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Synchronization of mobile chaotic oscillator networks

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We study synchronization of systems in which agents holding chaotic oscillators move in a two-dimensional plane and interact with nearby ones forming a time dependent network. Due to the uncertainty in observing other agents' states, we assume that the interaction contains a certain amount of noise that turns out to be relevant for chaotic dynamics. We find that a synchronization transition takes place by changing a control parameter. But this transition depends on the relative dynamic scale of motion and interaction. When the topology change is slow, we observe an intermittent switching between laminar and burst states close to the transition due to small noise. This novel type of synchronization transition and intermittency can happen even when complete synchronization is linearly stable in the absence of noise. We show that the linear stability of the synchronized state is not a sufficient condition for its stability due to strong fluctuations of the transverse Lyapunov exponent associated with a slow network topology change. Since this effect can be observed within the linearized dynamics, we can expect such an effect in the temporal networks with noisy chaotic oscillators, irrespective of the details of the oscillator dynamics. When the topology change is fast, a linearized approximation describes well the dynamics towards synchrony. These results imply that the fluctuations of the finite-time transverse Lyapunov exponent should also be taken into account to estimate synchronization of the mobile contact networks. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4962129]

Synchronization in time evolving networks has drawn much attention recently. Especially, mobile contact networks, in which the nodes move around and the network topology of interaction changes in time, are quite important for its wide range of potential applicability to chemotaxis, mobile communications, groups of animals, etc. Despite the importance of this topic, only some research has been performed so far. One important case that is needed to study is where the system contains irregularity in the internal dynamics. Here, we study the effect of noise on chaotic synchronization in the mobile contact networks. We find a transition between chaotic synchronization and desynchronized states for any motion rate. However, when the topology change is slow, we observe a switching between the quasi-synchronized (laminar) and desynchronized (burst) states close to the transition. We uncover that this switching is caused by large fluctuations of the transverse Lyapunov exponent due to a slow topology change. Our result suggests that small noise is already important because of these large fluctuations, and this effect has to be taken into account for constructing an efficient and robust mobile contact network. On the other hand, in the case of fast motion of agents, the fluctuations are relatively small, and the Lyapunov exponent asymptotically converges to that of the fast switching approximation (FSA).

I. INTRODUCTION

As already pointed out in Ref. 1, evolving complex networks, i.e., the ones whose topology changes in time, are one of the currently most important extensions of complex networks studies.[2] Various examples of such evolving networks can be found in person-to-person communication networks,[3–5] phone calls,[6] consensus problems,[7] and functional brain networks.[8] Among those evolving networks, there exists a certain class in which nodes can move around as agents and interact with the nearby ones. These networks have been called in the literature as moving neighborhood networks.[9] This particular type of time-dependent network, in which links are established by proximity, has been used in models of agents in search of consensus,[10] spreading dynamics,[11] motions of robots,[12] vehicles,[13] and groups of animals,[11,14] in which cooperative dynamics[12] emerges.

Among various possible interaction dynamics between agents, synchronization[15] is a fundamental type of emergence. We assume here that each agent has an oscillator, and that an interaction between nearby oscillators takes place. Although synchronization in the complex networks has been

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intensively studied mainly so far with fixed topology,[16,17] many examples where synchronization of mobile elements plays a crucial role have been reported recently, e.g., chemotaxis,[18] mobile *ad hoc* networks,[19] wireless sensor networks,[7] the expression of segmentation clock genes,[20,21] and the acoustic communication of frogs.[22]

Theoretically, the analysis of time dependent network topology is quite difficult and at the same time very challenging, and only few results have been reported so far. To our knowledge, the first attempt of a detailed analytical treatment was done by Skufca and Bollt,[9] who introduced the concept of the moving neighborhood networks and proposed the stability analysis based on the eigenvalues of the moving time average of the Laplacian matrices. One simplifying case is when the network topology changes faster than the time scale of oscillator dynamics.[23–26] The sufficient condition for the fast switching approximation has been proven for chaotic synchronization on time-dependent small world networks,[23] deterministically time-dependent networks,[26] and interacting random walkers on networks.[25] Another one is the analysis using the Fokker-Planck equation formalism when the population of agents is dense and arranged in a ring.[27]

In our recent studies,[28,29] we have introduced a conceptual model for synchronization of non-chaotic mobile oscillator networks where agents perform random walks in a two-dimensional (2D) plane. We have found two different mechanisms leading to synchronization, namely, local and global synchronization, depending on the parameter values. Our results suggest that the interplay between instantaneous topology, agent motion, and interaction rules plays an important role for synchronization of the mobile contact networks. This framework has been recently applied to integrate and fire oscillators.[30,31] In these studies, the internal dynamics is assumed to be regular and deterministic, neglecting noise which appears for many reasons, e.g., thermal fluctuation and error in observation of other agents.

In real systems, interaction is usually rather noisy and it can affect the global dynamics. However, such noise is not expected to produce a drastic effect when the dynamics of the units is regular. The situation changes completely when the internal dynamics has irregular properties, i.e., it has a chaotic behavior. In the general framework of synchronization, it is well known that synchronized behaviors are possible even if the internal dynamics is chaotic.[32,33] Therefore, it is important to study synchronization dynamics in such irregular systems in order to estimate the performance of a mobile network. Here, we analyze synchronization phenomena of the chaotic mobile oscillators with small random noise, which is unavoidable in realistic systems.

This paper is organized as follows. In Sec. II, we describe our model of mobile oscillator networks. We report the numerical simulation of the chaotic mobile oscillator network in Sec. III and show the novel type of synchronization transition and intermittent behavior of the synchronization error caused by the small noise in the parameter region where synchronization is linearly stable. In order to explain the intermittency and asymptotic dynamics close to complete synchronization, we introduce linearized equations in Sec. IV, and in Sec. IV A, we describe the conventional averaging which is valid when the topology change is very fast. The numerical results of the linearized equation are compared with those of the nonlinear equation in Sec. IV B, and we show that the linearized equation describes well the asymptotic dynamics near the synchronized regime. Section V is devoted to analyze the linearized dynamics in terms of the product of the instantaneous Laplacian matrices. Our results are summarized in Sec. VI.

II. MODEL

Our model consists of two parts: agents' dynamics (topology) and oscillator dynamics. The dynamics of agents is described by a random walk of N agents moving in a 2D space (size $L \times L$) with periodic boundary conditions. Each agent moves with the same speed v during time intervals of length τ_M. The angle of the ith agent's motion is $\xi_i(t_k) \in [0, 2\pi]$, and it changes randomly at discrete time steps t_k ($t_{k+1} - t_k = \tau_M$). The evolution of the ith agent's position is written as

$$\begin{aligned} x_i(t_k + \Delta t) &= x_i(t_k) + v \cos \xi_i(t_k)\Delta t \mod L, \\ y_i(t_k + \Delta t) &= y_i(t_k) + v \sin \xi_i(t_k)\Delta t \mod L, \end{aligned} \tag{1}$$

where $\Delta t \le \tau_M$. Note that the motion of the agents is diffusive.

We assume that each agent has an internal state $\varphi(t)$ which evolves with a chaotic map $F[\varphi(t)]$. Time update of oscillators takes place at each discrete time step τ_P. We assume that an oscillator i interacts with its nearby agent j with the coupling strength σ if the Euclidian distance $d_{ij}(t)$ between agents i and j is less than a threshold d

$$\varphi_i(t + \tau_P) = F\left[\varphi_i(t)\right] - \frac{\sigma}{N}\sum_{j=1}^{N} L_{ij}(t)F\left[\varphi_j(t)\right] + \zeta_i(t), \tag{2}$$

where $L_{ij}(t) = [k_i(t)\delta_{ij} - c_{ij}(t)]$ is the time dependent Laplacian matrix with $c_{ij}(t) = 1$ for $d_{ij}(t) < d$ ($i \ne j$) and $c_{ij}(t) = 0$ otherwise, and δ_{ij} denotes the Kronecker delta. Here, $k_i(t)$ represents the degree of the ith node, i.e., the number of oscillators that are connected with i. ζ_i is a noise term which has no time correlation and is uniformly distributed in $[0, \zeta_{\max}]$, where $\zeta_{\max} = 10^{-10}$ in the whole paper. Note that the noise is independent for different i. If we neglect the noise term, there exists a complete chaotic synchronized state $\varphi_i(t) = \varphi_0(t)$ for all i, where φ_0 is a chaotic orbit which satisfies $\varphi_0(t + \tau_P) = F[\varphi_0(t)]$ in Eq. (2). Since the noise is very small in our case, we expect that there exists a synchronized state in which the synchronization error, $\Delta\varphi(t) \equiv \sqrt{\sum_j (\varphi_j(t) - \bar{\varphi}(t))^2/N}$, is of the order of the noise intensity. This synchronized state is never observed if it is linearly unstable, but as we see below the linear stability is not a sufficient condition to observe a synchronized state.

In the present paper, we study the asymmetric tent map

$$F(\varphi_i(t)) = \begin{cases} \varphi_i(t)/a & \text{for } 0 \le \varphi_i(t) < a \\ (1 - \varphi_i(t))/(1 - a) & \text{for } a \le \varphi_i(t) \le 1 \end{cases} \tag{3}$$

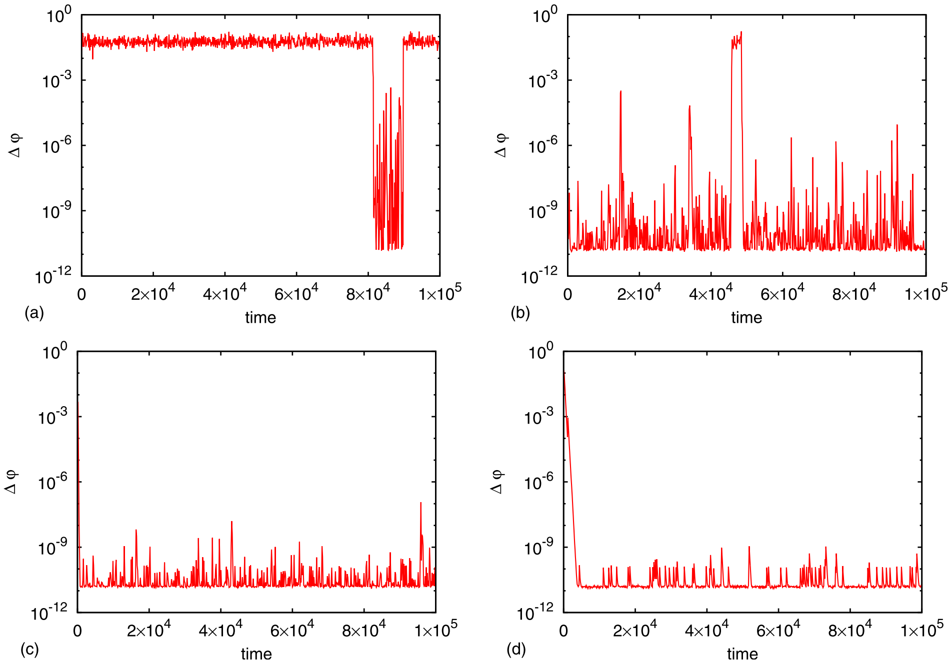

FIG. 1. Time evolution of the synchronization error $\Delta\varphi(t)$ for different τ_P. All results are for $d=32$. (a) $\tau_P=1$, where the system is not synchronized ($\Delta\varphi \sim 10^{-1}$), but sometimes the synchronization error can be very small. (b) $\tau_P=3$. The synchronized state lasts longer than (a). (c) $\tau_P=11$, and (d) $\tau_P=100$. After the initial transient, synchronization is achieved. The noise is not strong enough to break the synchronized state. Note that the FSA (see Sec. IV A) describes the dynamics leading to the synchronization for these cases.

for the internal dynamics. The Lyapunov exponent for the internal dynamics without interaction is analytically achieved as

$$\lambda = [-a\log a - (1-a)\log(1-a)]/\tau_p. \tag{4}$$

For more details about synchronization of the two asymmetric tent maps, see Ref. 15.

In the rest of the paper, we fix the parameter values as $a=0.94$, $L=100$, $\tau_M=1$, $v=5$, and $\sigma=1.7$ and vary τ_P and d, which are the relevant parameters characterizing the interaction.

III. NUMERICAL RESULTS

In this section, we present numerical results obtained by solving Eqs. (1) and (2). Figure 1 shows the time evolution of the overall synchronization error $\Delta\varphi(t)$ for different τ_P with $d=32$, where $\bar{\varphi}(t)=\sum_j \varphi_j(t)/N$. If φ_i distributes uniformly in $[0,1]$, we get $\Delta\varphi=(2\sqrt{3})^{-1}\approx 0.29$. Thus, $\Delta\varphi \sim \mathcal{O}(10^{-1})$ implies that the system is not synchronized. For $\tau_P=1$ (Fig. 1(a)), the system is not synchronized in most of the simulation time, but sometimes the synchronization error takes a small value which implies that all oscillators are almost synchronized. Such a switch between the quasi-synchronized (laminar) and non-synchronized (burst) states is observed typically close to a chaotic synchronization transition point (on-off intermittency),[34] because the chaotic oscillation stays near the synchronized state for a long time, but the mechanism observed here is different. This point will be discussed in Section IV. As τ_P is increased (Fig. 1(b)), we observe longer durations of the synchronized state. For larger τ_P, $\tau_P=11$ and $\tau_P=100$, the synchronization error decays exponentially from the initial stage and keeps the synchronized state. Therefore, we can conclude that the synchronization transition takes place in the τ_P region $3<\tau_P<11$.

Figure 2(a) plots $\langle\Delta\varphi\rangle$, the long-time average of $\Delta\varphi$ averaged over different 124 initial conditions of randomly distributed $\varphi_i(0)$ and position of agents. The result is plotted in the parameter regions where $\langle\Delta\varphi\rangle > 10^{-3}$ is satisfied after some initial transient. If this condition is satisfied, we regard that the system is not synchronized. This figure suggests that $\langle\Delta\varphi\rangle$ depends mainly on d in the non-synchronized region, and its dependence on τ_P is much smaller.

For the synchronized state, defined as the state in which $\langle\Delta\varphi\rangle < 10^{-3}$ is satisfied, a fast reaching of synchronization is often required for the application to real systems.[19,20,35,36] We plot the synchronization time T_s, i.e., the time at which $\Delta\varphi(T_s) < 10^{-3}$ is achieved for the first time (Fig. 2(b)), but eventually it can exit this regime. This result suggests that for a fixed d the synchronization time is larger for larger τ_P, whereas the system can be desynchronized for smaller τ_P. Therefore, there exists an optimal value of τ_P, where the fastest synchronization is obtained, in an intermediate region of τ_P.

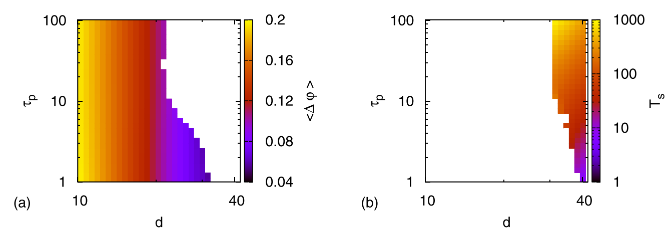

FIG. 2. Dependence on d and τ_P of (a) the average synchronization error $\langle\Delta\varphi\rangle$ in the non-synchronized regime and (b) the average synchronization time in the synchronized regime. We define those regions if all realizations of 124 different initial conditions settle down to the same state, and we did not plot in the intermediate parameter region of the bistable regime, where both the synchronized and non-synchronized states are observed.

IV. LINEARIZED EQUATION

Next, we study the linearization approximation in order to study the stability of the synchronized state. We set $\varphi_i(t) = \varphi_0(t) + \delta\varphi_i(t)$. If $\delta\varphi_i$ is small enough, we can linearize the evolution equation (2) with the Jacobian of $F[\varphi_0(t)]$, $\partial F[\varphi_0(t)]$, around the synchronized state $\varphi_0(t)$ as

$$\delta\varphi_i(t+\tau_P) = \partial F\left[\varphi_0(t)\right]\left[\delta\varphi_i(t) - \frac{\sigma}{N}\sum_{j=1}^{N} L_{ij}(t)\delta\varphi_j(t)\right], \tag{5}$$

or

$$\delta\varphi_i(t+\tau_P) = \begin{cases} \left[\delta\varphi_i(t) - \dfrac{\sigma}{N}\displaystyle\sum_{j=1}^{N} L_{ij}(t)\delta\varphi_j\right]\Big/ a & \text{for } 0 \le \varphi_0(t) < a \\ \left[-\delta\varphi_i(t) + \dfrac{\sigma}{N}\displaystyle\sum_{j=1}^{N} L_{ij}(t)\delta\varphi_j\right]\Big/ (1-a) & \text{for } a \le \varphi_0(t) \le 1. \end{cases} \tag{6}$$

Due to the zero row sum property of the time dependent Laplacian matrix L_{ij}, it always has at least one zero eigenvalue. $\delta\varphi_i$ parallel to the zero eigenvector corresponds to the perturbation along the synchronization manifold. The linear stability of the chaotic complete synchronization is estimated by the transverse Lyapunov exponents Λ for the non-zero eigenvalues of L_{ij} which are perpendicular to the synchronization manifold. The linear stability condition for the chaotic complete synchronization state is that all transverse Lyapunov exponents are negative.[32,37]

A. Fast switching approximation

Based on the linearized equation (5), we can apply the fast switching approximation (FSA).[23–26] It is valid under the following assumption: When the time scale of network variations is much shorter than that of the interaction dynamics, the instantaneous Laplacian matrix is replaced by a time averaged matrix whose elements are just the probability for a given link to exist, instead of 1's and 0's corresponding to real connections. This happens for large τ_P, in which case the correlations of an agent's position at two consecutive time steps t and $t+\tau_P$ are negligible. In such a situation, at each time step, the probability of a link to exist is just

$$\rho = \begin{cases} \pi d^2/L^2 & d \le \dfrac{L}{2} \\ \sqrt{4d^2 - L^2}/L + d^2\left[\pi - 4\cos^{-1}\left(\dfrac{L}{2d}\right)\right]\Big/ L^2 & \dfrac{L}{2} < d \le \dfrac{L}{\sqrt{2}} \\ 1 & d > \dfrac{L}{\sqrt{2}}. \end{cases} \tag{7}$$

From this expression, one can construct an average Laplacian matrix $\hat{L}_{ij} = (N\delta_{ij} - 1)\rho$. All non-zero eigenvalues of this average Laplacian are degenerated $\eta_i = \eta = N\rho\ (\forall i \ge 2)$. We then get the transverse Lyapunov exponent via FSA for $d \le L/2$ as

$$\Lambda_{FS} = \lambda + \frac{1}{\tau_p}\log\left(1 - \frac{\sigma\pi d^2}{L^2}\right), \tag{8}$$

by replacing L_{ij} in Eq. (5) with $\hat{L}_{ij}$ and considering the linear growth rate of the non-zero eigenmodes, where λ is the Lyapunov exponent of the internal dynamics given in Eq. (4). In the parameter set in this paper, $1 - \sigma\pi d^2/L^2 > 0$ holds for $d < d_0 = L/\sqrt{\sigma\pi} \approx 43.27$. Note that an oscillation arises when the interaction is too strong. For $1 - \sigma\pi d^2/L^2 < 0$, such an oscillation can be seen in Eq. (8) where the transverse Lyapunov exponent via FSA becomes complex. It is evident from the derivation of Eq. (8) that this happens in a discrete system where the coupling strength σ is not proportional to the time step. Since this is not a situation of physical interest, the numerical simulations have been performed for d values below d_0.

B. Comparison between linearized equation and asymptotic dynamics to synchronized state

Now, we compare the numerical results of the linearized equation with those of the original nonlinear equation when the state goes to the synchronized state. When the synchronization error is small, we expect that the dynamics is well approximated by the linearized equation (6). The upper panels of Fig. 3 show the time evolution of the average phase difference $\Delta\varphi(t)$ for different values of τ_P by numerically solving the full nonlinear equation (2); they are close-ups of

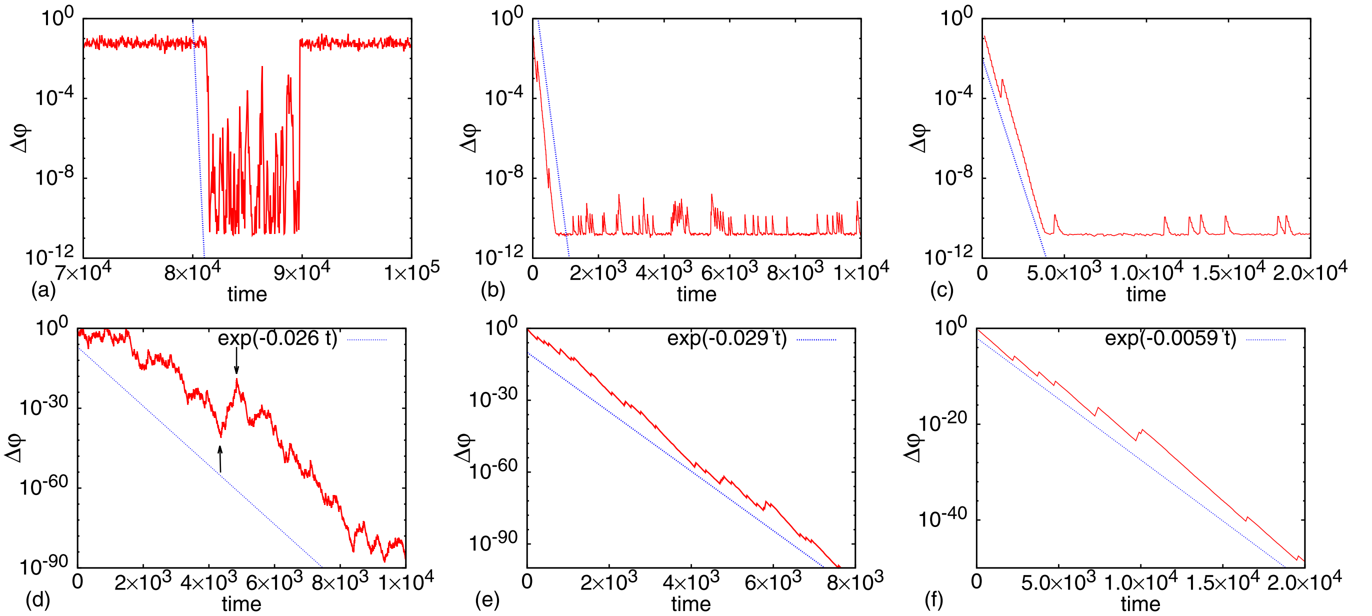

FIG. 3. Comparison of the time evolution of the synchronization error for the original nonlinear equation (2) versus the linearized equation (6) for $\tau_P = 1$ ((a) and (d)), $\tau_P = 11$ ((b) and (e)), and $\tau_P = 100$ ((c) and (f)). Upper panels are close-up of Fig. 1 for the corresponding parameter values from the nonlinear equation. Lower panels represent the results of the linearized equation for the same parameter values to the upper panels. The slope of the blue lines corresponds to the transverse Lyapunov exponent Λ. In panel (d), although Λ is negative, its fluctuation is so large that $\Delta\varphi$ can grow temporarily. The arrows indicate such an example of the temporal growth of $\Delta\varphi$.

Fig. 1. $\Delta\varphi$ evolves in time from $\Delta\varphi \sim 10^{-1}$ and saturates around the noise level $\Delta\varphi \sim \zeta_{\max} = 10^{-10}$.

We plot in the lower panels of Fig. 3 the results of the linearized model (6). Blue lines in the upper panels of Fig. 3 represent the exponential curves whose slope is estimated from the linearized equation in the lower panels. For small τ_P (Fig. 3(a)), fluctuations of the transverse Lyapunov exponent are large, and they can enhance the synchronization error $\zeta_{\max}^{-1}$ even if the long-time average of the exponent is negative. Since $\Delta\varphi \gtrsim \zeta_{\max}$ holds even when the system is synchronized, such large fluctuations cause the switching from laminar to burst states in the presence of small noise. For large τ_P, fluctuations of the transverse Lyapunov exponent are small and such a switch is not observed. Note that the synchronization achievement is faster for $\tau_P = 10$ than for $\tau_P = 100$.

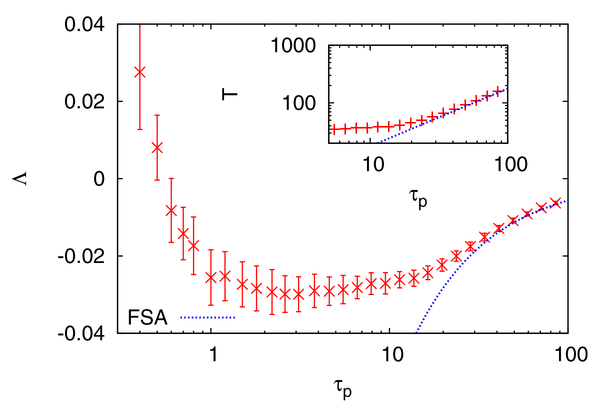

FIG. 4. Transverse Lyapunov exponent Λ for $d = 30$. Blue line represents the prediction of the fast switching approximation (8). In the inset, the characteristic time $T = 1/|\Lambda|$ is shown in a log-log scale. The average is taken over 124 realizations.

Figure 4 shows the dependence of the transverse Lyapunov exponent Λ on τ_P for $d = 30$. It is positive for $\tau_P \lesssim 0.5$, but negative for $\tau_P \gtrsim 0.5$. The dependence of Λ on d and τ_P is plotted in Fig. 5(a). For noise free non-chaotic systems,[28,29] Λ converges to a certain value for the small τ_P limit, but in the present case, where the internal dynamics is chaotic, it diverges. As we increase τ_P, the time scale of the topology change is faster than that of the oscillator dynamics and Λ is well approximated by FSA. In this case, we have $\Lambda \sim \tau_P^{-1}$ and it takes longer to be synchronized for larger τ_P. From these two asymptotics, we can predict that there exists an optimal parameter region which yields the fastest synchronization for intermediate τ_P values, which is indeed shown in Fig. 2(b).

In order to clarify that the switching between the laminar and burst states is caused by large fluctuations of the transverse Lyapunov exponent, we plotted the largest enhancement of the synchronization error $g_m \equiv \max_{t'} \Delta\varphi(t')/\Delta\varphi(t)$, where $t < t' < t + \tau$. Due to the noise term in the evolution equation, $\Delta\varphi$ stays around the noise strength ζ_0. Thus, if $\Delta\varphi(t')/\Delta\varphi(t)$ $(t < t')$ is as large as $\zeta_{\max}^{-1}$, the synchronization error can be $O(1)$ and the synchronized state breaks down even if it is

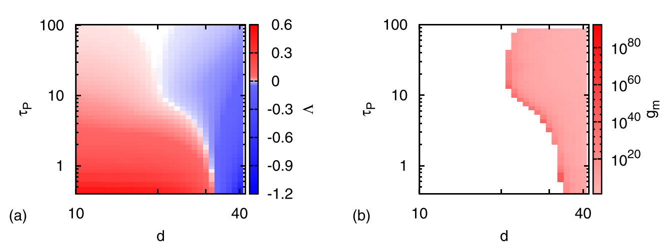

FIG. 5. (a)Transverse Lyapunov exponent Λ in the d-τ_P plane. (b) Largest growth of the synchronization error $g_m \equiv \max_{t'} \Delta\varphi(t')/\Delta\varphi(t)$. Note that g_m can be defined only if $\Lambda < 0$.

linearly stable. Figure 5(b) shows that large fluctuations are observed in the parameter region where the transverse Lyapunov exponent is negative but small.

V. SOLUTION OF THE LINEARIZED EQUATIONS

In order to get insight into the collective behavior of the set of mobile oscillators, we can proceed with the approximate linearized dynamics (5), since it describes very well the exponential decay of the average synchronization error to the synchronized state. This method has been recently proposed in Refs. 28 and 29 for the non-chaotic phase oscillators and shown that this method describes the asymptotic dynamics appropriately. Here, we give a full description of the method and extend it to the chaotic internal dynamics case.

We introduce the normal modes of the Laplacian matrix L, but in our current case, they are time dependent because the network topology (and hence L) changes with time. If $\theta_l(t)$ are the normal modes corresponding to the eigenvalue $\eta_l(t)$ at time t and $U_{jl}(t)$ is the orthogonal matrix of the transformation from the original coordinates to the normal ones, being its columns of the normalized eigenvectors of the Laplacian matrix, we can write

$$\varphi_j(t) = \sum_{l=1}^{N} U_{jl}(t)\theta_l(t), \tag{9}$$

at any time step. Multiplying both sides of Eq. (5) by the transpose $U^T_{li}(t+\tau_P)$ from the left, we get

$$\begin{aligned}\theta_l(t+\tau_P) &= \sum_{i,m} U^T_{li}(t+\tau_P)U_{im}(t)\partial F[\varphi_0(t)][1-\sigma\eta_m(t)]\theta_m(t)\\ &\equiv \sum_m O_{lm}(t)\partial F[\varphi_0(t)][1-\sigma\eta_m(t)]\theta_m(t),\end{aligned} \tag{10}$$

where $\partial F[\varphi_0(t)]$ stands for the Jacobian of F (see Eq. (5)). Note that $O_{lm}(t) = \sum_i U^T_{li}(t+\tau_P)U_{im}(t)$ is an orthogonal matrix with a unit determinant. Then, after an arbitrary number n of time steps, we get

$$\begin{aligned}\theta_{l_n}(t+n\tau_P) &= \prod_{q=0}^{n-1}\partial F[\varphi_0(t+q\tau_P)]\\ &\quad\times\left[\sum_{l_q=1}^{N} O_{l_{q+1}l_q}[1-\sigma\eta_{l_q}]\right]\theta_{l_0}(t),\end{aligned} \tag{11}$$

where l_q denotes the suffix corresponding to an eigenmode at time $t+q\tau_P$. The product of these matrices separately describes the transformation of the normal modes of the instantaneous networks by $O_{l_{q+1}l_q}$ and the decay or growth of each eigenmode by $\partial F[\varphi_0(t+q\tau_P)](1-\sigma\eta_{l_q})$.

The number of zero eigenvalues of an instantaneous Laplacian matrix is equal to the number of connected components. In our case, it is easy to understand that even if $L_{ij}(t)$'s have multiple zero eigenvalues, the product of the matrices can have less zero eigenvalues and eventually be reduced to a single one, because all random walkers meet in the finite system size if we wait for a long time. This implies that the system which is instantaneously disconnected is eventually in contact.[38]

Empirically, we find that, for large enough n, the eigenvalues of the product of matrices for n time steps (11) converge to certain values in such a way that the ith eigenmode shows an exponential growth or decay with the rate $\langle\partial F(\varphi_0)\rangle(1-\sigma\bar{\eta}_i)$, where $\bar{\eta}_i$ represent the eigenvalues of an effective Laplacian matrix $(0=\bar{\eta}_1\le\bar{\eta}_2\le\ldots\le\bar{\eta}_N)$. According to this behavior, the transverse Lyapunov exponent Λ will be

$$\Lambda = \lambda - \frac{\log(1-\sigma\bar{\eta}_2)}{\tau_P}. \tag{12}$$

Λ obtained with this method should agree with the numerical simulation even for smaller τ_P where the FSA does not hold. This was verified for the noise free non-chaotic system where $\lambda = 0$ holds.[28]

All the dynamical properties of the evolution of the system lie then in the product of these matrices, and our procedure could be generalized to describe the synchronization dynamics in the general case of the non-fixed topology systems where the Master Stability Function (MSF) formalism[17,39,40] is applicable. It is also important to note that our procedure is general and does not depend on the precise way the network is changed or has restrictions on the commutativity[41] or on the simultaneous triangularizability[42] of the graphs.

In our previous studies,[28,29] we have shown that the synchronization mechanism depends on the dominant time scales, namely, one of the oscillator dynamics and one of the motion of agents. For the case of the non-chaotic oscillators,[28,29] the time scale of the oscillator dynamics is different below and above the percolation transition $(d=d_c)$[43,44] of the instantaneous topology. We consider three asymptotic cases individually: derivation of FSA when the time scale of the topology change is much shorter than the oscillator dynamics. In the opposite case, deviation from FSA caused by local synchronization is studied for the two asymptotic cases $d \ll d_c$ and $d \gg d_c$. This procedure is an extension of the previous studies[28,29] to the case where the local dynamics shows the chaotic behavior.

A. Fast switching approximation

When the characteristic time scale of the agents' motion is much shorter than the signal interval, we obtain FSA if the motion of the agents is independent of each other. In such a case, the agents move a sufficiently long distance during τ_P, and $L_{ij}(t)$ and $L_{ij}(t+\tau_P)$ are regarded as independent. Thus, their eigenvalues, $\eta_l(t)$ and $\eta_l(t+\tau_P)$, are uncorrelated. We write $\eta_{k_q} = \langle\eta\rangle + \Delta_{k_q}$, where $\langle\eta\rangle$ is the average eigenvalue of the Laplacian matrices and $\langle\Delta_{k_q}\rangle = 0$. Then, we get $\prod_{q=0}^{n-1}(1-\sigma\eta_{k_q}) = (1-\sigma\langle\eta\rangle)^n\prod_{q=0}^{n-1}\xi_{k_q}$, where $\xi_{k_q} = [1-\frac{\sigma\Delta_{k_q}}{1-\sigma\langle\eta\rangle}]$. We can expect $\prod_{q=1}^{n}(1-\sigma\eta_{l_q}) \approx e^{n\langle\log(1-\sigma\eta)\rangle}$ for every possible combination of l_q's in Eq. (11) if we neglect the fluctuation of η's.

Since the average eigenvalue of the Laplacian matrix is the average degree, we get $\langle \eta \rangle = (N-1)\rho$, where ρ is defined by Eq. (7). Expanding the transverse Lyapunov exponent in powers of σ, we have

$$\Lambda = \lambda + \frac{\sigma}{\tau_P}(N-1)\rho + \mathcal{O}(\sigma^2), \tag{13}$$

which is equal to Λ_{FS} (8) plotted in Fig. 4, and which has been derived in a different way in Ref. 24, up to the lowest order in σ.

B. Multiple cluster local synchronization

We consider now the case for d below the percolation threshold d_c. There exist multiple disconnected clusters, and the characteristic time of the agent's motion is larger than that of synchronization inside the cluster. Thus, local synchronization is achieved.[28,29] If τ_P is small enough, the non-zero eigenmode $\theta_l(t)$ with $\eta_l(t) \neq 0$ vanishes before the network topology changes if its MSF is negative, implying that local synchronization is achieved. Even if we decrease the signal interval τ_P and hence increase the number of signals, we cannot get a further decrease in the synchronization error between the disconnected clusters, and the synchronization error can be enhanced for the chaotic internal dynamics case. Therefore, we expect that Λ converges to a finite value for $\tau_P \to 0$ if the internal dynamics is non-chaotic,[28,29] but it diverges for the chaotic internal dynamics (Fig. 4).

We assume that the MSF of all eigenmodes except the zero mode is negative. In such a case, $\theta_l(t)$ corresponding to a non-zero eigenmode $\eta_l(t) \neq 0$ vanishes before the topology changes. Therefore, the transition from the zero eigenmodes to the non-zero ones dominates the synchronization dynamics. We introduce

$$\bar{O}_{ij} \equiv \begin{cases} O_{ij} & \text{if } \eta_i = 0 \quad \text{and } \eta_j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Then, the evolution of the zero eigenmodes is approximately written as

$$\theta_{l_n}(t+n\tau_P) \approx \langle \partial F(\varphi_0) \rangle^n \prod_{q=1}^{n} \left\{ \sum_{l_q} \bar{O}_{l_{q+1} l_q} \right\} \theta_{l_0}(t). \tag{15}$$

It is assumed in this approximation that the non-zero eigenmode decays before the topology changes, and the transition from a non-zero mode to zero mode has been neglected. Note that $\bar{O}_{l_{q+1} l_q}$'s are no longer orthogonal.

The product of matrices in (15) can be diagonalized, and the ith eigenmode is expected to show an exponential growth or decay with the rate $\langle \partial F(\varphi_0) \rangle (1 - \overline{\Lambda_i})$, where $\overline{\Lambda_i}$ represent the eigenvalues of the product of $\bar{O}_{ij}$ $(0 = \overline{\Lambda_1} \le \overline{\Lambda_2} \le \ldots \le \overline{\Lambda_N})$. The transverse Lyapunov exponent for this approximation Λ_{ML} can then be estimated as $\Lambda_{ML} = \lambda - \log(1 - \overline{\Lambda_2})/\tau_P$. It is important to note that Λ_{ML} does not depend on τ_P if the τ_P is small enough. The reason is as follows. If the network topology is unchanged between time t and $t + \tau_P$, $\bar{O}_{ij}$ is 1 if $\eta_i = \eta_j = 0$ and is 0 otherwise. Once local synchronization is achieved, interactions do not contribute to the decay rate, and Λ_{ML} does not depend on τ_P for the non-chaotic oscillator case. This result also implies the divergence of the transverse Lyapunov exponent as $\Lambda \propto \tau_P^{-1}$ $(\tau_P \to 0)$ if $\lambda > 0$.

C. Single cluster local synchronization

For $d \gg d_c$, the motion is slow, and hence, the whole network is almost always connected which implies that $\eta_2(t)$ is finite. If the network topology remains unchanged for a long time, the η_l $(l \ge 3)$ decay much faster than η_2 before topology changes. Therefore, we can approximate the transverse Lyapunov exponent using the average of the second smallest eigenvalue as

$$\Lambda_{SL} = \lambda + \frac{\langle \log(1 - \sigma\eta_2) \rangle}{\tau_P}. \tag{16}$$

This is a good approximation for d sufficiently above the percolation threshold.[28] The inequality $\langle \eta_2 \rangle \le \langle \eta \rangle$ implies that $\Lambda_{SL} \ge \Lambda_{FS}$ holds. For $d \approx L$, every non-zero eigenvalue degenerates and $\langle \eta_2 \rangle \approx \langle \eta \rangle = (N-1)\rho$ holds. This implies that we recover FSA for larger d.

VI. CONCLUSIONS

In this paper, we have studied synchronization of the mobile oscillators which are irregular in the internal dynamics. We have found here a synchronization transition which was not observed in Ref. 28 by changing the parameter values relevant to the interaction between agents. Moreover, we uncovered a switching between the laminar and burst states close to the transition point. We have found that it is caused by the large fluctuations of the transverse Lyapunov exponent, and it can happen even if the synchronized state is linearly stable. This exhibits deviation from the fast switching approximation.[23–26] This can be induced already by a very small noise. Therefore, it is very important to take into account this effect in applications, because small noise is inevitable in any real system.

Our study suggests that very small noise can be enhanced by a chaotic dynamics and break synchronization. Due to the noisy dynamics, there exists an optimal parameter value of τ_P in the intermediate region: if τ_P is too small, synchronization is unstable, and the synchronization time is long for large τ_P. This is a result that we have to taken into account for the application to construction of synchronous devices of the noisy mobile oscillators.

In this study, the agents' motion is homogeneous and independent of that of other agents. We will approve that such a synchronization transition is possible in real systems whose agent dynamics is inhomogeneous in forthcoming studies.

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