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Detecting early-warning signals in periodically forced systems with noise

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Early-warning signals for imminent regime shifts in multi-stable systems are highly desirable because it is often difficult to revert a system to the previous state once a transition has occurred. In this paper, two indicators, the phase lag and amplitude difference of the system's response, are extended to detect early-warning signals of a periodically driven, bistable complex system with noise. Our results show that both indicators can announce a regime shift of a complex system with small noise, namely, the critical point of the regime shift near a bifurcation point of the corresponding deterministic system. However, they fail to early indicate the regime shift in the case of large noise where the shift is far from the original bifurcation point. Based on the moment-expanding scheme, we reduce a large noise to a small one, and then both indicators work well again. We illustrate this approach via a parameterized lake eutrophication model verified by data. The regime shift to eutrophication could be detected in advance by studying the phase lag and amplitude difference of phosphorus concentrations. A basic statistical test is performed for the robustness of the proposed indicators. This approach provides a theoretical basis to prevent ecological environment deteriorations. *Published by AIP Publishing.* https://doi.org/10.1063/1.5012129

Multi-stable dynamical systems are often under the influence of uncertain disturbances, which may have a delicate or even profound effect on regime shifts. Early-warning signals of regime shifts have been mostly considered under small noise. However, a considerable noise strength cannot be avoided often due to the complex structure of the environment. In this paper, early-warning signals of a regime shift under different noise levels are detected by two indicators, phase lag and amplitude difference, which are verified via a basic statistical hypothesis test. Taking a lake eutrophication model as an example, two indicators are generalized to get a much earlier warning regime shift induced even by large noise.

INTRODUCTION

Regime shifts between alternative stable states have been observed in multi-stability dynamical systems[1–3] and they occur when a threshold is crossed, often without strong pre-indicators.[4–8] These drastic, sudden, and often irreversible changes can cause significant losses that affect human economies and societies, e.g., systemic market crashes in global finance,[9,10] abrupt shifts of ocean circulation, or climate in the Earth system[11,12] and catastrophic shifts of rangelands, and fish populations or wildlife populations in an ecosystem.[4,13,14] Finding signals that announce regime shifts is therefore a challenging issue for complex systems science in theory and practice.

Avoiding unintentional regime shifts is widely regarded as a significant part of nonlinear dynamics in virtue of rather small changes shown before a regime shift is impending. Moreover, models of dynamical systems, ranging from ecosystems to financial markets and climate, are usually not accurate enough to reliably predict when such regime shifts may occur. Interestingly, it now appears that certain generic symptoms may occur in a wide class of dynamical systems as they approach a regime shift and these early-warning signals can be measured in advance of regime shifts,[5] including changing autocorrelation,[15] variance,[16] skewness,[17] conditional heteroscedasticity,[18] spectral ratio,[19] and others.[20] Various indicators have been successfully provided early warnings to paleoclimate time series,[21] Indian summer monsoon,[22] lab experiments on plankton,[23,24] and a whole-lake food web experiment.[25] The subsequent interference of suitable management strategies, in some cases, has reduced the incidence of surprising regime shifts.[26] However, these indicators are only investigated in systems with noise, while the effect of periodic forcing on these systems has almost not been considered. In real dynamical systems, the influence from environmental periodic change often cannot be ignored.[27–29] It is important to identify early-warning signals of periodically forced systems with noise. Furthermore, it is a common problem that earlier regime shifts induced by strong fluctuations has not been considered in existing indicators. For solving those problems, we intend to introduce early-warning indicators of regime shifts through analyzing the change of the solution of a system with periodic force and noise. It is known that the phase and amplitude of the system's response can be often found in periodically forced systems.[30,31] The limitation for varying phase and amplitude to announce regime

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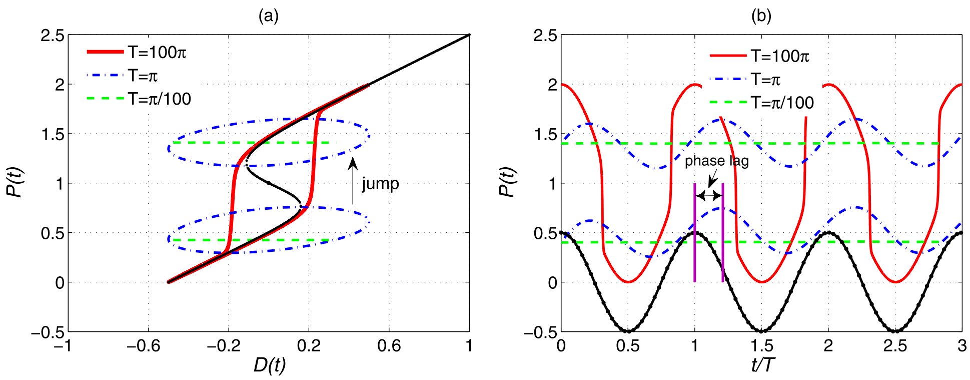

FIG. 1. The dynamics of system (1) for three different values of T and fixed $\alpha = \frac{1}{2}$ and $\sigma = 0$. (a) $P(t)$ is plotted against $D(t)$, and the black line is the equilibrium phosphorus concentration. (b) $P(t)$ is plotted against t/T and the black line is $D(t)$.

shifts with large noise has been discussed in Ref. 32. In the present paper, the phase lag and amplitude difference will be extended to announce regime shifts induced by different noise levels. We do not consider situations where the noise is large enough to destroy the structure of bistability. Here, a small noise means that the critical point of the regime shift is near a bifurcation point of the corresponding deterministic system and the shift can be announced based on the system's response and the forcing. A large noise means that the shift takes place earlier than the original bifurcation point and the obvious relation between the system's response and the forcing disappears in this case.

In this paper, two early-warning indicators, namely, the phase lag and amplitude difference, are introduced to detect regime shifts in periodic, noise-induced complex systems, especially the case of large noise. We address this problem via an ecological model of the lake eutrophication that has been verified by data.[33] This model includes both periodical recycling and random disturbance of phosphorus associated with the regime shift. Based on the linear dynamics approximation,[32] we first analyze the approximate solution of this system only with a periodic force. Our results show that there are some differences in phase and amplitude between the system's response and the periodic forcing. Then, we calculate the phase lag and amplitude difference of the system's response when the critical point is near the bifurcation point of the deterministic system. We find that there is a significant rising of the phase lag and amplitude difference before the critical point. However, like existing indicators, the phase lag and amplitude difference fail to indicate the earlier regime shift induced by large noise, because the difference of the phase and amplitude of the response against the forcing vanishes due to strong random fluctuations. Then, a scheme of the moment expansion[34–36] is applied to reduce the noise levels, and both indicators work again. Finally, a detailed examination about the increase in the phase lag and amplitude difference as a critical transition is given. Note that the rising phase lag and amplitude difference are detected by studying an empirical model of eutrophication, and did not require detailed knowledge of the actual ecosystem dynamics.

MODEL

We investigate early-warning signals of a periodically forced dynamical system with noise by taking the paradigmatic lake eutrophication model as a case study.[33,37,38] This model represents key features of a broad class of regime shifts and has been extensively used to study early-warning signals and the potential for abrupt transitions. A dynamical model subjected to fluctuations in periodical recycling as well as in random inputting of phosphorus associated with regime shifts is defined as follows:

$$\frac{dP(t)}{dt} = \alpha - sP + r\frac{P^n}{P^n + 1^n} + D(t) + \eta(t), \tag{1}$$

where $P(t)$ is the phosphorus concentration, α is the phosphorus input rate (control parameter), r represents the maximum recycling rate ($r = 1$), s is the phosphorus loss rate ($s = 1$), n is the exponent that describes the recycling relationship to phosphorus concentration ($n = 8$), and t denotes time. The periodic forcing function $D(t)$ is given by $D(t) = D_a\cos(\omega t)$, $D_a = 1/2$ is a constant, $\omega = \frac{2\pi}{T}$ is the angular frequency, and T is the period of the forcing. $\eta(t)$ is a zero-mean Gaussian white noise with intensity σ.

The dynamics of model (1) with three different T are shown in Fig. 1. The red line is for $T = 100\pi$, the result indicates that the periodic force has a small effect on system (1) and almost all dynamics comes from $\alpha - sP + r\frac{P^n}{P^n+1^n}$. In the presence of $T = \pi/100$ (green line), there are also two stable attractors and the dynamics induced by the forcing dominate the behaviors of system (1). The blue line is the intermediate regime, $T = \pi$, the phase of system (1) lags the periodic forcing, and now the dynamics of system (1) can be represented by a linear combination of cosines when we take an intermediate regime. Hence, a relationship between the system's response and the forcing can be established based on Fourier series. Since the dominance in Fourier series is the linear part, the linear dynamics approximation is firstly used for analyzing the approximate dynamics of system (1). Then, the potential of this method is verified below.

With $\sigma = 0$, Eq. (1) reduces to a deterministic case

$$\frac{dP(t)}{dt} = \alpha - sP + r\frac{P^n}{P^n + m^n} + D(t) = f(P). \tag{2}$$

We now Taylor expand $f(P)$ to the first order around $\bar{P}$, where $\bar{P} = \frac{1}{T}\int_{t_s}^{T+t_s} P(t)dt$ is the mean of the steady state of system (2). Based on the method of the constant variation and the linear dynamics approximation,[32] we find that for large t system (2) will be settled into the orbit

$$P(t) = b\tau + \frac{D_a\tau}{\sqrt{1+\omega^2\tau^2}}\cos(\omega t + \varphi), \tag{3}$$

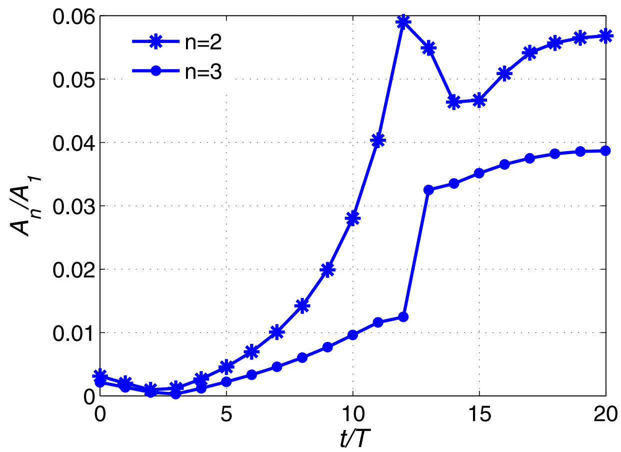

FIG. 2. The ratio of the second ($n = 2$) and third ($n = 3$) harmonic amplitudes to the fundamental harmonic amplitude A_n/A_1.

where $b = f(\bar{P}) - \frac{\partial f}{\partial P}|_{P=\bar{P}}\bar{P}, \tau = -1/\frac{\partial f}{\partial P}|_{P=\bar{P}}$ and $\varphi = \arctan(\omega\tau)$.

Equation (3) further shows that the phase of system (2) lags $D(t)$ and the amplitude of system (2) is amplified by the factor $\frac{\tau}{\sqrt{1+\omega^2\tau^2}}$ corresponding to $D(t)$.

As described next, the potential of the linear dynamics approximation will be tested by writing the state variable as a Fourier series.

Through the Fourier series, the full nonlinear response $P(t)$ of system (2) can be written as

$$P(t) = \sum_{n=0}^{N} A_n \cos(\omega_n t + \phi_n),$$

where N represents the total number of sinusoidal functions, $\omega_n = \frac{2\pi n}{T}$ are angular frequencies, A_n are amplitudes, and ϕ_n denotes phases. It is known that the 2nd ($n = 2$) and 3rd order ($n = 3$) are the largest harmonics compared with the linear response A_1. Therefore, the ratios A_n/A_1 ($n = 2, 3$) will play a key role to quantify how large these nonlinear effects are.

In Fig. 2, the ratios of the second (dotted line) and third (solid line) harmonic amplitudes to the fundamental harmonic amplitude are shown. Since the ratios $A_n/A_1 < 10^{-1}$,[32] the linear approximation can effectively help us to understand the approximate dynamics of system (2) and to obtain some obvious characteristics when a regime shift is impending.

According to the above analysis, we find that the phase of the system's response obviously lags $D(t)$. In addition to an amplification of the amplitude of system (1), from the equilibrium concentration in Fig. 1, a jump appears when the state transition is impending. In what follows, we will use the phase lag and amplitude difference to identify early-warning signals for regime shifts in system (1).

For the deterministic system (2), the phase lag at a certain α is taken as the average of the summation of the phase lag in M cycles, as follows:

$$\text{phase lag} = \frac{\sum_{i=1}^{M}[\arg\max\{P(t), t \in [(i-1) \times T, i \times T]\} - 2\pi \times i] \times \Delta t}{M}, \tag{4}$$

where $t \in [0, M \times T]$ is the length of the time series of $P(t)$, M is the total number of cycles of a realization, argmax denotes the abscissa at the maximum of $P(t)$, and Δt represents the time step.

Similarly, the definition of the amplitude difference is

$$\text{amplitude difference} = \frac{\sum_{i=1}^{M}[\max(P(t), t \in [(i-1) \times T, i \times T]) - \min(\langle P(t)\rangle, t \in [(i-1) \times T, i \times T])]}{M}. \tag{5}$$

Here, we take $M = 20$ so that it can assure the accuracy of the result.

For the stochastic system (1), the definitions of the phase lag and amplitude difference can be written as

$$\text{phase lag} = \frac{\sum_{j=1}^{N}\left[\frac{\sum_{i=1}^{M}[\arg\max(\langle P(t)\rangle, t \in [(i-1) \times T, i \times T]) - 2\pi \times i] \times \Delta t}{M}\right]}{N} \tag{6}$$

and

$$\text{amplitude difference} = \frac{\sum_{j=1}^{N}\left[\frac{\sum_{i=1}^{M}[\max(\langle P(t)\rangle, t\in[(i-1)\times T, i\times T]) - \min(\langle P(t)\rangle, t\in[(i-1)\times T, i\times T])]}{M}\right]}{N}, \quad (7)$$

where $\langle\cdot\rangle$ is the operator for calculating the mean of the random variable $P(t)$ in the presence of noise and N means that the Eqs. (4) or (5) is calculated N times and then averaged for an accurate result. Here, we take $N = 100$ so that it can assure the accuracy of the stochastic result.

RESULTS AND DISCUSSIONS

Two cases will be considered in this paper. Firstly, the deterministic system (2) and the random system (1) with small noise are taken into account, where the proposed indicators can be used directly. But in the presence of large noise, an increase cannot be observed for the phase lag and amplitude difference in the vicinity of critical points. Therefore, secondly the moment-expanding scheme is adopted to deal with large noise.

Detecting early-warning signals of regime shift under different noise levels

In this part, we calculate the phase lag and amplitude difference of the system's response to investigate typical properties prior to a regime shift.

The dynamics of system (2) with α are shown in Fig. 3. The results in Figs. 3(a) and 3(b) suggest that the varying concentrations of the total phosphorus show a phenomenon of alternative states and a regime shift occurs around the cycle $\alpha = 0.570$. Note that some phase lag and varying amplitude of the periodic system against the forcing $D(t)$ are shown more clearly. In Figs. 3(c) and 3(d), the amplitude difference and phase lag of the system's response are plotted with α. Remarkable differences are evident in the proximity of a regime shift of these two indicators. Such an increase in variability should be a general feature of lakes before a critical point of regime shift to eutrophication. Therefore, the phase lag and amplitude difference are useful early-warning signals of a regime shift under a periodically driven, bistable system.

In real dynamical systems, environmental fluctuations (e.g. nutrient input and rainfall) are usually uncertain and unavoidable. Noise is an important factor that should be considered in modeling. We have found that the phase lag and amplitude difference are two effective indicators for detecting early-warning signals of periodic bistable systems in the above sections. However, these two indicators will work or not when we take environmental perturbations into consideration. The robustness of them will be verified below.

Figure 4 shows the dynamics of system (1) with α and different noise intensities. For a relatively small intensity $\sigma = 0.0001$, the response of system (1) in Figs. 4(a) and 4(b) is accompanied with slight fluctuations against the periodic system in Figs. 3(a) and 3(b) and the equilibrium concentration of the random system (1) can hardly be distinguished from the deterministic system (2). It means that the regime shift of the random system (1) also occurs around $\alpha = 0.570$. In the presence of $\sigma = 0.001$, an earlier transition occurs around $\alpha = 0.563$. Adopting the same method as above, the phase lag and amplitude difference are plotted in Figs. 4(c), 4(d), 4(g), and 4(h), respectively. Our results show that an increase occurs before the critical points, both indicators can act as early-warning signals of shifts in these two cases. For a larger noise intensity $\sigma = 0.01$, however, the phase lag between the system's response and the forcing disappears and the amplitude varies with larger fluctuations. Meanwhile, a much earlier regime shift takes place around $\alpha = 0.532$ [Figs. 4(i) and 4(j)]. Furthermore, the phase lag and amplitude difference are calculated using the same method as shown in Figs. 4(k) and 4(l). Now, these two indicators cannot reflect and announce an earlier regime shift. To make two indicators more general, we need to overcome this limitation.

Detecting early-warning signals of regime shift under large noise

In order to extend both indicators successful to more general cases and in particular realize early-warning signals detection under large noise, we need to reduce the level of the noise substantially through some methods. Generally, a stochastic system with large noise can be approximated by a second order moment system with small noise. We can use the moment-expanding scheme to transform the original data (state-variables) with large noise into new data (distribution-variables) but with small noise.[36] However, it is noteworthy that here we do not consider the situation that the noise is large enough to destroy the structure of bistability. Next, we will take $\sigma = 0.01$ as an example of large noise to identify early-warning signals using both indicators again.

Based on this approximation of the moment expansion, system (1) with large noise can be expressed by a second order moment system with small noise. Since the moment expansion is based on the polynomial form of the deterministic part, two important points need to be emphasized in transforming the stochastic system (1): (i) $D(t)$ is a constant when solving

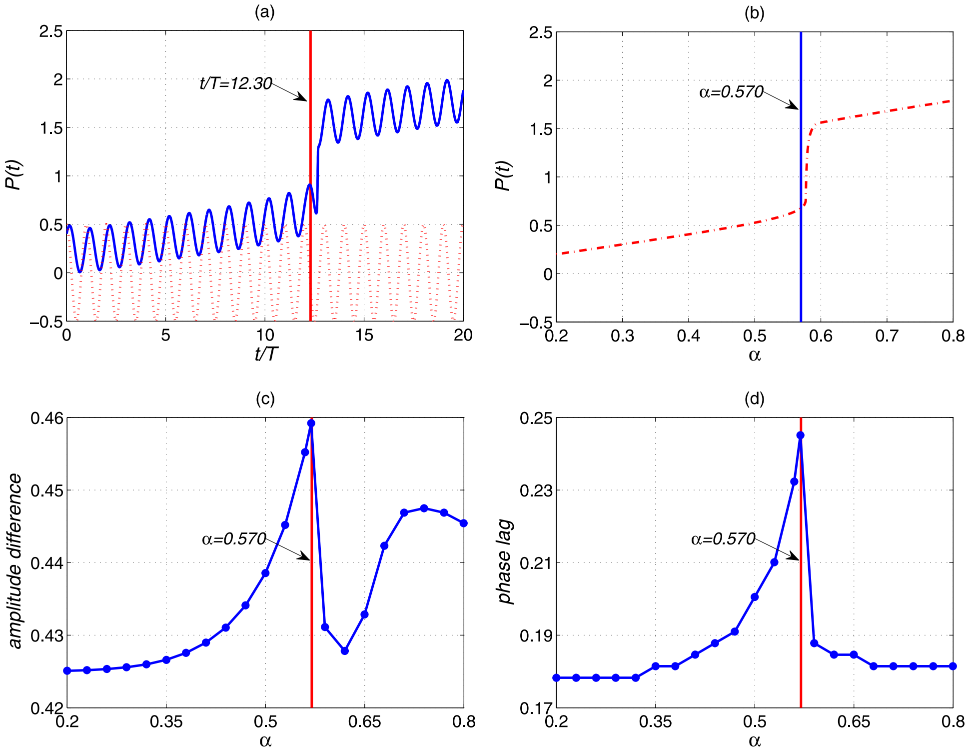

FIG. 3. The dynamics of system (2) with $T = \pi$ and α. (a) The system's response (solid line) and $D(t)$ (dotted line) against cycles t/T. (b) The equilibrium concentration of system (2) plotted against α. The amplitude difference (c) and phase lag (d) are calculated.

the moments of $P(t)$; (ii) Because

$$f(P) = \alpha - sP + r\frac{P^n}{P^n + 1^n} = \alpha - sP + r - r\frac{1}{P^n + 1^n}, s = r = 1, n = 8$$

is not a polynomial function, we can expand $\frac{1}{P^8+1}$ with respect to P about point $P = 1$ in a Taylor series given below.

$$\frac{1}{P^8+1} = f(1) + \frac{f'(1)}{1!}(P-1) + \frac{f''(1)}{2!}(P-1)^2 + \frac{f'''(1)}{3!}(P-1)^3 + \cdots \approx -\frac{13}{2} + 26P - 29P^2 + 10P^3.$$

Let $E(P)$ represent the expectation of P. Through the process of moment closure, i.e.,

$$E(P^3) = 3m_1m_2 + m_1^3,$$
$$E(P^4) = 3m_2^2 + 6m_1^2m_2 + m_1^4,$$

we get the 2-dimensional moment equation of system (1) under large noise as follows:

$$\begin{aligned}\frac{dm_1(t)}{dt} &= \alpha + \frac{15}{2} - 27m_1(t) + 29m_2(t) + 29{m_1}^2(t) \\ &\quad - 30m_1(t)m_2(t) - 10m_1(t)^3 + D(t) + \xi_1(t), \\ \frac{dm_2(t)}{dt} &= -54m_2(t) + 116m_1(t)m_2(t) - 60{m_2}^2(t) \\ &\quad - 60{m_1}^2(t)m_2(t) + \sigma + \xi_2(t),\end{aligned} \tag{8}$$

where m_1 and m_2 are the first and second order central moment, respectively. σ is the intensity of Gaussian white noise $\eta(t)$. $\xi_1(t)$ and $\xi_2(t)$ are small noises transformed from large noise through the moment-expanding scheme. They are within the range of noise that does not induce a regime shift earlier. Clearly, the first order central moment $m_1(t)$ approximately represents the response of the original state P in system (1). Therefore, in the following, the phase lag and amplitude difference will be calculated based on the response of $m_1(t)$ in system (8).

In Fig. 5, the dynamics of system (8) with α are plotted. Obviously, as shown in Figs. 5(a) and 5(b), the noise levels of $m_1(t)$ and $m_2(t)$ are clearly smaller than those of the original in Fig. 4 and a regime shift of system (8) occurs around $\alpha = 0.533$. It almost equals to the critical point of system (1) under large noise shown in Figs. 4(i) and 4(j). Thus, for this moment system, the phase lag and amplitude difference are sensitive to the regime shift again due to small noise as shown in Figs. 5(c) and 5(d). Both indicators work again on the new data in system (8).

Based on the moment expansion, the difference between the critical points of regime shifts of the reconstructed high-dimensional system and system (1) driven by the large noise ($\sigma = 0.01$) is within the error range which is less than 10^{-3}. Therefore, detecting early-warning signals of system (1) under large noise can be transformed to detecting the reconstructed high-dimensional system under small noise. Here, the phase lag and amplitude difference can be used to detect early-warning signals in this higher-dimensional periodically forced system.

Verification of the increase for two indicators

It is known that the phase lag and amplitude difference are established by analyzing the dynamic phenomenon and relationship between the system's response and the periodic forcing. Therefore, for quantifying prediction success of these

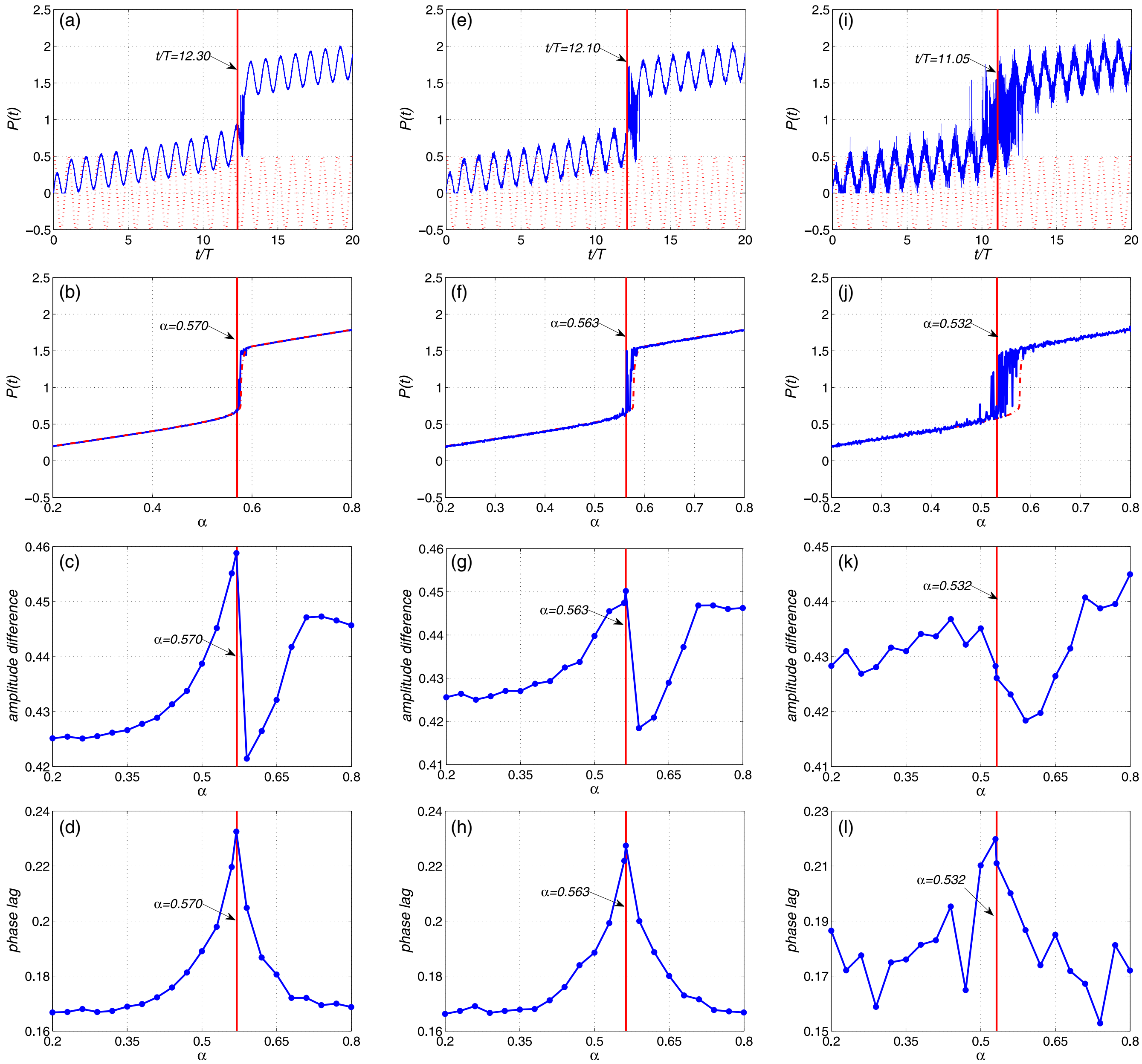

FIG. 4. The dynamics of system (1) with $\sigma = 0.0001$ [(a)–(d)], $\sigma = 0.001$ [(e)–(h)], $\sigma = 0.01$ [(i)–(l)], and α. [(a), (e), and (i)] The system's response (solid line) and $D(t)$ (dotted line) against t/T. [(b), (f), and (j)] The equilibrium concentration (blue line) of Eq. (1) plotted against α, the red line is the equilibrium concentration of Eq. (1) without noise. The early-warning indicators of the amplitude difference [(c), (g), and (k)], and phase lag [(d), (h), and (l)] are calculated, respectively.

two indicators, a basic statistical hypothesis test to test the existence of the increase is given.

Through Eq. (3), we get the following results:

1. The response of system (2) lags the $D(t)$ by
$$\varphi = \arctan(\omega\tau). \tag{9}$$
2. The amplitude of system (2) is amplified corresponding to $D(t)$ by
$$C = \frac{D_a\tau}{\sqrt{1+\omega^2\tau^2}}. \tag{10}$$

where $\tau = -1/\frac{\partial f}{\partial P}\big|_{P=\bar{P}}$ changes with the mean state $\overline{P}$.

By the derivatives of Eqs. (9) and (10) with respect to τ, we obtain
$$\frac{d\varphi}{d\tau} = \frac{\omega}{1+\omega^2\tau^2} > 0$$
and
$$\frac{dC}{d\tau} = \frac{D_a}{\sqrt{1+\omega^2\tau^2}} - \frac{D_a}{\sqrt{1+\omega^2\tau^2}} \times \frac{\omega^2\tau^2}{1+\omega^2\tau^2} > 0.$$
Then, Eqs. (9) and (10) are monotonically increasing versus the variable τ. Furthermore, the amplitude difference and the phase lag of system (2) with respect to $D(t)$ increase gradually, since τ is the e folding time which increases by the Euler number e.[32]

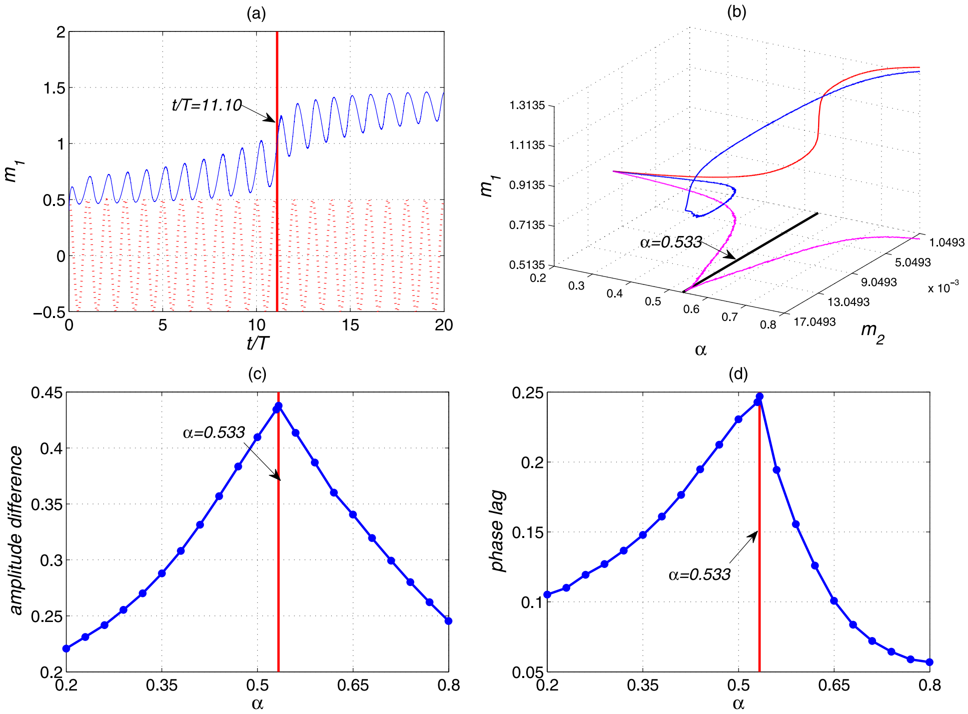

FIG. 5. The dynamics of system (8) with α. (a) m_1 (solid line) and $D(t)$ (dotted line) against t/T. (b) The moment-variables of Eq. (8) plotted against α. The amplitude difference (c) and phase lag (d) of Eq. (8) are calculated. It can be seen that two indicators work again.

In Figs. 4(a), 4(b), 4(e), and 4(f), the phase lag can be clearly observed when system (1) is induced by small noise. The amplitude difference of the average response is not affected even when the response is accompanied by small fluctuations. In fact, these phenomena can be found in many periodically driven bistable complex systems with small noise.[30,31] Hence, the phase lag and amplitude difference are robust to identify state transitions under small noise.

For a larger noise intensity, the phase lag becomes less obvious and the system response is accompanied with large fluctuations as shown in Figs. 4(i) and 4(j), i.e., the characteristics of Eqs. (9) and (10) are almost destroyed. Although the moment-expanding scheme is used to transform the original data with large noise into the new data with small noise, the dimension of the system will increase and the approximate expressions of the phase lag and amplitude also cannot be obtained. Based on the definition of the monotonically increasing functions, an idea about the statistical hypothesis procedure is constructed to test the robustness of both indicators.

Let $x_1, x_2, \cdots x_m, \cdots x_n, m < n$ be a time series of the phase lag obtained from Eqs. (4) or (6), and x_m is the value of the corresponding critical transition and n is the length of the series. Then, the indicator function I is defined as

$$I_i := \begin{cases} 1 & \textit{if } x_{i+1} - x_i > 0, \\ 0 & \textit{if } x_{i+1} - x_i \le 0, \end{cases} \quad i = 1, 2, \cdots m-1.$$

Here, we take $i < m-1$ to detect an increase of the phase lag when a state transition is imminent.

The sum that satisfies $x_{i+1} - x_i > 0$ is

$$k = \sum_{i=1}^{m-1} I_i.$$

Therefore, the probability of an increase in the time series of the phase lag is given as

$$p = \frac{k}{m-1}.$$

Now, we consider the following Null-hypothesis: there is no increase in the phase lag when a transition is impending. If this hypothesis is rejected, $1 - p < z$ needs to be satisfied, where z is a critical probability value, which is generally set as $z = 5\%$ in the hypothesis test. By comparing the value of $1 - p$ with the given value 5%, the robustness of the phase lag can be tested. However, it is remarkable that small enough time steps and enough sample points are needed to make p more accurate.

Here, we calculate the phase lag of system (8) and reject the hypothesis through the introduced method. The shift occurring at $t/T = 11.10$ of system (8) can be obtained numerically firstly. Then, we take the time step as $\Delta t = 0.001$ in $t \in [0, 20\pi]$ and a series $x_0, x_1, \cdots, x_{34872}, \cdots, x_{62831}$ of the phase lag is obtained from Eq. (6), where $m = 34872, n = 62831$. The probability of an increase near x_{34872} is

$$p = \frac{k = \sum_{i=1}^{34871} I_i}{34871} = 0.98.$$

Then,

$$1 - p = 2\% < 5\%.$$

The Null-hypothesis is rejected and our result is accepted and plotted in Fig. 5(d). The robustness of the amplitude difference can also be verified by the above mentioned similar steps.

Hence, the phase lag and amplitude difference can be used to detect early warning signals of state transitions in further periodically forced bistable systems with different noise levels.

CONCLUSIONS

In the present paper, regime shifts in periodically forced systems with different noise levels are identified via the paradigmatic lake eutrophication model as a case study. When a regime shift of a periodically forced complex system with small noise intensities occurs, the early-warning signals can be detected by the phase lag and amplitude difference of the system's response. However, when the system is perturbed by large noise, the phase and the amplitude of the system's response varies with larger fluctuations and the shift occurs earlier than the original bifurcation point. In this case, both indicators are invalid. To overcome this limitation, the moment-expanding scheme is used to transform the original data with large noise into new data but with small noise and both indicators are extended. Furthermore, a basic statistical test is performed for the robustness of the proposed indicators. Therefore, taking an empirical lake model as an example, the rising phase lag and amplitude difference when a state transition is imminent may provide an early warning in ecosystem management. The data-based forecast of regime shifts in practical systems and the perfect statistical analysis of prediction success will be carried out in our further works.

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