\begin{aligned}\frac{dy_{ki}(t)}{dt} &= -d_{ki}y_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} a_{pjki}(y_{ki}(t))f_{pj}(y_{ki}(t)) + I_{ki}\\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))g_{pj}(y_{pj}(t-\tau_{pjki}(t))) + U_{ki}(t),\\ &\qquad t \geq \sigma,\ \ i \in I,\ \ k \in K\,.\end{aligned} \tag{39}$$

Then, the error system $e_{ki}(t)$ is given as $e_{ki}(t) = y_{ki}(t) - \alpha(t)x_{ki}(t-\sigma)$. By the theories of differential inclusion and set-valued map, we get the following synchronization error system:

$$\frac{de_{ki}(t)}{dt} = \frac{dy_{ki}(t)}{dt} - \alpha(t)\frac{dx_{ki}(t-\sigma)}{dt} - \frac{d\alpha(t)}{dt}x_{ki}(t-\sigma)\,, \quad t \geq \sigma\,. \tag{40}$$

Combining with the systems (38) and (39), we obtain

$$\begin{aligned}\frac{de_{ki}(t)}{dt} \in &-\{d_{ki}y_{ki}(t) - d_{ki}x_{ki}(t-\sigma)\alpha(t)x_{ki}(t-\sigma)\}\\ &+ \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\{co[a_{pjki}(y_{ki}(t))]f_{pj}(y_{pj}(t))\\ &- co[a_{pjki}(x_{ki}(t-\sigma)))]\alpha(t)f_{pj}(x_{pj}(t-\sigma))\}\\ &+ \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\{co[b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))]g_{pj}(y_{pj}(t-\tau_{pjki}(t)))\\ &- co[b_{pjki}(x_{ki}(t-\tau_{pjki}(t-\sigma)))]\alpha(t)g_{pj}(x_{pj}(t-\tau_{pjki}(t-\sigma)))\}\\ &+ (1-\alpha(t))I_{ki} - \frac{d\alpha(t)}{dt}x_{ki}(t-\sigma) + U_{ki}(t), \quad t \geq \sigma\,. \end{aligned} \tag{41}$$

Due to Assumptions (A1) and (A2), the system (40) can reach exponentially lag function projective stable under the following adaptive controller:

$$U_{ki}(t) = U_{ki_1}(t) + U_{ki_2}(t) + U_{ki_3}(t) + U_{ki_4}(t)\,, \tag{42}$$

where

$$U_{ki_1} = \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[A_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\sigma)) - (1+\Delta)L_a - A_{pjki}f_{pj}(\alpha(t)x_{pj}(t-\sigma))]\,,$$

$$\begin{aligned}U_{ki_2} = &\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[B_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\tau_{pjki}(t)-\sigma))\\ &- B_{pjki}f_{pj}(\alpha(t-\tau_{pjki}(t))x_{pj}(t-\tau_{pjki}(t)-\sigma)) - (1+\Delta)L_b]\,,\end{aligned}$$

$$U_{ki_3} = (\alpha(t)-1)I_{ki} + \dot{\alpha}(t)x_{ki}(t-\sigma), \quad U_{ki_4} = -\eta_{ki}(t)e_{ki}(t)\,,$$

where $\eta_{ki}(t)$ is the update controlling strength. Based on the hybrid adaptive controller (43), the following results can be derived.

Theorem 2. *According to Assumptions* (A1) *and* (A2), *the systems* (38) *and* (39) *are exponentially lag function projective synchronized with control inputs* (43) *and updated by the following law*:

$$\dot{\eta}_{ki}(t) = \omega_{ki}|e_{ki}(t)|^{q}e^{\varepsilon(t-\sigma)}, \quad t \leq \sigma\,, \tag{43}$$

where $\omega_{ki} > 0$.

According to the hybrid adaptive controller and the updated law, we draw the conclusion that ensure the synchronization between systems (38) and (39)

$$q(\varepsilon - d_{ki} + m_{ki}) + \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} \left[(q-1)(A_{pjki} + B_{pjki}) + \frac{r_{pj}}{r_{ki}} A_{pjki} (L^f_{pjki})^q + \frac{r_{pj}}{r_{ki}(1-\tau_0)} B_{pjki} (L^g_{pjki})^q \right] < 0 \,, \tag{44}$$

which $r_{pj} > 0$, $r_{ki} > 0$, and $m_{ki} < 0$.

Now we consider a Lyapunov function as

$$V_{ki}(t) = r_{ki} \sum_{\substack{k=1,\\ p\neq k}}^{m} \sum_{i=1}^{n_k} \Bigg[|e_{ki}(t)|^q \exp\{q\varepsilon(t-\sigma)\} + \frac{\exp\{q\varepsilon\tau\}}{1-\tau_0} \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} B_{pjki} \int_{t-\tau_{pjki}(t)}^{t} |G_{pj}(e_{pj}(s))|^q \times \exp\{q\varepsilon(s-\sigma)\} ds + \frac{q}{2\omega_{ki}} (\eta_{ki}(t) + m_{ki})^2 \Bigg] \,, \tag{45}$$

which $G_{pj}(e_{pj}(t)) = g_{pj}(y_{pj}(t)) - g_{pj}(\alpha(t) x_{pj}(t-\sigma))$.

According to Assumptions (A1), (A2) and Lemma 1, by calculating the upper right derivation $D^+V(t)$ of $V(t)$ along with the solution to Eq. (40), we obtain

$$\begin{aligned} \frac{d}{dt} V_{ki}(t) = \sum_{\substack{k=1,\\ p\neq k}}^{m} \sum_{i=1}^{n_k} r_{ki} \Bigg\{ & q|e_{ki}(t)|^{q-1} \exp\{q\varepsilon(t-\sigma)\} \theta_{ki} \frac{d}{dt} e_{ki}(t) \\ & + |e_{ki}(t)|^q q\varepsilon \exp\{q\varepsilon(t-\sigma)\} \\ & + \frac{\exp\{q\varepsilon\tau\}}{1-\tau_0} \sum_{\substack{p=1,\\ p\neq k}}^{m} \sum_{j=1}^{n_p} B_{pjki} [G_{pj}(e_{pj}(t))|^q \exp\{q\varepsilon(t-\sigma)\} \\ & - G_{pj}|(e_{pj}(t-\tau_{pjki}(t)))|^q \exp\{q\varepsilon(t-\tau_{pjki}(t)-\sigma)\} (t-\tau_{pjki}(t))'] \\ & + q(\eta_{ki}(t) + m_{ki}) |e_{ki}(t)|^q \exp\{q\varepsilon(t-\sigma)\} \Bigg\} \,. \end{aligned} \tag{46}$$

Calculating the Clarkes generalized gradient of absolute value of function $|e_{ki}(t)|$ by Lemmas 1 and 2, we conclude that $\theta_{ki}e_{ki}(t) = |e_{ki}(t)|$. Then we have

$$\begin{aligned}\theta_{ki}\frac{d}{dt}e_{ki}(t) = \frac{d|e_{ki}(t)|}{dt} \\ &= (-d_{ki} - \eta_{ki}(t))|e_{ki}(t)| + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} A_{pjki}L_{pj}^{f}|e_{pj}(t)| \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjki}L_{pj}^{g}|e_{pj}(t-\tau_{pjki}(t))|\,. \end{aligned} \tag{47}$$

Due to Assumption (A1), we deduce that

$$\begin{aligned}&\frac{\exp\{q\varepsilon\tau\}}{1-\tau_0}\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjki}[G_{pj}(e_{pj}(t))|^{q}\exp\{q\varepsilon(t-\sigma)\} \\ &\quad - G_{pj}|(e_{pj}(t-\tau_{pjki}(t)))|^{q}\exp\{q\varepsilon(t-\tau_{pjki}(t)-\sigma)\}(t-\tau_{pjki}(t))'] \\ &\leq \frac{\exp\{q\varepsilon\tau\}}{1-\tau_0}\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjki}[(L_{pj}^{g})^{q}|e_{pj}(t)|^{q}\exp\{q\varepsilon(t-\sigma)\} \\ &\quad - (L_{pj}^{g})^{q}|e_{pj}(t-\tau_{pjki}(t))|^{q}\exp\{q\varepsilon(t-\tau-\sigma)\}(1-\tau_0)]\,. \end{aligned} \tag{48}$$

Based on the above discussion

$$\begin{aligned}\frac{d}{dt}V_{ki}(t) &\leq \sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k} r_{ki}\Bigg\{q|e_{ki}(t)|^{q-1}\exp\{q\varepsilon(t-\sigma)\}\Bigg[(-d_{ki}-\eta_{ki}(t))|e_{ki}(t)| \\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} A_{pjki}L_{pj}^{f}|e_{pj}(t)| + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjki}L_{pj}^{g}|e_{pj}(t-\tau_{pjki}(t))|\Bigg] \\ &\quad + \exp\{q\varepsilon(t-\sigma)\}\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjki}\Bigg[(L_{pj}^{g})^{q}|e_{pj}(t)|^{q}\frac{\exp\{q\varepsilon\tau\}}{1-\tau_0} \\ &\quad - (L_{pj}^{g})^{q}|e_{pj}(t-\tau_{pjki}(t))|^{q}\Bigg] + q\varepsilon|e_{ki}(t)|^{q}\exp\{q\varepsilon(t-\sigma)\} \\ &\quad + q(\eta_{ki}(t)+m_{ki})|e_{ki}(t)|^{q}\exp\{q\varepsilon(t-\sigma)\}\Bigg\}\,. \end{aligned} \tag{49}$$

Combining with Eqs. (46)–(49), we have

$$\frac{d}{dt}V_{ki}(t) \le \sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k} r_{ki}\Bigg\{q\exp\{q\varepsilon(t-\sigma)\}\Bigg[(-d_{ki}-\eta_{ki}(t)+\varepsilon)|e_{ki}(t)|^q$$
$$+\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} A_{pjki}L^f_{pj}|e_{pj}(t)\|e_{ki}(t)|^{q-1}$$
$$+\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjki}L^g_{pj}|e_{pj}(t-\tau_{pjki}(t))\|e_{ki}(t)|^{q-1}\Bigg]$$
$$+\exp\{q\varepsilon(t-\sigma)\}\Bigg[\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjki}(L^g_{pj})^q|e_{pj}(t)|^q\frac{\exp\{q\varepsilon\tau\}}{1-\tau_0}$$
$$-\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} B_{pjki}(L^g_{pj})^q|e_{pj}(t-\tau_{pjki}(t))|^q\Bigg]$$
$$+q(\eta_{ki}(t)+m_{ki})|e_{ki}(t)|^q\exp\{q\varepsilon(t-\sigma)\}\Bigg\}. \tag{50}$$

According to Young inequality $ab \le \frac{1}{\beta_1}a^{\beta_1}+\frac{1}{\beta_2}b^{\beta_2}$, in which $a>0, b>0, \beta_1>1, \frac{1}{\beta_1}+\frac{1}{\beta_2}=1$, we have

$$L^f_{pj}|e_{pj}(t)\|e_{ki}(t)|^{q-1} \le \frac{1}{q}(L^f_{pj})^q|e_{pj}(t)|^q+\frac{q-1}{q}|e_{ki}(t)|^q,$$
$$L^g_{pj}|e_{pj}(t-\tau_{pjki}(t))\|e_{ki}(t)|^{q-1} \le \frac{1}{q}(L^g_{pj})^q|e_{pj}(t-\tau_{pjki}(t))|^q+\frac{q-1}{q}|e_{ki}(t)|^q. \tag{51}$$

Then we conclude

$$\frac{d}{dt}V_{ki}(t) \le \sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k} r_{ki}\Bigg\{\exp\{q\varepsilon(t-\sigma)\}\Bigg[q(-d_{ki}-\eta_{ki}(t)+\varepsilon)|e_{ki}(t)|^q$$
$$+\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[A_{pjki}(L^f_{pj})^q|e_{pj}(t)|^q+A_{pjki}(q-1)|e_{ki}(t)|^q]$$

$$+ \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[B_{pjki}L_{pj}^{g}|e_{pj}(t-\tau_{pjki}(t))|^q + B_{pjki}(q-1)|e_{ki}(t)|^q]\Bigg]$$

$$+ \exp\{q\varepsilon(t-\sigma)\}\left[\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}B_{pjki}(L_{pj}^{g})^q|e_{pj}(t)|^q\frac{\exp\{q\varepsilon\tau\}}{1-\tau_0}\right.$$

$$\left.- \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}B_{pjki}(L_{pj}^{g})^q|e_{pj}(t-\tau_{pjki}(t))|^q\right]$$

$$\left.+ q(\eta_{ki}(t)+m_{ki})|e_{ki}(t)|^q\exp\{q\varepsilon(t-\sigma)\}\right\}. \tag{52}$$

Due to the above discussion, we get

$$\frac{d}{dt}V_{ki}(t) = \sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k}r_{ki}\exp\{q\varepsilon(t-\sigma)\}|e_{ki}(t)|^q\left\{q(\varepsilon+m_{ki}-d_{ki})\right.$$

$$+ \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}(q-1)(B_{pjki}+A_{pjki}) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\left[\frac{r_{pj}}{r_{ki}}A_{pjki}(L_{ki}^{f})^q\right.$$

$$\left.\left.+ \frac{r_{pj}}{r_{ki}}B_{pjki}(L_{ki}^{g})^q\frac{\exp\{q\varepsilon\tau\}}{1-\tau_0}\right]\right\}. \tag{53}$$

Then we can choose a small $\varepsilon > 0$ such that

$$q(\varepsilon - d_{ki} + m_{ki}) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\left[(q-1)(A_{pjki}+B_{pjki})\right.$$

$$\left.+ \frac{r_{pj}}{r_{ki}}A_{pjki}(L_{pj}^{f})^q + \frac{r_{pj}}{r_{ki}(1-\tau_0)}B_{pjki}(L_{pj}^{g})^q\right] < 0\,. \tag{54}$$

Thus we have

$$V_{ki}(t) \leq 0\,, \tag{55}$$

and together with Eq. (45) we obtain

$$V_{ki}(t) \leq V_{ki}(\sigma)\,, \tag{56}$$

for $t \geq \sigma$.

Based on Eq. (45), we obtain that

$$
\begin{aligned}
V_{ki}(t) &\geq \sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k} r_{ki}\exp\{q\varepsilon(t-\sigma)\}|e_{ki}(t)|^q \\
&\geq \min_{i\in I,k\in K} r_{ki}\exp\{q\varepsilon(t-\sigma)\}\sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k}|e_{ki}(t)|^q\,. \qquad (57)
\end{aligned}
$$

Then we have

$$
\min_{i\in I,k\in K} r_{ki}\exp\{q\varepsilon(t-\sigma)\}\sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k}|e_{ki}(t)|^q \leq V_{ki}(t)\leq V_{ki}(\sigma)\,, \qquad (58)
$$

where

$$
\begin{aligned}
V_{ki}(\sigma) = r_{ki}\sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k}\Bigg[&|e_{ki}(\sigma)|^q + \frac{q}{2\omega_{ki}}(\eta_{ki}(\sigma)+m_{ki})^2 \\
&+\frac{\exp\{q\varepsilon\tau\}}{1-\tau_0}\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}B_{pjki}\int_{\sigma-\tau_{pjki}(\sigma)}^{\sigma}|G_{pj}(e_{pj}(s))|^q\exp\{q\varepsilon(s-\sigma)\}ds\Bigg] \\
\leq \sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k}\Bigg\{&\Bigg[\max_{i\in I,k\in K} r_{ki} + (L^g_{pj})^q\tau\frac{\exp\{q\varepsilon\tau\}}{1-\tau_0}\sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p} r_{ki}B_{pjki}\Bigg] \\
&\sup_{t-\sigma\leq t\leq\sigma}[|\phi_{ki}(t)-\alpha(t)\varphi_{ki}(t-\sigma)|]^q + \frac{q}{2\omega_{ki}}(\eta_{ki}(\sigma)+m_{ki})^2\Bigg\} \triangleq N^*\,.
\end{aligned}
\qquad (59)
$$

Thus for $t\geq\sigma$, there exists a positive constant M^* satisfy

$$
N^* \leq M^*\|\phi_{ki}(t)-\alpha(t)\varphi_{ki}(t-\sigma)\|^q\,. \qquad (60)
$$

Together with the above discussion, we obtain the following inequality:

$$
\sum_{\substack{k=1,\\ p\neq k}}^{m}\sum_{i=1}^{n_k}|e_{ki}|^q \leq M^*\exp\{-\varepsilon(s-\sigma)\}\|\phi_{ki}(t)-\alpha(t)\varphi_{ki}(t-\sigma)\|^q\,. \qquad (61)
$$

By Definition 2, we get

$$
\|y_{ki}(t)-\alpha(t)x_{ki}(t-\sigma)\| \leq M\exp\{-\varepsilon^*(s-\sigma)\}\|\phi_{ki}(t)-\alpha(t)\varphi_{ki}(t-\sigma)\|\,, \qquad (62)
$$

where $M=(M^*)^{\frac{1}{q}}$ and $\varepsilon^*=\varepsilon\backslash q$. The proof is completed. □

Remark 6. The results of Theorem 2 can be easily extended to the MMAMNNs without delay as shown below. Here, we give a corollary as the special case.

If $\sigma = 0$ in system (41), that is to say the error system becomes the form such as $e_{ki}(t) = y_{ki}(t) - \alpha(t)x_{ki}(t)$, then we have the following corollary:

Corollary 2. *We define the synchronization error* $e_{ki}(t)$ *as follows*: $e_{ki}(t) = y_{ki}(t) - \alpha(t)x_{ki}(t)$. *According to the theories of set-valued map and differential inclusion, we get the synchronization error system for* $t \geq 0$

$$\frac{de_{ki}(t)}{dt} = \frac{dy_{ki}(t)}{dt} - \alpha(t)\frac{dx_{ki}(t)}{dt} - \frac{d\alpha(t)}{dt}x_{ki}(t)\,. \tag{63}$$

Combining with the systems (38) and (39) with $\sigma = 0$, we design the hybrid adaptive controller as follows:

$$U_{ki}(t) = U_{ki_1}(t) + U_{ki_2}(t) + U_{ki_3}(t) + U_{ki_4}(t)\,, \tag{64}$$

where

$$\begin{aligned} U_{ki_1} &= \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[A_{pjki}\alpha(t)f_{pj}(x_{pj}(t)) - (1+\Delta)L_a - A_{pjki}f_{pj}(\alpha(t)x_{pj}(t))]\,,\\ U_{ki_2} &= \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}[B_{pjki}\alpha(t)f_{pj}(x_{pj}(t-\tau_{pjki}(t)))\\ &\quad - B_{pjki}f_{pj}(\alpha(t-\tau_{pjki}(t))x_{pj}(t-\tau_{pjki}(t)) - (1+\Delta)L_b]\,,\\ U_{ki_3} &= (\alpha(t)-1)I_{ki} + \dot{\alpha}(t)x_{ki}(t),\\ U_{ki_4} &= -\eta_{ki}(t)e_{ki}(t)\,. \end{aligned}$$

Under the following updated law

$$\dot{\eta}_{ki}(t) = \omega_{ki}|e_{ki}(t)|^q e^{\varepsilon(t)}, \quad t \leq 0\,, \tag{65}$$

where $\omega_{ki} > 0$.

We can derive the following synchronization criterion which guarantee the error system (63) get exponentially function projective stable

$$\begin{aligned} &q(\varepsilon - d_{ki} + m_{ki}) + \sum_{\substack{p=1,\\ p\neq k}}^{m}\sum_{j=1}^{n_p}\left[(q-1)(A_{pjki} + B_{pjki}) + \frac{r_{pj}}{r_{ki}}A_{pjki}(L^f_{pjki})^q\right.\\ &\qquad \left. + \frac{r_{pj}}{r_{ki}(1-\tau_0)}B_{pjki}(L^g_{pjki})^q\right] < 0\,, \end{aligned} \tag{66}$$

which r_{pj}, r_{ki} are positive constants, and $m_{ki} < 0$.

Proof. This proof can be derived directly by taking $\sigma = 0$ in Theorem 2. Thus it is omitted here. □

Remark 7. In many practical systems, it is generally fulfilling to treat the lag factor σ as a variable $\sigma(t)$. Consequently, how to extend the approaches in this paper to time-varying lag synchronization will be our next research.

4. Numerical Simulation

In this section, several numerical examples are given to illustrate the efficiency of our theoretical results.

Example 1. Consider the following MMAMNNs system with three nodes as the driven system, for $i = 1, j = 1, n_k = 1, n_p = 1, k = 1, 2, 3$.

$$\begin{aligned}\frac{dx_{ki}(t)}{dt} &= -d_{ki}x_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{3}\sum_{j=1}^{n_p} a_{pjki}(x_{ki}(t))f_{pj}(x_{ki}(t))\\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{3}\sum_{j=1}^{n_p} b_{pjki}(x_{ki}(t-\tau_{pjki}(t)))g_{pj}(x_{pj}(t-\tau_{pjki}(t)))\\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{3}\sum_{j=1}^{n_p} c_{pjki}(x_{ki}(t))\int_{t-\mu_{pjki}(t)}^{t} h_{pj}(x_{ki}(s))ds + I_{ki}\,. \end{aligned} \tag{67}$$

The corresponding response system is given by

$$\begin{aligned}\frac{dy_{ki}(t)}{dt} &= -d_{ki}y_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{3}\sum_{j=1}^{n_p} a_{pjki}(y_{ki}(t))f_{pj}(y_{ki}(t))\\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{3}\sum_{j=1}^{n_p} b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))g_{pj}(y_{pj}(t-\tau_{pjki}(t)))\\ &\quad + \sum_{\substack{p=1,\\ p\neq k}}^{3}\sum_{j=1}^{n_p} c_{pjki}(y_{ki}(t))\int_{t-\mu_{pjki}(t)}^{t} h_{pj}(y_{ki}(s))ds + U_{ki}(t) + I_{ki}\,. \end{aligned} \tag{68}$$

We design the parameters of the drive system (67) and the response system (68) as follows: $I_{ki} = (0,0,0)^T, x_{ki}(0) = (-0.15, 0.35, -0.55)^T, y_{ki}(0) = (0.1, 1.71, 0.11)^T, \tau_{pjki} = 0.5+0.5\sin(t), \mu_{pjki} = 0.5+0.5\cos(t), f(x) = h(x) = \sin(|x|), g(x) = \tanh(|x|)$.

Then, according to the achievements of Theorem 1 and Corollary 1, we design the suitable parameters as following:

$$d_{11}(x_{11}(t)) = 1, \quad d_{21}(x_{21}(t)) = 2, \quad d_{31}(x_{31}(t)) = 3,$$

$$a_{2111}(x_{11}(t)) = \begin{cases} -0.92, & x_{11}(t) \le 0\,,\\ -0.74, & x_{11}(t) > 0\,, \end{cases} \quad a_{3111}(x_{11}(t)) = \begin{cases} -1.45, & x_{11}(t) \le 0\,,\\ -1.57, & x_{11}(t) > 0\,, \end{cases}$$

$$c_{2111}(x_{11}(t)) = \begin{cases} -2, & x_{11}(t) \le 0\,, \\ -1.5, & x_{11}(t) > 0\,, \end{cases} \quad a_{3111}(x_{11}(t)) = \begin{cases} -0.8, & x_{11}(t) \le 0\,, \\ -1.2, & x_{11}(t) > 0\,, \end{cases}$$

$$a_{1121}(x_{21}(t)) = \begin{cases} -1, & x_{21}(t) \le 0, \\ -1.6, & x_{21}(t) > 0, \end{cases} \quad a_{3121}(x_{21}(t)) = \begin{cases} -1.4, & x_{21}(t) \le 0\,, \\ -1.5, & x_{21}(t) > 0\,, \end{cases}$$

$$c_{1121}(x_{21}(t)) = \begin{cases} -1.2, & x_{21}(t) \le 0\,, \\ -2.4, & x_{21}(t) > 0\,, \end{cases} \quad c_{3121}(x_{21}(t)) = \begin{cases} -3, & x_{21}(t) \le 0\,, \\ -2, & x_{21}(t) > 0\,, \end{cases}$$

$$a_{1131}(x_{31}(t)) = \begin{cases} 1, & x_{31}(t) \le 0\,, \\ 1.6, & x_{31}(t) > 0\,, \end{cases} \quad a_{2131}(x_{31}(t)) = \begin{cases} -0.1, & x_{31}(t) \le 0\,, \\ -0.9, & x_{31}(t) > 0\,, \end{cases}$$

$$c_{1131}(x_{31}(t)) = \begin{cases} 1, & x_{31}(t) \le 0\,, \\ 1.2, & x_{31}(t) > 0\,, \end{cases} \quad c_{2131}(x_{31}(t)) = \begin{cases} 2, & x_{31}(t) \le 0\,, \\ 1.5, & x_{31}(t) > 0\,, \end{cases}$$

$$b_{2111}(x_{11}(t - \tau_{2111}(t)) = \begin{cases} 1, & x_{11}(t - \tau_{2111}(t)) \le 0\,, \\ 2, & x_{11}(t - \tau_{2111}(t)) > 0\,, \end{cases}$$

$$b_{3111}(x_{11}(t - \tau_{3111}(t))) = \begin{cases} -4.2, & x_{11}(t - \tau_{3111}(t)) \le 0\,, \\ -4, & x_{11}(t - \tau_{3111}(t)) > 0\,, \end{cases}$$

$$b_{1121}(x_{21}(t - \tau_{1121}(t))) = \begin{cases} -1.8, & x_{21}(t - \tau_{1121}(t)) \le 0\,, \\ -1.4, & x_{21}(t - \tau_{1121}(t)) > 0\,, \end{cases}$$

$$b_{3121}(x_{21}(t - \tau_{3121}(t))) = \begin{cases} -1.6, & x_{21}(t - \tau_{3121}(t)) \le 0\,, \\ -1, & x_{21}(t - \tau_{3121}(t)) > 0\,, \end{cases}$$

$$b_{1131}(x_{31}(t - \tau_{1131}(t))) = \begin{cases} -4, & x_{31}(t - \tau_{1131}(t)) \le 0\,, \\ -1.4, & x_{31}(t - \tau_{1131}(t)) > 0\,, \end{cases}$$

$$b_{2131}(x_{31}(t - \tau_{2131}(t))) = \begin{cases} -1.9, & x_{31}(t - \tau_{1131}(t)) \le 0\,, \\ -1, & x_{31}(t - \tau_{1131}(t)) > 0\,. \end{cases}$$

Figure 2 shows that the drive system (67) has a limit cycle in the case of the above-mentioned parameters. Taking the control gain $\eta_{11} = 1532$, $\eta_{21} = 1320$, $\eta_{31} = 1500$, in the controller (18), Lipschitz constants $L^f_{pj} = L^g_{pj} = L^h_{pj} = 1$, $e = 2.718$, $\tau = \mu = 1$, $\varepsilon = 1$. $\alpha(t)$ is scaling factor of projective synchronization. Get together the above mentioned parameters with the condition (21), we calculate that

(i) $1 + \eta_{11} - 1 - (0.92 + 1.57) + (2 + 4.2) \times 2.718 + (2 + 1.2) \times 2.718 > 0$

$\eta_{11} > 28.0392\,,$

(ii) $2 + \eta_{21} - 1 - (1.6 + 1.5) + (1.8 + 1.6) \times 2.718 + (2.4 + 3) \times 2.718 > 0$

$\eta_{21} > 31.4544\,,$

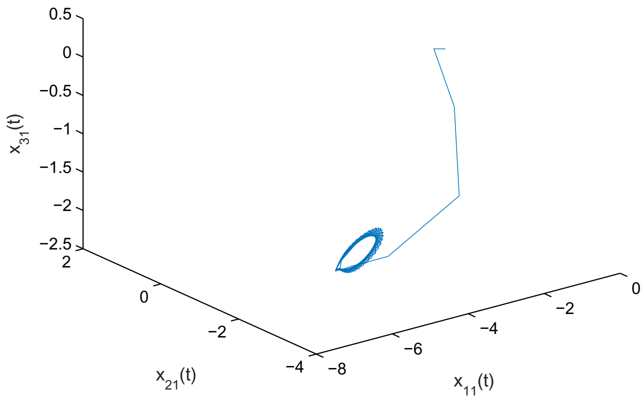

Fig. 2. Limit cycle of system (67) with initial value $x_{ki}(0) = (-0.15, 0.35, -0.55)^T$.

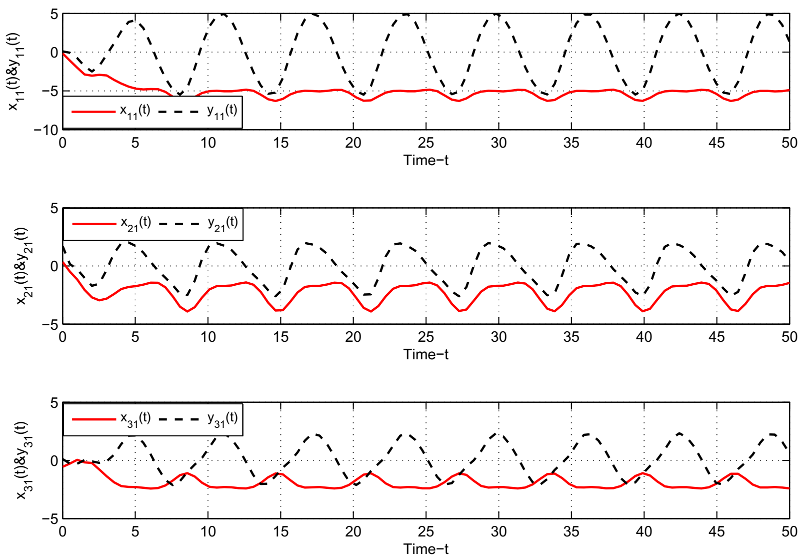

Fig. 3. The state trajectories of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = \sin(t)$, $\sigma = 0.6$.

(iii) $3 + \eta_{31} - 1 - (1.6 + 0.9) + (4 + 1.9) \times 2.718 + (2 + 1.2) \times 2.718 > 0$

$\eta_{31} > 25.2338\,.$

Figures 3–6 show the state trajectories when $\alpha(t)$ is chosen different values. It can be seen that Figs. 3–5 illustrate the lag functional projective synchronization, lag complete synchronization, and lag anti-synchronization, respectively. And Fig. 6 demonstrates the lag functional projective synchronization error. It says that the scaling factor $\alpha(t)$ is crucial to the types of the synchronization. In order to demonstrate the reasonableness of Corollary 1, we let $\alpha(t) = \sin(t), \sigma = 0$. Figure 7 illustrates the normal function projective synchronization between the systems (67) and (68).

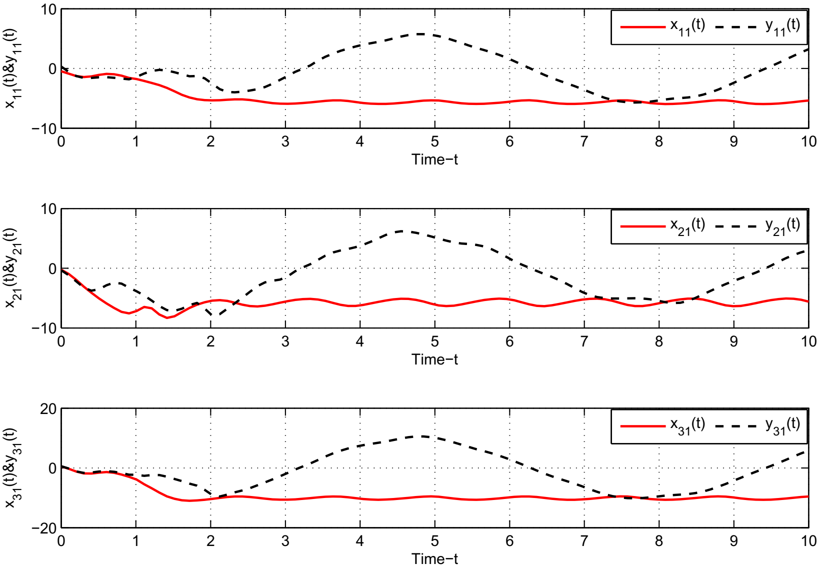

Fig. 4. The state trajectories of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = 1$, $\sigma = 0.6$.

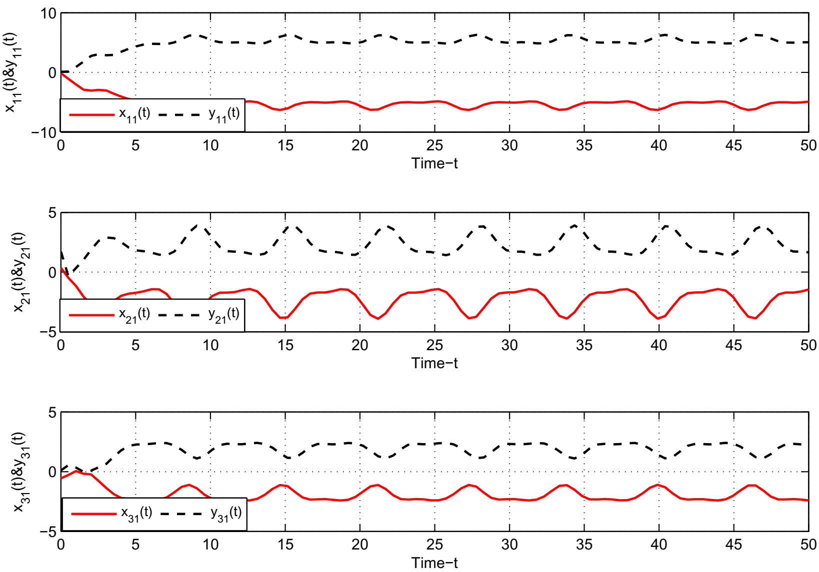

Fig. 5. The state trajectories of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = -1$, $\sigma = 0.6$.

In this example, we also represent the phase diagrams of the drive-response systems when $\alpha(t) = \sin(t), 1, -1$ as shown in Figs. 8–10.

Example 2. Consider the following MMAMNNs system with three nodes as the driven system for, $i = 1, j = 1, n_k = 1, n_p = 1, k = 1, 2, 3$.

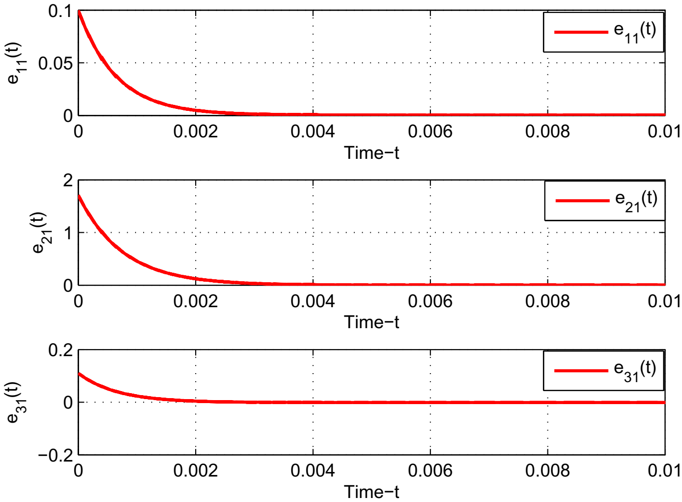

Fig. 6. The synchronization error curves of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = \sin(t)$, $\sigma = 0.6$.

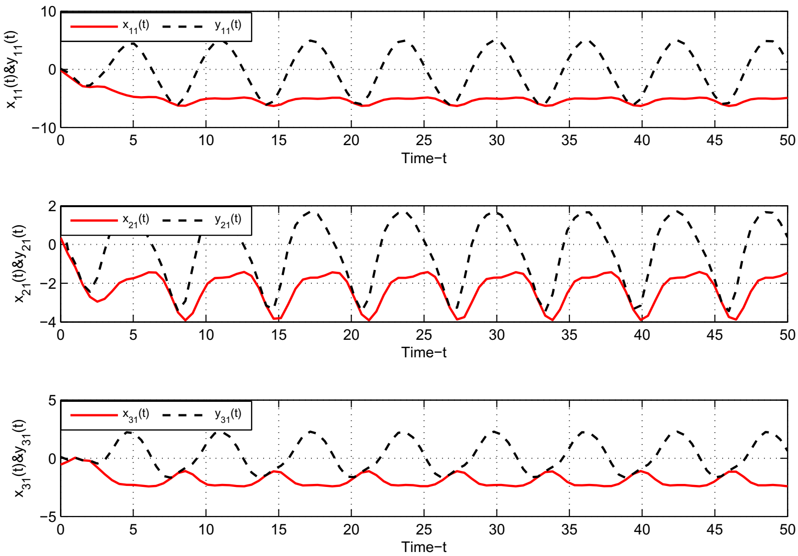

Fig. 7. The state trajectories of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = \sin(t)$, $\sigma = 0$.

$$\frac{dx_{ki}(t)}{dt} = -d_{ki}x_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{3} \sum_{j=1}^{n_p} a_{pjki}(x_{ki}(t)) f_{pj}(x_{ki}(t))$$

$$+ \sum_{\substack{p=1,\\ p\neq k}}^{3} \sum_{j=1}^{n_p} b_{pjki}(x_{ki}(t - \tau_{pjki}(t))) g_{pj}(x_{pj}(t - \tau_{pjki}(t))) + I_{ki}\,, \qquad (69)$$

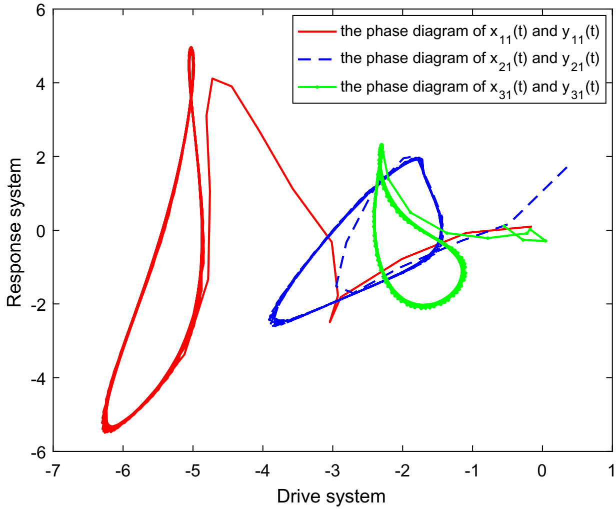

Fig. 8. (Color online) The phase diagram of the drive-response systems (67), (68) when $\alpha(t) = \sin(t)$, $\sigma = 0.6$.

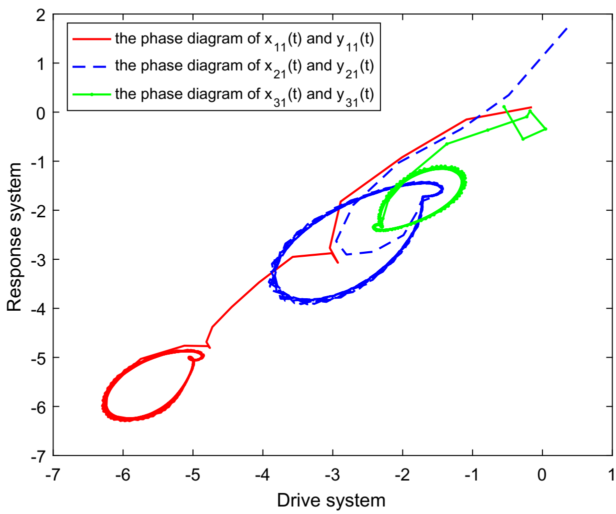

Fig. 9. (Color online) The phase diagram of the drive-response systems (67), (68) when $\alpha(t) = 1$, $\sigma = 0.6$.

and the corresponding response system is presented by

$$\frac{dy_{ki}(t)}{dt} = -d_{ki}y_{ki}(t) + \sum_{\substack{p=1,\\ p\neq k}}^{3}\sum_{j=1}^{n_p} a_{pjki}(y_{ki}(t))f_{pj}(y_{ki}(t)) + \sum_{\substack{p=1,\\ p\neq k}}^{3}\sum_{j=1}^{n_p} b_{pjki}(y_{ki}(t-\tau_{pjki}(t)))g_{pj}(y_{pj}(t-\tau_{pjki}(t))) + U_{ki}(t) + I_{ki}\,. \tag{70}$$

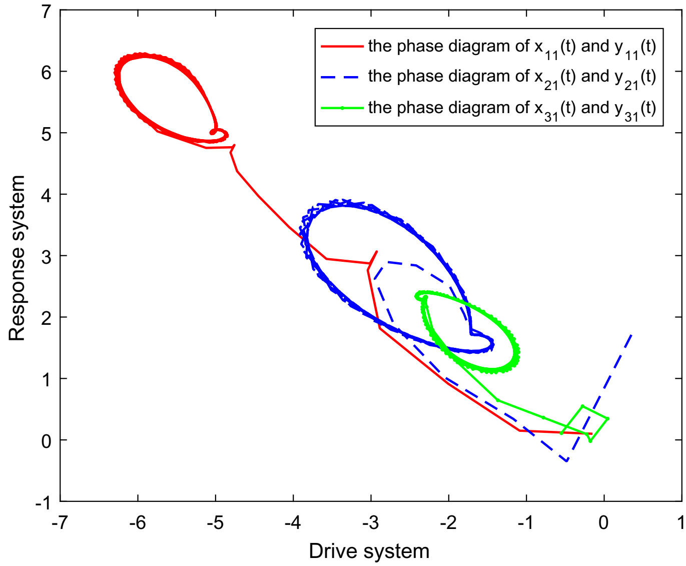

Fig. 10. (Color online) The phase diagram of the drive-response systems (67), (68) when $\alpha(t) = -1$, $\sigma = 0.6$.

All parameters of the drive system (69) and the response system (70) are selected as $I_{ki} = [0; 0; 0]^T$, $x_{ki}(0) = (-0.46, -0.25, 0.54)^T$, $y_{ki}(0) = (0.31, -0.33, 0.58)^T$, $\tau_{pjki} = 0.5 + 0.5\sin(t)$, $\mu_{pjki} = 0.5 + 0.5\cos(t)$, $f(x) = h(x) = \sin(|x|)$, $g(x) = \tanh(|x|)$.

And we define the parameters of the systems are demonstrated as

$$d_{11}(x_{11}(t)) = d_{21}(x_{21}(t)) = d_{31}(x_{31}(t)) = 1\,,$$

$$a_{2111}(x_{11}(t)) = \begin{cases} -3, & x_{11}(t) \le 0\,, \\ 1, & x_{11}(t) > 0\,, \end{cases} \quad a_{3111}(x_{11}(t)) = \begin{cases} 2, & x_{11}(t) \le 0\,, \\ -4, & x_{11}(t) > 0\,, \end{cases}$$

$$a_{1121}(x_{21}(t)) = \begin{cases} -1, & x_{21}(t) \le 0\,, \\ -5, & x_{21}(t) > 0\,, \end{cases} \quad a_{3121}(x_{21}(t)) = \begin{cases} -8, & x_{21}(t) \le 0\,, \\ -8, & x_{21}(t) > 0\,, \end{cases}$$

$$a_{1131}(x_{31}(t)) = \begin{cases} -5, & x_{31}(t) \le 0\,, \\ -9, & x_{31}(t) > 0\,, \end{cases} \quad a_{2131}(x_{31}(t)) = \begin{cases} -6, & x_{31}(t) \le 0\,, \\ -8, & x_{31}(t) > 0\,, \end{cases}$$

$$b_{2111}(x_{11}(t - \tau_{2111}(t))) = \begin{cases} 2, & x_{11}(t - \tau_{2111}(t)) \le 0\,, \\ 5, & x_{11}(t - \tau_{2111}(t)) > 0\,, \end{cases}$$

$$b_{3111}(x_{11}(t - \tau_{3111}(t))) = \begin{cases} -8, & x_{11}(t - \tau_{3111}(t)) \le 0\,, \\ -10, & x_{11}(t - \tau_{3111}(t)) > 0\,, \end{cases}$$

$$b_{1121}(x_{21}(t-\tau_{1121}(t))) = \begin{cases} -5, & x_{21}(t-\tau_{1121}(t)) \le 0\,, \\ 3, & x_{21}(t-\tau_{1121}(t)) > 0\,, \end{cases}$$

$$b_{3121}(x_{21}(t-\tau_{3121}(t))) = \begin{cases} -6, & x_{21}(t-\tau_{3121}(t)) \le 0\,, \\ 4, & x_{21}(t-\tau_{3121}(t)) > 0\,, \end{cases}$$

$$b_{1131}(x_{31}(t-\tau_{1131}(t))) = \begin{cases} -9, & x_{31}(t-\tau_{1131}(t)) \le 0\,, \\ -4, & x_{31}(t-\tau_{1131}(t)) > 0\,, \end{cases}$$

$$b_{2131}(x_{31}(t-\tau_{2131}(t))) = \begin{cases} -7, & x_{31}(t-\tau_{1131}(t)) \le 0\,, \\ 8, & x_{31}(t-\tau_{1131}(t)) > 0\,. \end{cases}$$

Taking the control gain $\omega_{11} = 1$, $\omega_{21} = 2$, and $\omega_{31} = 3$. In the controller (43), Lipschitz constants $L^f_{pj} = L^g_{pj} = L^h_{pj} = 1, \tau = \mu = 1, e = 2.718, \varepsilon = 1, \tau_0 = 0.5$. $\alpha(t)$ is scaling factor of projective synchronization. Get together the above-mentioned parameters with the condition (45), we calculate that

$$\begin{aligned} \text{(i)}\quad & 2 \times (1 + m_{11} - 1) + (5 + 3) + (10 + 4) + (3 + 4) \\ & + 2.718 \times 2.718 \div 0.5 \times (5 + 10) < 0 \\ & m_{11} < -125.31\,, \\ \text{(ii)}\quad & 2 \times (1 + m_{21} - 1) + (5 + 5) + (6 + 8) + (6 + 5) \\ & + 2.718 \times 2.718 \div 0.5 \times (5 + 6) < 0 \\ & m_{21} < -98.76\,, \\ \text{(iii)}\quad & 2 \times (1 + m_{31} - 1) + (9 + 9) + (8 + 8) + (9 + 8) \\ & + 2.718 \times 2.718 \div 0.5 \times (9 + 8) < 0 \\ & m_{31} < -151.09\,. \end{aligned}$$

Figures 11–13 show the state trajectories when $\alpha(t)$ is chosen different values. Under the adaptive hybrid controller (64), the synchronous error of the systems (69) and (70) can converge to zero, which is shown in Fig. 14.

During the process of proving Corollary 2, we redefine the initial values such that $x_{ki}(0) = (-0.15, -0.35, -0.55)^T, y_{ki}(0) = (0.1, 1.71, 0.11)^T$. With the parameters mentioned above, Fig. 15 demonstrates that Corollary 2 is reasonable under the controller (64).

Remark 8. By contrast, we can get that during the process of synchronization, the gain of the feedback controller in Example 1 is so large that will cause some waste in the practical, but the control gains of the adaptive controller in Example 2 is more flexible and general.

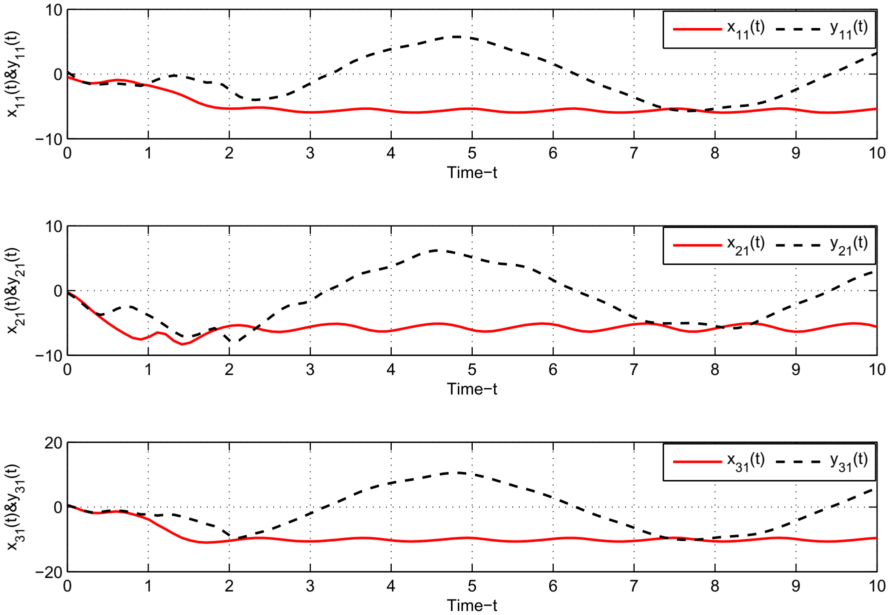

Fig. 11. The state trajectories of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = \sin(t)$, $\sigma = 0.6$.

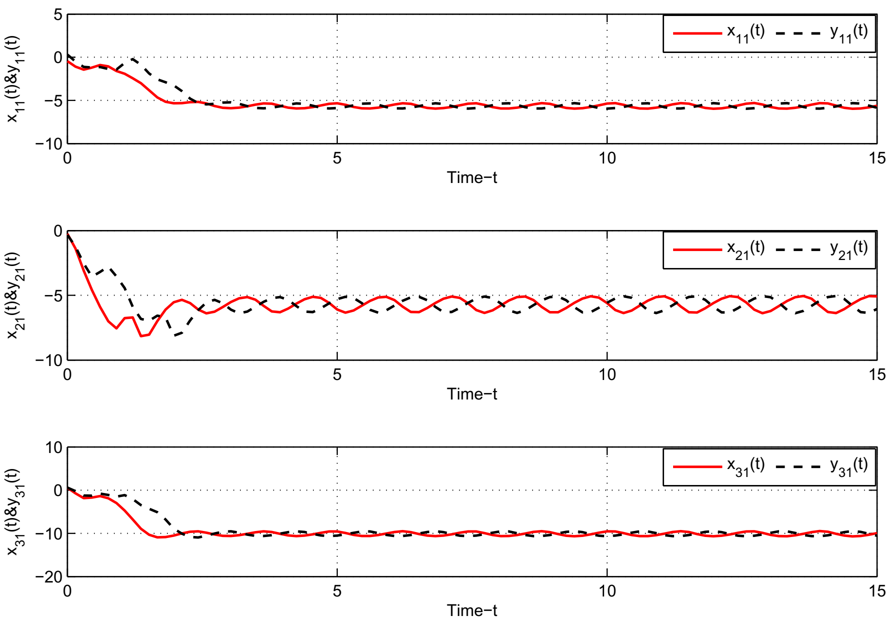

Fig. 12. The state trajectories of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = 1$, $\sigma = 0.6$.

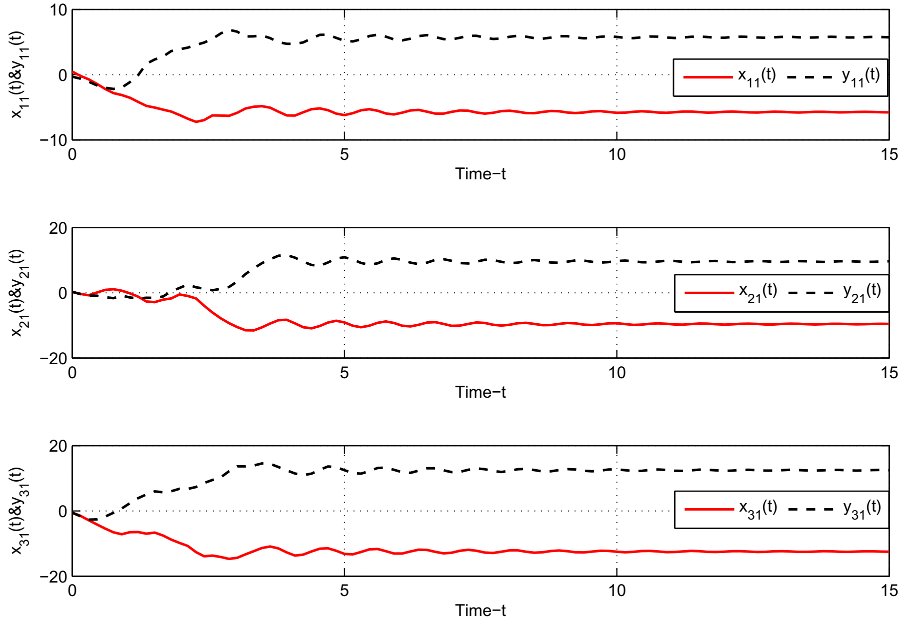

Fig. 13. The state trajectories of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = -1$, $\sigma = 0.6$.

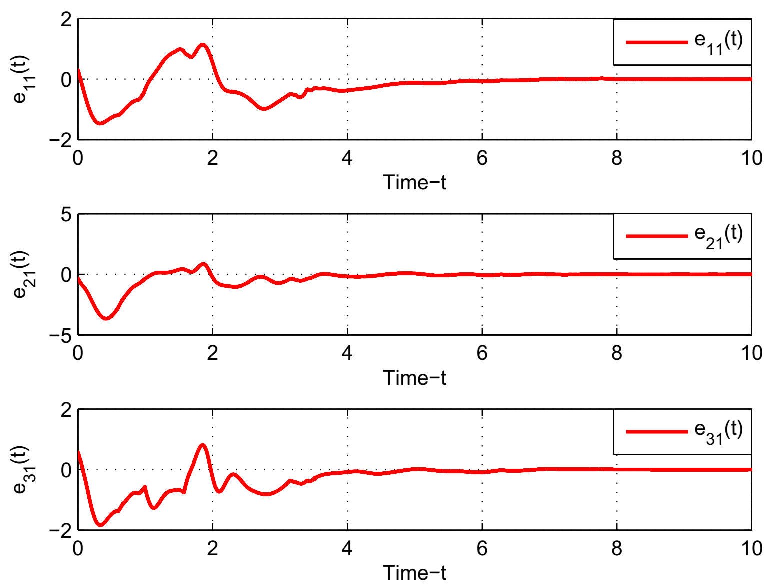

Fig. 14. The synchronization error curves of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = \sin(t)$, $\sigma = 0.6$.

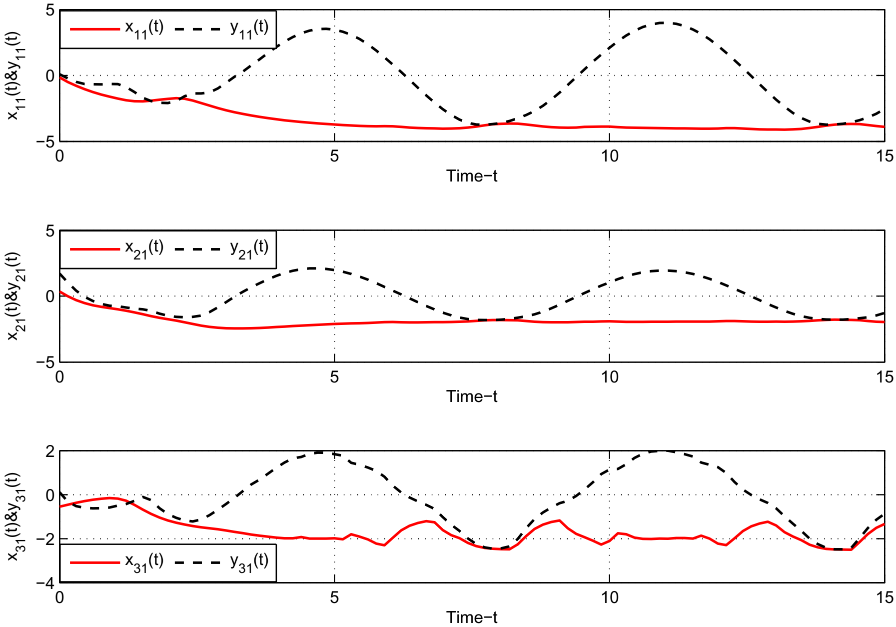

Fig. 15. The state trajectories of $x_{ki}(t)$, $y_{ki}(t)$ when $\alpha(t) = \sin(t)$, $\sigma = 0$.

5. Conclusion

This paper mainly considered the exponential lag function projective synchronization of MMAMNNs with mixed time-varying delays. With the assistance of the theories of the set-valued mapping, the mathematical model of memristor, differential inclusions, we proposed the definition of exponential lag function projective synchronization. In addition, according to the characteristic of the memristor, we also discussed the effect of parameter mismatched between the drive and response systems. Based on the results, we designed two kinds of hybrid controllers, they are more suitable and practicable than the traditional controllers to the MMAMNNs. The effectiveness of the proposed approach had been illustrated by numerical simulation.

The future work mainly includes the following aspects: (i) MMAMNNs is a new and challenging topic, looking for more complex and practical memristive associative memory model is our further work. Since the MMAMNNs can be treated as a discontinuous switched system, it is necessary to employ a more preferable mathematical method to study. (ii) How to apply the results in practice, such as the associative memory of brain-like, mass storage,[63] machine learning, and so on. In summary, the memristive associative memory neural networks still have a lot of problems worthy of further study.

Acknowledgments

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