Distributed Robust Synchronization of Dynamical Networks with Stochastic Coupling

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***Abstract*—This paper deals with the problem of robust adaptive synchronization of dynamical networks with stochastic coupling by means of evolutionary algorithms. The complex networks under consideration are subject to: 1) the coupling term in a stochastic way is considered; 2) uncertainties exist in the node's dynamics; 3) pinning distributed synchronization is also considered. By resorting to Lyapunov function methods and stochastic analysis techniques, the tasks to get the distributed robust synchronization and distributed robust pinning synchronization of dynamical networks are solved in terms of a set of inequalities, respectively. The impacts of degree information, stochastic coupling and uncertainties on synchronization performance, i. e., mean control gain and convergence rate, are derived theoretically. The potential conservativeness for the distributed robust pinning synchronization problem is solved by means of an evolutionary algorithm-based optimization method, which includes a constraint optimization evolutionary algorithm and a convex optimization method and aims at improving the traditional optimization methods. Simulations are provided to illustrate the effectiveness and applicability of the obtained results.**

***Index Terms*—Synchronization/Consensus, Complex dynamical networks, Stochastic coupling, Evolutionary algorithms.**

I. Introduction

The past decades have seen a tremendous upsurge in the research efforts toward the intrinsic features of complex networks and multi-agent systems. Complex networks have found applications in various fields as communication networks, genetics regulatory networks, social networks, neuronal networks, and the Internet [1]–[4]. Among them, synchronization and cooperative control have attracted unprecedented attention of the physics and control communities [5]–[13] in view of their wide applications in various emerging fields such as chemical reactions, information consensus, power grids, formation control in robots and flights, etc [14].

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For a variety of biological, physical and social networks, a typical problem is cooperative control and regulating the dynamics of coupled networked systems to a desired state by means of a small fraction of inputs owing to the reduction of mean control gain. Therefore, pinning control/controllability of complex networks or multi-agent systems has been attracting recurrent research interests [15]–[20]. In [21], [22], adaptive pinning synchronization was investigated and some useful criteria were proposed to ensure synchronization of complex dynamical networks. Specially, it was found that the underlying network topology can affect the convergence rate and the terminal mean control gain [21]. In [23], distributed pinning synchronization of stochastic coupled neural networks under controller missing was studied by casting the problem into a convex optimization problem. However, due to the difficulty in a mathematical derivation, there are still basic points for improving the above mentioned results in particular: 1) how to determine driver nodes for obtaining better global synchronizability or controllability of complex networks; 2) the pinning synchronization of complex networks with parameter uncertainties and stochastic coupling has been widely overlooked in the current literature, despite their importance in practice.

In fact, the limited energy, computational power, and internal and external factors will inevitably lead to deterministic and stochastic disturbances that are rather challenging in investigating pinning synchronization of complex networks. Firstly, modelling errors are usually used for describing dynamics of complex networks, since they can account for the occurrence of unstable fluctuations of message transmissions through the networks and the estimation of the variance from statistical tests for identification of the network parameters, etc. Secondly, the network coupling could occur in a stochastic way, and stochastic disturbances could appear in both the coupling term and the overall networks caused by noisy environments [24]. Therefore, synchronization of complex networks with uncertainties or stochastic coupling has attracted increasing attention during the past few years [25]. Unfortunately, these mentioned results are based on linear matrix inequalities (LMIs), whose main focus is on presenting criteria to ensure under what kinds of conditions the synchronization of complex networks can be achieved. It still remains unclear how to characterize the synchronization performance, such as mean control gain and convergence rate when uncertainties and stochastic coupling are included.

An evolutionary algorithm (EA) is a generic population-based meta-heuristic optimization algorithm, which is inspired by principles of biological evolution, such as reproduction, mutation, recombination, and selection. Recently, local controllability of complex networks, including determining driver nodes and designing control gains, has been investigated by means of single objective optimization EAs [26] and constraint optimization EAs [27], respectively. However, the investigated problem is *local* controllability instead of a *global* one and the measures of controllability are confined to complex networks without any disturbances, which substantially limits the application of the presented results. To the best of authors' knowledge, up to now, very little research effort has been made to distributed robust (pinning) synchronization of uncertain networked systems with stochastic coupling. It is important to emphasize that the optimization for such problems in this paper is based on EAs and convex optimization methods. Therefore, the main purpose of this paper is to investigate the distributed (pinning) synchronization problem of dynamical networks with stochastic coupling and to unveil the relationship between stochastic coupling and synchronization performance, where an EA-based approach is utilized to solve the addressed problem.

In this paper, we focus on the distributed robust (pinning) synchronization problem for networked systems with stochastic coupling, which is solved by an EA-based optimization method. The impacts of uncertainties and stochastic coupling on synchronization performance are also analyzed theoretically. The main contributions of this paper can be listed as follows: (1) intensive stochastic analysis is performed to establish a unified framework for robust distributed (pinning) synchronization of dynamical networks that provides the simultaneous presence of parameter uncertainties as well as stochastic coupling; (2) effects of uncertainties and stochastic coupling on synchronization performance are derived in a theoretical way, in which some information such as degree information and edge number are used to analyze the theoretical results on synchronization based on graph theory; (3) the obtained results for distributed robust (pinning) synchronization of uncertain networked systems with stochastic coupling are solved in terms of an EA-based convex optimization method.

The remainder of this paper is organized as follows. In Section II, the problem addressed is formulated and some preliminaries are briefly outlined. In Section III, the main results are given for the distributed robust (pinning) synchronization of networked systems. In Section IV, an EA-based optimization algorithm is introduced for solving the presented synchronization criteria. In Section V, one numerical example is given to demonstrate the effectiveness of the obtained results. In Section VI, some concluding remarks are provided. In Section VII, proofs of the main theorems are presented.

Notations: In this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ indicate, respectively, the n-dimensional Euclidean space and the set of all $n\times m$ real matrices. The Kronecker product of matrices $X\otimes Y\in\mathbb{R}^{mp\times nq}$, where $X\in\mathbb{R}^{m\times n}$ and $Y\in\mathbb{R}^{p\times q}$. $|.|_c$ denotes the cardinality. $|.|$ is the absolute value. $\|.\|$ is the Euclidean vector norm in $\mathbb{R}^n$. $\lambda_{\max}(.)$ is the maximum eigenvalue of a matrix. Let a graph be $\mathcal{G}=[\mathcal{V},\mathcal{E}]$, where $\mathcal{V}=\{1,...,N\}$ stands for the vertex set and $\mathcal{E}=\{e(i,j)\}$ is the edge set. $\mathcal{N}_i$ represents the neighborhood of vertex i in the sense $\mathcal{N}_i=\{j\in\{\mathcal{V}:e(i,j)\in\mathcal{E}\}$. The graph $\mathcal{G}$ is supposed to be connected, undirected and simple. Let $L=[a_{ij}]_{i,j=1}^N$ be the Laplacian matrix of the graph $\mathcal{G}$, which is defined as: for any pair $i\neq j, a_{ij}=a_{ji}=-1$ if $e(i,j)\in\mathcal{E}$; otherwise, $a_{ij}=a_{ji}=0$. $a_{ii}=-\sum_{j=1,j\neq i}^N a_{ij}$ is the degree of vertex i $(i\in\mathcal{V})$. For $\mathcal{T}\subset\mathcal{V}$, all vertices $\mathcal{V}\backslash\mathcal{T}$ can be accessible from the vertex set $\mathcal{T}$, i.e., for any vertex i in $\mathcal{V}\backslash\mathcal{T}$, there exists at least one vertex $j\in\mathcal{T}$ such that a path between vertices i and j exists. $\delta_{\mathcal{T}}(\cdot)$ is the characteristic function of the set $\mathcal{T}$, i.e., $\delta_{\mathcal{T}}(i)=1$ if $i\in\mathcal{T}$; otherwise, $\delta_{\mathcal{T}}(i)=0$. l is the element number of the finite set $\mathcal{T}$ composed of the vertices to be controlled. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a complete probability space, where Ω represents a sample space, $\mathcal{F}$ is a σ-algebra and $\mathcal{P}$ is a probability measure. $\mathbb{E}\{.\}$ stands for the expectation and Prob$\{.\}$ denotes the probability of an event.

II. Problem formulation and preliminaries

In this section, the problem addressed is formulated and some preliminaries about the dynamical model and evolutionary algorithms are briefly outlined. In this paper, complex networks with stochastic coupling are considered as follows:

$$\begin{aligned} dx_i(t) &= [(A+\triangle A(t))x_i(t)+f(x_i,t)+m(x_i,t)\\ &\quad +c\sum_{j\in\mathcal{N}_i}(x_j(t)-x_i(t))]dt\\ &\quad +\sum_{j\in\mathcal{N}_i}(g(x_j,t)-g(x_i,t))dv(t), i\in\mathcal{V}, \end{aligned} \quad (1)$$

where $x_i(t)=[x_{i1}(t),x_{i2}(t),...,x_{in}(t)]^T\in\mathbb{R}^n$ $(i\in\mathcal{V})$ is the state vector; $A>0$ is the system matrix and $\triangle A(t)$ represents the uncertainty in the linear part satisfying $\|\triangle A(t)\|\leq\iota$; $f(x_i,t)=[f_1(x_i,t),...,f_n(x_i,t)]^T$ and $g(x_i,t)=[g_1(x_i,t),...,g_n(x_i,t)]^T$ are continuous nonlinear functions; $m(x_i,t)=[m_1(x_i,t),...,m_n(x_i,t)]^T$ is the uncertain nonlinearity [28]; c is the global coupling strength; $v(t)$ is one-dimensional Brownian motion defined on $(\Omega,\mathcal{F},\mathcal{P})$ satisfying $\mathbb{E}\{dv(t)\}=0$, and $\mathbb{E}\{[dv(t)]^2\}=dt$. According to Gershgorin's disk theorem [29], all the eigenvalues of L corresponding to graph $\mathcal{G}$ satisfy the following relationship $0=\lambda_1(L)\leq\lambda_2(L)\leq...\leq\lambda_N(L)$. In addition, $\lambda_2(L)>0$, since $\mathcal{G}$ is connected and undirected.

Here, we define the following set $\mathcal{T}$:

$$\begin{cases} \mathcal{T}=\mathcal{V}, & \text{if all the nodes in } \mathcal{V} \text{ are controlled,}\\ \mathcal{T}\subset\mathcal{V}, & \text{if a fraction of nodes in } \mathcal{V} \text{ are controlled.}\end{cases} \quad (2)$$

The pinning controllers $u_i(t)$ are used in the set of driver nodes for achieving distributed synchronization in mean square of (1):

$$\begin{aligned} dx_i(t) &= [(A+\triangle A(t))x_i(t)+f(x_i,t)+m(x_i,t)\\ &\quad +u_i(t)+c\sum_{j\in\mathcal{N}_i}(x_j(t)-x_i(t))]dt\\ &\quad +\sum_{i\in\mathcal{N}_i}(g(x_j,t)-g(x_i,t))dv(t), i\in\mathcal{T},\\ dx_i(t) &= [(A+\triangle A(t))x_i(t)+f(x_i,t)+m(x_i,t)\\ &\quad +c\sum_{j\in\mathcal{N}_i}(x_j(t)-x_i(t))]dt\\ &\quad +\sum_{j\in\mathcal{N}_i}(g(x_j,t)-g(x_i,t))dv(t), i\notin\mathcal{T}. \end{aligned} \quad (3)$$

The distributed controllers $u_i(t)$ are designed as follows:

$$u_i(t) = \sum_{j\in\mathcal{N}_i} \epsilon_i(t)(x_j(t) - x_i(t)), i \in \mathcal{T}, \quad (4)$$

where $\epsilon_i(t)$ is the control gain of the ith node and is updated according to the following equation:

$$d\epsilon_i(t) = \alpha_i[\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))]^T [\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))]dt, i \in \mathcal{T}, \quad (5)$$

where $\alpha_i > 0$ and $\alpha_i \in [\check{\alpha}, \hat{\alpha}] \subseteq [0, 1]$.

Remark 1. In reality, it is unavoidable that modeling errors occur in the process of constructing multi-agent systems and complex networks. Modeling errors may arise from fluctuations of information transmission among the nodes, some inconsistency induced by the discretization process, or the estimation variance from statistical tests for the identification of the network parameters. In order to characterize modelling errors, a natural and efficient way is to use parameter uncertainties to stand for modeling errors. Here, we consider distributed robust synchronization under norm bounded uncertainties and the effects of parameter uncertainties on synchronization performance will be analyzed theoretically. Compared with previous works on robust synchronization of complex networks [25], [30], the influence of parameter uncertainties on synchronization will be investigated in the following.

Remark 2. In system (3), the nonlinear stochastic coupling term $\sum_{j\in\mathcal{N}_i}(g(x_j(t))-g(x_i(t)))dv(t)$ is to describe stochastic effects of the information transmission among the nodes and reflects nonlinear properties in the communication. Although synchronization of complex networks with stochastic coupling has been investigated in [31], the impacts of stochastic coupling on synchronization performance still remains unclear due to the mathematical difficulty, despite its importance in practice. In addition, previous works on pinning synchronization of networked systems have not taken into account parameter uncertainties and stochastic coupling [16], [18]–[23] and thus our model here renders more practical factors. In the following, we will shorten such a gap by investigating robust distributed (pinning) synchronization of uncertain networked systems with stochastic coupling in (3).

Remark 3. Model (3) is general, since it includes parameter uncertainties, deterministic and stochastic coupling into one unified model. Different from the models in [10], [23], the stochastic coupling and the uncertainty term are included in our model and the synchronization of system (3) is investigated by means of an EA-based algorithm. Hence, our model is more general to describe uncertainties or noise information, since we aim at considering various disturbances by utilizing parameter uncertainties and stochastic coupling.

Remark 4. Usually, "pinning synchronization" is referred as controlling the states of networks to the isolated node, such as a chaotic system, a periodic solution or a equilibrium [7], [20], [32]. According to the definition of [19], "pinning controllability" means that the states of networks are forced to a desired state. In this paper, the desired state is the consensus value of the nodes, which is like the concept of conventional synchronization or consensus in [8], [14]. By injecting distributed controllers to the driver nodes in networks, synchronization can be finally reached. In order to differ from the usual "pinning synchronization" in [7], [20], [32], we refer pinning synchronization in our paper as "pinning distributed synchronization" if only a subset of nodes in networks are injected with distributed controllers to achieve conventional synchronization.

III. Conditions and upper bounds for distributed robust synchronization of dynamical networks

In this section, the distributed synchronization (pinning) synchronization of the dynamical network in (3) is investigated under $\mathcal{T}$. Upper bounds of $\mathcal{C}$ and $\mathcal{S}$ are derived for $\mathcal{T} = \mathcal{V}$, where $\mathcal{C}$ and $\mathcal{S}$ are provided in the following to quantify the synchronization performance:

$$\mathcal{C} = \frac{1}{N}\sum_{i\in\mathcal{V}} \epsilon_{i,\infty}, \quad (6)$$

and

$$\mathcal{S} = \mathbb{E}\int_0^\infty \frac{1}{N-1}\sum_{i\in\mathcal{V}}[x_i(t) - \bar{x}(t)]^T[x_i(t) - \bar{x}(t)]dt, \quad (7)$$

where $\bar{x}(t) = \frac{1}{N}x(t)$ and $\epsilon_{i,\infty} = \lim_{t\to\infty}\mathbb{E}\{\epsilon_i(t)\}$. Apparently, a good synchronization performance indicates a high convergence rate (a small $\mathcal{S}$) and low mean control gain (a small $\mathcal{C}$), as shown in (6) and (7).

A. Assumptions and definitions

The following assumptions, lemmas and definitions are necessary to derive our main results.

Assumption 1. The functions $f(x_i, t)$, $g(x_i, t)$ and $m(x_i, t)$ are said to be Lipschitz continuous with respect to t if there exists positive constants h_1, h_2 and h_3 such that the following inequalities hold for all $x_i, x_j \in \mathbb{R}^n$:

$$\begin{aligned} &\|f(x_i, t) - f(x_j, t)\| \le h_1\|x_i - x_j\|, \\ &\|g(x_i, t) - g(x_j, t)\| \le h_2\|x_i - x_j\|, \\ &\|m(x_i, t) - m(x_j, t)\| \le h_3\|x_i - x_j\|, i, j \in \mathcal{V}. \end{aligned} \quad (8)$$

Assumption 2. [21] The vector-valued continuous function $f(x, t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is said to be uniformly decreasing if there exist $\vartheta > 0 \in \mathbb{R}$ and $\Delta > 0 \in \mathbb{R}$ such that

$$\begin{aligned} & (x - y)^T[f(x, t) - f(y, t) - \vartheta(x - y)] \\ \le\ & -\Delta(x - y)^T(x - y), \end{aligned} \quad (9)$$

holds for all $x, y \in \mathbb{R}^n$ and $t \ge 0$.

Assumption 3. $f(0, t) = 0$, $g(0, t) = 0$ and $m(0, t) = 0$.

Lemma 1. [33] Let $\mathcal{G}$ be a simple graph. $d(u)$ is the degree of vertex $u \in \mathcal{V}$ and $m(u)$ is the average of the degrees of the vertices adjacent to u. $\mathcal{G}$ contains M edges. Assume that by δ_1 and δ_2 the maximum and minimum degrees of $\mathcal{G}$, respectively.

Sort the degree of $\mathcal{G}$ as $d_1 \geq d_2 \geq ... \geq d_N$. Then, the following inequalities hold:

1. $\lambda_N(L) \leq \max\{d(u)+d(v)|(u,v) \in \mathcal{E}\}$,
2. $\lambda_N(L) \leq d_N + \sqrt{(d_N - \frac{1}{2})^2 + \sum_{i\in\mathcal{V}} d_i(d_i - d_N)} + \frac{1}{2}$, where the equality if and only if $\mathcal{G}$ is a regular bipartite graph.
3. $\lambda_N(L) \leq \max\{d(u)+m(u)|u \in \mathcal{V}\}$,
4. $\lambda_N(L) \leq \frac{\delta_2 - 1 + \sqrt{(\delta_2-1)^2 + 8(\delta_1^2 + 2M - (N-1)\delta_2)}}{2}$, where the equality if and only if $\mathcal{G}$ is a regular bipartite graph.
5. $\lambda_N(L) \leq \max\{d(u)+d(v) - |\mathcal{N}_u \cap \mathcal{N}_v|_c|(u,v) \in \mathcal{E}\}$. (10)

Lemma 2. [33] Let $\mathcal{G}$ be a simple graph. Denote by d_k the kth largest degree of $\mathcal{G}$. Then, the following inequality holds:

$$\lambda_2(L) \geq d_{N-1} - N + 3. \tag{11}$$

Definition 1. Let $x_i(t)(1 \leq i \leq N)$ be a solution of the uncertain complex network with stochastic coupling in (3), where $x_i(0) = (x_1^0, x_2^0, \ldots, x_n^0)$. If there exists a nonempty subset $\Omega \subseteq \mathbb{R}^n$, with $x_i(0) \in \Omega(1 \leq i \leq N)$, such that $x_i(t) \in \mathbb{R}^n$ for all $t \geq t_0, 1 \leq i \leq N$,

$$\lim_{t\to\infty} \mathbb{E}\|x_i(t) - x_j(t)\|^2 = 0, i, j \in \mathcal{V},$$

then the uncertain complex network with stochastic coupling in (3) is said to achieve distributed synchronization in mean square.

B. Distributed synchronization of uncertain dynamical networks with stochastic coupling under $\mathcal{T} = \mathcal{V}$

Theorem 1. For $\mathcal{T} = \mathcal{V}$, suppose that the graph $\mathcal{G}$ is connected and $f(.,t)$, and $g(.,t)$ satisfy Assumptions 1-3. If the following inequality holds:

$$[(\lambda_{\max}(A) + \iota + \vartheta + h_3)I_N - aL - cL + \frac{h_2^2}{4}L^2]L \leq 0, \tag{12}$$

where a is a positive constant, then the uncertain network with stochastic coupling in (3) under (4) and (5) will be globally synchronized in mean square.

Proof: See the appendix. ■

Following Theorem 1, we have the following corollary by enlarging the noise term in stochastic coupling.

Corollary 1. For $\mathcal{T} = \mathcal{V}$, suppose that the graph $\mathcal{G}$ is connected and $f(.,t)$, and $g(.,t)$ satisfy Assumptions 1-3. If the following inequality holds:

$$\begin{aligned}&[(\lambda_{\max}(A) + \iota + \vartheta + h_3)I_N \\ &- aL - cL + \frac{h_2^2}{4}\lambda_N(L)L]L \leq 0,\end{aligned} \tag{13}$$

where a is a positive constant, then the uncertain network with stochastic coupling in (3) under (4) and (5) will be globally synchronized in mean square.

From Corollary 1, if we utilize Lemma 1, we can have the following corollary.

Corollary 2. For $\mathcal{T} = \mathcal{V}$, suppose that the graph $\mathcal{G}$ is connected and $f(.,t)$, and $g(.,t)$ satisfy Assumptions 1-3. If the following inequality holds:

$$[(\lambda_{\max}(A) + \iota + \vartheta + h_3)I_N - aL - cL + \frac{h_2^2}{4}\phi L]L \leq 0, \tag{14}$$

where $\phi = \frac{\delta_2 - 1 + \sqrt{(\delta_2-1)^2 + 8(\delta_1^2 + 2M - (N-1)\delta_2)}}{2}$, a is a positive constant, then the uncertain network with stochastic coupling in (3) under (4) and (5) will be globally synchronized in mean square.

Remark 5. It should be mentioned that ϕ in Corollary 2 can be replaced by using other terms of the right hand of the inequalities in Lemma 1. For example, one can set $\phi = d_N + \sqrt{(d_N - \frac{1}{2})^2 + \sum_{i\in\mathcal{V}} d_i(d_i - d_N)} + \frac{1}{2}$, $\phi = \max\{d(u) + m(u)|u \in \mathcal{V}\}$ or $\phi = \max\{d(u)+d(v) - |\mathcal{N}_u \cap \mathcal{N}_v|_c|(u,v) \in \mathcal{E}\}$. From Lemma 1 and Corollary 2, it can be seen that the properties of networks such as the degree information, the number of edges and the degree of neighbors can heavily affect the synchronization results. From Lemma 1, one can conjecture whether the conditions are satisfied by knowing some statistical information of the dynamical networks. In the following, we will also illustrate the effects of the properties of networks on synchronization performance.

By utilizing the matrix decomposition theory [29], one has the following theorem from Theorem 1.

Theorem 2. For $\mathcal{T} = \mathcal{V}$, suppose that the graph $\mathcal{G}$ is connected and $f(.,t)$, and $g(.,t)$ satisfy Assumptions 1-3. If the following inequality holds:

$$\begin{aligned}&\lambda_{\max}(A) + \iota + \vartheta + h_3 - a\lambda_i(L) \\ &- c\lambda_i(L) + \frac{h_2^2}{4}\lambda_i^2(L) < 0, i \in \mathcal{V}\backslash\{1\},\end{aligned} \tag{15}$$

where a is a positive constant, then the uncertain network with stochastic coupling in (3) under (4) and (5) will be globally synchronized in mean square.

Proof: See the appendix. ■

C. Distributed synchronization of uncertain dynamical networks with stochastic coupling under $\mathcal{T} \subset \mathcal{V}$

In the following, we will investigate the pinning distributed synchronization of the uncertain dynamical network in (3) with stochastic coupling.

Theorem 3. For $\mathcal{T} \subset \mathcal{V}$, suppose that the graph $\mathcal{G}$ is connected and $f(.,t)$, and $g(.,t)$ satisfy Assumptions 1-3. If the following inequality holds:

$$[(\lambda_{\max}(A) + \iota + \vartheta + h_3)I_N - aL\mathcal{I} - cL + \frac{h_2^2}{4}L^2]L \leq 0, \tag{16}$$

where a is a positive constant, $\mathcal{I}(i,j) = \begin{cases} 1, & \text{if } \ i = j, \delta_{\mathcal{T}}(i) = 1 \\ 0, & \text{else} \end{cases}$, then the uncertain network with stochastic coupling in (3) under (4) and (5) will be globally synchronized in mean square.

Proof: See the appendix. ■

Remark 6. In Theorem 1 and Theorem 2, the mean square synchronization problem is investigated for the uncertain complex dynamical network with stochastic coupling in (3) in terms of inequalities that can be readily solved by using convex optimization algorithms [34]. It is worth mentioning that once an adequate complex network is established in Theorem 1 and in Theorem 2, and the corresponding parameters are identified, we can analyze the synchronization problem of uncertain complex networks with stochastic coupling by simply checking the feasibility of the inequalities. In the past decade, convex optimization methods have gained much research attention and their efficiency has been shown without tuning additional parameters.

Remark 7. In Theorem 2, the singular matrix $\mathcal{I}$ is used to denote the fixed pinning set, i. e., if the element is 1 then the node is selected as a driver node; otherwise, the node is just a follower. Usually, statistical methods from the complex networks theory are employed to construct $\mathcal{I}$, such as degree-based methods, betweenness centrality-based methods and closeness-based methods [26]. In [23], after determining the driver nodes by using degree-based methods, the criteria are converted into a convex optimization problem. Although it is convenient to apply, the selection of driver nodes suffers from unavoidable conservativeness. Actually, the selection of driver nodes is a combinatorial optimization problem and thus it is naturally a NP-hard problem. Therefore, in order to select the driver nodes with accuracy, enhancing controllability is now becoming a hot topic in both physics and control communities [19], [20], [22]. In addition, Theorem 3 is a little bit difficult to check. How to simplify the conditions by using other tools is a future research topic in the near future.

Remark 8. In order to handle the selection of driver nodes, *local* controllability of complex networks was investigated by means of evolutionary algorithms (EAs) [26], [27]. The selection of driver nodes and the design of control gains are converted into single objective optimization problems [26] and constraint optimization problems [27], respectively. Nevertheless, the optimization problem is composed of two parts: a combinatorial optimization problem and a continuous optimization problem. The design of control gains is a continuous optimization problem, which increases the complexity of the problem and reduces the accuracy of EAs. Fortunately, an alternative way is to design an adaptive controller and an updating law to reach synchronization without additionally adjusting control gains. In this sense, adaptive pinning control is a suitable way to deal with controllability of networks [20], [22], [23].

D. Upper bounds of $\mathcal{C}$ and $\mathcal{S}$ of uncertain dynamical networks with stochastic coupling under $\mathcal{T} = \mathcal{V}$

Theorem 4. If all assumptions and conditions in Corollary 1 are satisfied, then when $\epsilon_i(0) = 0, (\forall i \in \mathcal{V})$, an upper bound of the mean control gain $\mathcal{C}$ is as follows:

$$\mathcal{C} \leq \hat{\mathcal{C}} = \begin{cases} \mathbb{E}\Big\{2\mathfrak{F} + \sqrt{\dfrac{2q_0\hat{\alpha}}{N}}\Big\}, & \text{if } \mathfrak{F} \geq 0, \\ \mathbb{E}\Big\{\sqrt{\dfrac{2q_0\hat{\alpha}}{N}}\Big\}, & \text{else}, \end{cases} \tag{17}$$

where

$$\begin{cases} \mathfrak{F} = \dfrac{\lambda_{\max}(A) + \iota + \vartheta + h_3}{\lambda_2(L)} \\ \qquad + \dfrac{(h_2^2\lambda_N(L) - 4c)}{4} + a(\tilde{\alpha} - 1), \\ q_0 = \dfrac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\|e_{ij}(0)\|^2, \\ \tilde{\alpha} = \dfrac{\hat{\alpha}}{\check{\alpha}}. \end{cases} \tag{18}$$

An upper bound of $\mathcal{S}$ is

$$\mathcal{S} \leq \hat{\mathcal{S}} = \begin{cases} \mathbb{E}\Big\{\dfrac{N}{(N-1)\lambda_2^2(L)\check{\alpha}}\Big[2\mathfrak{F} + \sqrt{\dfrac{2q_0\hat{\alpha}}{N}}\Big]\Big\}, \\ \text{if } \mathfrak{F} \geq 0, \\ \mathbb{E}\Big\{\dfrac{N}{(N-1)\lambda_2^2(L)\check{\alpha}}\sqrt{\dfrac{2q_0\hat{\alpha}}{N}}\Big\}, \text{else}. \end{cases} \tag{19}$$

Proof: See the appendix. ■

By utilizing Lemma 2, one can have the following theorem.

Theorem 5. If all assumptions and conditions in Corollary 1 are satisfied, then when $\epsilon_i(0) = 0, (\forall i \in \mathcal{V})$, an upper bound of the mean control gain $\mathcal{C}$ is as follows:

$$\mathcal{C} \leq \hat{\mathcal{C}} = \begin{cases} \mathbb{E}\Big\{2\mathfrak{F} + \sqrt{\dfrac{2q_0\hat{\alpha}}{N}}\Big\}, & \text{if } \mathfrak{F} \geq 0, \\ \mathbb{E}\Big\{\sqrt{\dfrac{2q_0\hat{\alpha}}{N}}\Big\}, & \text{else}, \end{cases} \tag{20}$$

where

$$\begin{cases} \mathfrak{F} = \dfrac{\lambda_{\max}(A) + \iota + \vartheta + h_3}{\lambda_2(L)} + \dfrac{(h_2^2\phi - 4c)}{4} \\ \qquad + a(\tilde{\alpha} - 1), \\ \phi = \dfrac{\sqrt{(\delta_2 - 1)^2 + 8(\delta_1^2 + 2M - (N-1)\delta_2)}}{2} \\ \qquad + \dfrac{(\delta_2 - 1)}{2}, \\ q_0 = \dfrac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\|e_{ij}(0)\|^2, \\ \tilde{\alpha} = \dfrac{\hat{\alpha}}{\check{\alpha}}. \end{cases} \tag{21}$$

An upper bound of $\mathcal{S}$ is

$$\mathcal{S} \leq \hat{\mathcal{S}} = \begin{cases} \mathbb{E}\Big\{\dfrac{N}{(N-1)\lambda_2^2(L)\check{\alpha}}\Big[2\mathfrak{F} + \sqrt{\dfrac{2q_0\hat{\alpha}}{N}}\Big]\Big\}, \\ \text{if } \mathfrak{F} \geq 0, \\ \mathbb{E}\Big\{\dfrac{N}{(N-1)\lambda_2^2(L)\check{\alpha}}\sqrt{\dfrac{2q_0\hat{\alpha}}{N}}\Big\}, \text{else}. \end{cases} \tag{22}$$

Remark 9. From Theorem 4, it is observed that stochastic coupling greatly influences the upper bounds of $\mathcal{C}$ and $\mathcal{S}$. In addition, the norm of uncertainties has effects on synchronization performance. Like Remark 5, one can also replace ϕ as follows: $\phi = d_N + \sqrt{(d_N - \frac{1}{2})^2 + \sum_{i\in\mathcal{V}} d_i(d_i - d_N) + \frac{1}{2}}$, $\phi = \max\{d(u) + m(u)|u \in \mathcal{V}\}$ or $\phi = \max\{d(u) + d(v) - |\mathcal{N}_u \cap \mathcal{N}_v|_c|(u,v) \in \mathcal{E}\}$. In addition, if $\lambda_2 \geq d_{N-1} - N + 3 > 0$ in Lemma 2, one can also have $\mathfrak{F} = \frac{\lambda_{\max}(A) + \iota + \vartheta + h_3}{d_{N-1} - N + 3} + \frac{(h_2^2\phi - 4c)}{4} + a(\tilde{\alpha} - 1)$ in Theorem 5. By this way, the upper bound of $\mathcal{C}$ and $\mathcal{S}$ can be obtained without knowing the eigenvalues of $\mathcal{G}$. The advantage for this is that we can estimate the upper bounds of mean control gain and convergence rate by only knowing partial information of the networks. For example, if one does not get the global coupling matrix and it is impossible to calculate the eigenvalues of $\mathcal{G}$. Fortunately, it is still achievable to estimate the upper bounds of mean control gain and convergence rate if one has the degree information for each node.

IV. Optimization methods for solving distributed robust (pinning) synchronization of networked systems with stochastic coupling

In this section, we will present two algorithms for solving the criteria for distributed robust (pinning) synchronization of networked systems with stochastic coupling in (3). The first one is aimed at presenting a convex optimization method for solving the criteria in Theorem 1 under $\mathcal{T} = \mathcal{V}$. The latter one is to present an EA-based optimization approach for solving the criteria in Theorem 3 under $\mathcal{T} \subset \mathcal{V}$.

A. Optimization problems for distributed (pinning) synchronization of networked systems with stochastic coupling in (3)

In order to measure the optimization results, we consider the transformation of the criteria in Theorems 1 and 3 into the following optimization problems. Taking into the criteria in (15) and (16), the optimization problems can be formulated as follows, respectively:

$$\begin{cases} [(\lambda_{\max}(A) + \iota + \vartheta + h_3)I_N - aLI_N - cL \\ +\frac{h_2^2}{4}L^2]L \leq 0, \\ \text{Assumption 2 should be satisfied,} \end{cases} \tag{23}$$

and

$$\begin{aligned} &\min \quad c \\ &\text{subject to} \\ &\begin{cases} [(\lambda_{\max}(A) + \iota + \vartheta + h_3)I_N - aL\mathcal{I} - cL \\ +\frac{h_2^2}{4}L^2]L \leq 0, \\ \text{Assumption 2 should be satisfied,} \end{cases} \end{aligned} \tag{24}$$

where $\mathcal{I}(i,j) = \begin{cases} 1, & \text{if} \quad i = j, \delta_{\mathcal{T}}(i) = 1 \\ 0, & \text{else} \end{cases}$.

For Theorem 1, the convex optimization method can be employed to solve (23). However, for Theorem 3, (24) can be solved by the convex optimization method once $\mathcal{I}$ is fixed. As mentioned in the introduction and main results, how to determine $\mathcal{I}$ is the key for solving (24). If $\mathcal{I}$ is chosen satisfactorily, the conservativeness of the results will be reduced. In the following, an EA-based algorithm will be adopted to solve the problem of distributed robust pinning synchronization of networked systems with stochastic coupling in (3), in which the convex optimization method is embedded into the framework of EAs.

B. An improved dynamic hybrid framework

Here, we adopt an improved dynamic hybrid framework (IDyHF) in [27] to solve the distributed robust pinning algorithm in (24). IDyHF is used to select driver nodes characterized by $\mathcal{I}$ and deal with the constraints in (24).

An improved dynamic hybrid framework (IDyHF) was proposed in [27]. IDyHF is a constraint optimization evolutionary algorithm (COEA), which is composed of a search approach and a constraint handling technique.

Algorithm 1 An EA-based optimization method in [35]

Begin

Generate a random population $P_n(n = 0)$ with SP individuals and $D = l + 1$ in Ψ. One individual from the population P_0 is initialized by the degree information [23]. Set $f_e = 0, n = 0$. /*The first l dimension is to represent $\mathcal{I}$, which follows the encoding scheme in [27]. The $l + 1$ dimension is to denote c*/

Calculate the number of feasible solutions (NFS) in P_n

while $f_e \leq f_{e,\max}$ **do**

Compute the objective value c and the constraint violation Σ according to (24) /*Σ can be computed by using Matlab and Yalmip [34] to make the inequalities satisfied*/

$\chi = \frac{SP-NFS}{SP}$; /*Calculate the portion of infeasible solution in P_n*/

$P_n =$ IDyHF(P_n) /*Update the solutions according to IDyHF*/

Update f_e;

$n = n + 1$.

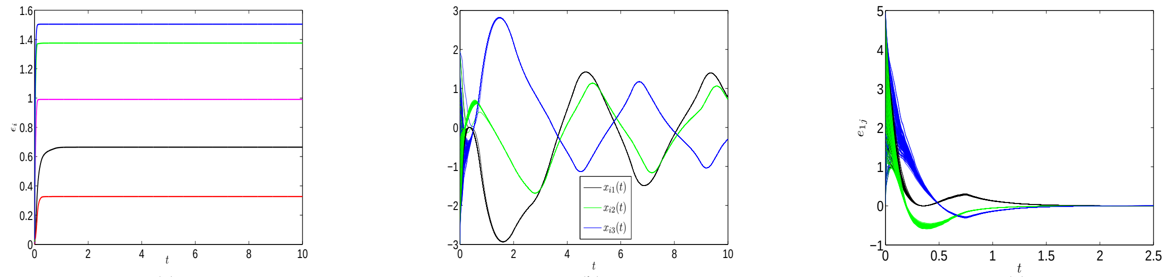
end while

End

C. An EA-based optimization method for distributed robust pinning synchronization of networked systems with stochastic coupling

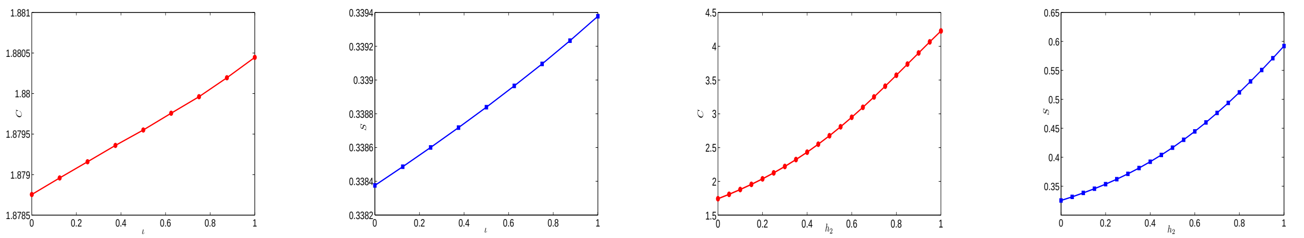
The main motivation of improving IDyHF by presenting an EA-based optimization method is that IDyHF suffers from its inefficiency in matrix computation of the constraints of (24) [35]. The convex optimization method has gained increasing attention due to its capabilities of dealing with matrix computation. Here, we utilize a hybrid optimization method [35], in which these two methods are combined into one unified optimization framework and thus their advantages are combined too.

Remark 10. It is also worth mentioning that the results in [27] have the following deficiencies: 1) the controllability is a local one and it is difficult to extend the results to the model with either uncertainties or stochastic coupling; 2) IDyHF is used to detect driver nodes and design the coupling strengths



(a) (b) (c)

Fig. 1. Synchronization results with parameter uncertainties and stochastic coupling when $l = 5$ and $N = 100$. (a) Control gains; (b) state trajectories; (c) synchronization errors.



(a) (b) (c) (d)

Fig. 2. The impacts of the bound of uncertainties and the intensity of stochastic coupling on $\mathcal{C}$ and $\mathcal{S}$. (a) The impact of ι on $\mathcal{C}$; (b) the impact of ι on $\mathcal{S}$; (c) the impact of h_2 on $\mathcal{C}$; (d) the impact of h_2 on $\mathcal{S}$.

between the states in networks and the desired state, which renders occupation of unnecessary computation resources. Different from [27], parameter uncertainties and stochastic coupling are taken into account in the model of this paper. In addition, adaptive control is used to design coupling strengths and thus coupling strengths are tuned adaptively, which makes the results more applicable.

Remark 11. In Algorithm 1, we initialize the population P_n by using the information of degree. Therefore, the method here will perform not worse than the method in [23]. In addition to the advantages pointed out in Remark 2 over the results without considering parameter uncertainties and stochastic coupling in [21]–[23], our results present a unified framework to deal with pinning synchronization of uncertain networks with stochastic coupling, and choosing driver nodes from the perspective of hybrid optimization including convex optimization and artificial intelligence.

V. Examples

In this section, one example is given to demonstrate the effectiveness of the proposed criteria and optimization methods.

The complex network is composed of an identical Hopfield neural network on each node [36]:

$$\begin{aligned} dx_i(t) &= [(A+\triangle A(t))x_i(t)+f(x_i,t)+m(x_i,t)+u_i(t) \\ &\quad + c\sum_{j\in\mathcal{N}_i}(x_j(t)-x_i(t))]dt \\ &\quad + \sum_{i\in\mathcal{N}_i}(g(x_j,t)-g(x_i,t))dv(t), i\in\mathcal{T}, \\ dx_i(t) &= [(A+\triangle A(t))x_i(t)+f(x_i,t)+m(x_i,t) \\ &\quad + c\sum_{j\in\mathcal{N}_i}(x_j(t)-x_i(t))]dt \\ &\quad + \sum_{j\in\mathcal{N}_i}(g(x_j,t)-g(x_i,t))dv(t), i\notin\mathcal{T}, \end{aligned} \tag{25}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t)]^T$, $f(x_i,t) = -Cx_i + Hh(x_i)$, A, C and H are given as follows:

$$A = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$H = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{pmatrix},$$

and $\triangle A(t) = \mathrm{diag}(|\iota\sin(t)|, |\iota\sin(t)|, |\iota\sin(t)|)$. Therefore, $\|\triangle A(t)\| \leq \iota$. Here, we choose $\iota = 0.0005$. $m_i(x_i,t) = \mathrm{diag}(h_3\tanh(x_{i1}), h_3\tanh(x_{i2}), h_3\tanh(x_{i3}))$. h_3 is set as $h_3 = 0.0005$. The nonlinear function $h(x_i) = [h(x_{i1}), h(x_{i2}), h(x_{i3})]^T$ is picked as $h(x_i) = \frac{(|x_i+1|-|x_i-1|)}{2}$. The function $g(x_i)$ in the stochastic coupling term is $g(x_i) = h_2 x_i(t)$. We set $h_2 = 0.2$. The α_i in $u_i(t)$ are chosen as $\alpha_i = 0.5$. The simulation time is set as $T = 10$. The step size of our algorithm is chosen as 0.005. The connecting matrix considered here is a scale-free network [2]. The growth starts from three nodes and no edges. At each step, a new node with three edges is added to the existing network. Repeating this method, we will generate a scale-free network, which satisfies "connected" condition. In order to satisfy Assumption 2 in (24), the following inequality should be feasible [37]:

$$\begin{bmatrix} 2(C+\vartheta) - \Phi * W^2 - 2\Delta & -H \\ -H^T & \Phi \end{bmatrix} < 0, \tag{26}$$

where $W = 1$.

We consider the distributed robust pinning synchronization of networked systems with stochastic coupling in (3), i. e., $\mathcal{I} \subset \mathcal{V}$. The parameter setting for IDyHF follows [27]. The initial interval for the population of IDyHF is $(0, N+1)$ for the first l dimension and $(0, \rho)$ for the last dimension, $\rho = 10$ when $N = 100$. The maximum number of fitness evaluation is $f_{e,\max} = D * \xi$, where D is the dimension size of the problem and $\xi = 1000$ is an adjustable parameter for balancing the tradeoff between complexities and accuracies. The running times of the EA-based algorithm are 20 times.

When $N = 100$ and the number of pinned nodes $l = 5$, the result of optimizing c achieved by the convex optimization method adopted in [10], [23] is 5.92, in which $\mathcal{I}$ is selected according the descending degree information. However, by means of the EA-based optimization method, the mean result of optimizing c is 3.72 and the minimum result of optimizing c is 3.67. The results indicate that the EA-based optimization method is reliable and even more accurate than the convex optimization method in [23]. The corresponding feasible solution for the minimum c is

$$\begin{aligned} & a = 36.4948, \Delta = 5.5936, \vartheta = 0.0146, \\ & \Psi = \mathrm{diag}\{6.6888, 7.3906, 5.7143\}. \end{aligned} \tag{27}$$

The adaptive control gains, synchronization errors and state trajectories are plotted in Fig. 1, which further validates the effectiveness of our main results.

In the following, the impacts of uncertainties and the intensity of stochastic coupling on $\mathcal{C}$ and $\mathcal{S}$ are illustrated by simulations. We use the same network as above and all the nodes are injected with distributed controllers. For showing the impacts of uncertainties the intensity of stochastic coupling on $\mathcal{C}$ and $\mathcal{S}$, we vary ι and h_2, respectively. The results are shown in Fig. 2. We find that increasing the bound of uncertainties and the intensity of stochastic coupling, $\mathcal{C}$ and $\mathcal{S}$ increase accordingly. The simulations verify the theoretical results in Theorem 4 well.

VI. Conclusions

In this paper, distributed robust (pinning) synchronization was investigated for a class of complex networks with parameter uncertainties and stochastic coupling. By employing the Lyapunov functional stability theory and the stochastic analysis technique, it was verified that such distributed robust (pinning) synchronization can be ensured in mean square sense if a set of matrix inequalities are solvable. Upper bounds of mean control gain and convergence rate were derived which show the effects of degree information, parameter uncertainties and stochastic coupling on the synchronization performance. The presented distributed robust (pinning) synchronization criteria were solved by a mixed optimization algorithm, which is based on a constraint optimization evolutionary algorithm. The obtained results were illustrated by a simulation example.

In the end, it is worth providing some future works. One should extend the results into the case of directed networks, which is more practical in applications. It is also of great importance to design controllers with fixed control gains for reducing the mean control gain by employing the optimal control theory [38].

VII. Appendix

The following proof is based on the results of [10], [21], [23], [28], [39].

A. Proof of Theorem 1

Proof: Let $e_{ij} = x_i - x_j, \forall i, j \in \mathcal{V}$. Define $x = [x_1^T, ..., x_N^T]^T \in \mathbb{R}^{nN}$, $y = [y_1^T, ..., y_N^T]^T \in \mathbb{R}^{nN}$ and $z = [z_1^T, ..., z_N^T]^T \in \mathbb{R}^{nN}$, where $y_i = \sum_{j\in\mathcal{N}_i} e_{ji}$, $z_i = \sum_{j\in\mathcal{N}_i} \tilde{g}_{ji}$ and $\tilde{g}_{ji} = g(x_j,t) - g(x_i,t)$. Take the Lyapunov candidate as follows:

$$V(t) = \frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T e_{ij} + \sum_{i\in\mathcal{V}} \frac{1}{2\alpha_i}(\epsilon_i(t) - a)^2, \tag{28}$$

where a is a positive constant to be determined.

By the Itô-differential formula [39] and the Appendix, the operator $\mathscr{L}$ is computed according to (3):

$$\begin{aligned} \mathscr{L}V(t) \; = \; & \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T \Big\{ (A + \triangle A(t))(x_i - x_j) \\ & + f(x_i,t) - f(x_j,t) + m(x_i,t) - m(x_j,t) \\ & + c\sum_{k\in\mathcal{N}_i}(x_k - x_i) - c\sum_{r\in\mathcal{N}_j}(x_r - x_j) \Big\} \\ & + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T \Big\{ \epsilon_i(t)[\sum_{k\in\mathcal{N}_i}(x_k - x_i)] \\ & - \epsilon_j(t)[\sum_{r\in\mathcal{N}_j}(x_r - x_j)] \Big\} \\ & + \sum_{i\in\mathcal{V}}(\epsilon_i(t) - a)[\sum_{j\in\mathcal{N}_i} e_{ij}]^T[\sum_{j\in\mathcal{N}_i} e_{ij}] \\ & + \frac{1}{4} z^T (L \otimes I_n) z. \end{aligned} \tag{29}$$

The following equalities or inequalities are true:

$$
\begin{aligned}
&\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T e_{ij} = x^T(L\otimes I_n)x,\\
&\sum_{i\in\mathcal{V}}(\sum_{j\in\mathcal{N}_i} e_{ij})^T(\sum_{j\in\mathcal{N}_i} e_{ij}) = x^T(L^2\otimes I_n)x,\\
&\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T\epsilon_i(t)\sum_{k\in\mathcal{N}_i} e_{ki}\\
=\ & -\sum_{i\in\mathcal{V}}\epsilon_i(t)[\sum_{j\in\mathcal{N}_i} e_{ij}]^T[\sum_{j\in\mathcal{N}_i} e_{ij}],\\
&\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T[c\sum_{k\in\mathcal{N}_i} e_{ki} - c\sum_{r\in\mathcal{N}_j} e_{rj}]\\
=\ & -c\sum_{i\in\mathcal{V}}[\sum_{j\in\mathcal{N}_i} e_{ij}]^T[\sum_{j\in\mathcal{N}_i} e_{ij}]\\
=\ & -cx^T(L^2\otimes I_n)x,\\
&\frac{1}{4}z^T(L\otimes I_n)z\\
\le\ & \frac{h_2^2}{4}y^T(L\otimes I_n)y\\
=\ & \frac{h_2^2}{4}x^T(L^3\otimes I_n)x. \qquad (30)
\end{aligned}
$$

Utilizing the fact of $\|\triangle A(t)\| \le \iota$, Assumptions 1 and 2, (29) and (30), we have

$$
\begin{aligned}
&\mathscr{L}V(t)\\
\overset{(11)}{\le}\ & -\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\Delta e_{ij}^T e_{ij} + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\vartheta e_{ij}^T e_{ij}\\
& + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}(\lambda_{\max}(A)+\iota+h_3)e_{ij}^T e_{ij}\\
& -(a+c)\sum_{i\in\mathcal{V}}[\sum_{j\in\mathcal{N}_i} e_{ij}]^T[\sum_{j\in\mathcal{N}_i} e_{ij}]\\
& + \frac{h_2^2}{4}x^T(L^3\otimes I_n)x\\
\overset{(14)}{\le}\ & -\Delta x^T(L\otimes I_n)x. \qquad (31)
\end{aligned}
$$

Therefore, it follows from (31) that $\mathbb{E}\mathscr{L}V(t) \le 0$. According to Theorems 2.2 and 2.3 of [39] and the mean square stability of the Lyapunov function in [40], the distributed robust synchronization of uncertain networked systems with stochastic coupling in (3) can be achieved in mean square. This completes the proof. ■

B. Proof of Theorem 2

Proof: According to (31) of Theorem 2, we have

$$
\begin{aligned}
&\mathscr{L}V(t)\\
\le\ & -\Delta x^T(L\otimes I_n)x + x^T\{[(\lambda_{\max}(A)+\iota+\vartheta+h_3)I_N\\
& -aL - cL + \frac{h_2^2}{4}L^2]L\otimes I_n\}x. \qquad (32)
\end{aligned}
$$

There exists a unitary matrix U such that $L = U\Lambda U^T$ [10], [29], where $\Lambda = \text{diag}\{\lambda_1(L), \lambda_2(L), \ldots, \lambda_N(L)\} = \text{diag}\{0, \lambda_2(L), \ldots, \lambda_N(L)\}$, $U = [u_1, u_2, \ldots, u_N]$, and $u_1 = 1/\sqrt{N}[1, 1, \ldots, 1]^T$. We consider the transformation $w(t) = (U^T\otimes I_n)x(t) = [w_1^T(t), w_2^T(t), \ldots, w_N^T(t)]^T$, where $w_i(t) \in \mathbb{R}^n$ $(i \in \mathcal{V})$. Therefore, we have

$$
\begin{aligned}
&x^T[(\lambda_{\max}(A)+\vartheta+\iota+h_3)L\otimes I_n]x\\
=\ & w^T(U^T\otimes I_n)[(\lambda_{\max}(A)+\vartheta+\iota\\
& +h_3)L\otimes I_n](U\otimes I_n)w\\
=\ & \sum_{i=2}^N w_i^T[(\lambda_{\max}(A)+\vartheta+\iota+h_3)\lambda_i(L)]w_i. \qquad (33)
\end{aligned}
$$

Similarly, one gets

$$
\begin{aligned}
&-ax^T(L^2\otimes I_n)x\\
=\ & -aw^T(U^T\otimes I_n)(L^2\otimes I_n)(U\otimes I_n)w\\
=\ & -aw^T(U^TL^2U\otimes I_n)w\\
=\ & -a\sum_{i=2}^N\lambda_i^2(L)w_i^Tw_i, \qquad (34)
\end{aligned}
$$

$$
-cx^T(L^2\otimes I_n)x = -c\sum_{i=2}^N\lambda_i^2(L)w_i^Tw_i, \qquad (35)
$$

and

$$
\frac{h_2^2}{4}x^T(L^3\otimes I_n)x = \frac{h_2^2}{4}\sum_{i=2}^N\lambda_i^3(L)w_i^Tw_i. \qquad (36)
$$

Combining (32)-(36) yields that

$$
\begin{aligned}
&x^T\{[(\lambda_{\max}(A)+\vartheta+\iota+h_3)I_N\\
& -aL - cL + \frac{h_2^2}{4}L^2]L\otimes I_n\}x\\
=\ & \sum_{i=2}^N w_i^T\lambda_i(L)[\lambda_{\max}(A)+\vartheta+\iota+h_3-a\lambda_i(L)\\
& -c\lambda_i(L) + \frac{h_2^2}{4}\lambda_i^2(L)]w_i \le 0. \qquad (37)
\end{aligned}
$$

Therefore, it follows from (32) and (37) that $\mathbb{E}\mathscr{L}V(t) \le 0$. Similar to Theorem 1, the robust distributed synchronization of uncertain networked systems with stochastic coupling in (3) can be achieved in mean square. This completes the proof. ■

C. Proof of Theorem 3

Proof: Consider the following Lyapunov candidate:

$$
V(t) = \frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^Te_{ij} + \sum_{i\in\mathcal{T}}\frac{1}{2\alpha_i}(\epsilon_i(t)-a)^2, \qquad (38)
$$

where a is a positive constant to be determined.

The operator $\mathscr{L}$ is calculated as follows:

$$\begin{aligned}\mathscr{L}V(t) &= \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} e_{ij}^T\Big\{(A+\triangle A(t))(x_i-x_j)\\ &+(f(x_i,t)-f(x_j,t))+(m(x_i,t)-m(x_j,t))\\ &+c\sum_{k\in\mathcal{N}_i}(x_k-x_i)-c\sum_{r\in\mathcal{N}_j}(x_r-x_j)\Big\}\\ &+\frac{1}{2}\sum_{i\in\mathcal{T}}\sum_{j\in\mathcal{N}_i} e_{ij}^T\Big\{\epsilon_i(t)[\sum_{k\in\mathcal{N}_i}(x_k-x_i)]\\ &-\epsilon_j(t)[\sum_{r\in\mathcal{N}_j}(x_r-x_j)]\Big\}\\ &+\sum_{i\in\mathcal{T}}(\epsilon_i(t)-a)[\sum_{j\in\mathcal{N}_i}e_{ij}]^T[\sum_{j\in\mathcal{N}_i}e_{ij}]\\ &+\frac{1}{4}z^T(L\otimes I_n)z. \end{aligned}\tag{39}$$

Utilizing the fact of $\|\triangle A(t)\|\le\iota$, Assumptions 1 and 2 and (39), we have

$$\begin{aligned}&\mathscr{L}V(t)\\ \le\ & -\Delta x^T(L\otimes I_n)x+x^T\{[(\lambda_{\max}(A)+\iota+\vartheta+h_3)I_N\\ &-aL\mathcal{I}-cL+\frac{h_2^2}{4}L^2]L\otimes I_n\}x\\ \le\ & -\Delta x^T(L\otimes I_n)x,\end{aligned}\tag{40}$$

where $\mathcal{I}(i,j)=\begin{cases}1, & \text{if}\quad i=j,\delta_{\mathcal{T}}(i)=1\\ 0, & \text{else}\end{cases}$.

Therefore, it follows from (40) that $\mathbb{E}\mathscr{L}V(t)\le 0$. Similar to Theorem 1, the robust distributed pinning synchronization of uncertain networked systems with stochastic coupling in (3) can be achieved in mean square. This completes the proof. ■

D. Proof of Theorem 4

Proof: By carrying out integration of (5), the following equality holds:

$$\{\sum_{i\in\mathcal{V}}\int_0^\infty d\epsilon_i(t)\}=\sum_{i\in\mathcal{V}}\int_0^\infty\alpha_i[\sum_{j\in\mathcal{N}_i}e_{ij}]^T[\sum_{j\in\mathcal{N}_i}e_{ij}]dt.\tag{41}$$

Therefore, $\mathcal{C}$ can be calculated according to (6) and (41):

$$\mathcal{C}=\mathbb{E}\{\frac{1}{N}\int_0^\infty x^T(L\Theta L\otimes I_n)xdt\},\tag{42}$$

where $\Theta=\text{diag}\{\alpha_1,...,\alpha_N\}$.

According to Corollary 1, one has

$$\begin{aligned}\mathbb{E}\mathscr{L}V &\le \mathbb{E}\{x^T(t)[(\lambda_{\max}(A)+\iota+\vartheta+h_3)I_N-aL\\ &-cL+\frac{h_2^2}{4}\lambda_N(L)L]Lx(t)\}.\end{aligned}\tag{43}$$

Now we aim to show that the following inequality holds:

$$\begin{aligned}&[(a+c-\frac{h_2^2}{4}\lambda_N(L))-\frac{\lambda_{\max}(A)+\iota+\vartheta+h_3}{\lambda_2(L)}]L^2\\ \le\ &[(a+c-\frac{h_2^2}{4}\lambda_N(L))L\\ &-(\lambda_{\max}(A)+\iota+\vartheta+h_3)I_N]L.\end{aligned}\tag{44}$$

Pick p_i be the eigenvector of L associated with the eigenvalue $\lambda_i(L)$ sorted by $0=\lambda_1(L)\le\lambda_2(L)\le\lambda_3(L)\le...\le\lambda_N(L)$. For any $p\in\mathbb{R}^N$, p can be written as $p=\sum_{i\in\mathcal{V}}r_ip_i,(i\in\mathcal{V})$. The eigenvectors are chosen such that they correspond to the same eigenvalue with multiplicity such that $p_1,...,p_N$ compose an orthogonal standard basis of $\mathbb{R}^N$. We get $p_i^Tp_j=0,\forall i\ne j$.

Note that

$$\begin{aligned}&p^T\Big\{[(a+c-\frac{h_2^2}{4}\lambda_N(L))-\frac{\lambda_{\max}(A)+\iota+\vartheta+h_3}{\lambda_2(L)}]L^2\\ &-[(a+c-\frac{h_2^2}{4}\lambda_N(L))L\\ &-(\lambda_{\max}(A)+\iota+\vartheta+h_3)I_N]L\Big\}p\\ =\ &\sum_{i\in\mathcal{V}}p_i^Tp_i\Big[(a+c-\frac{h_2^2}{4}\lambda_N(L)\\ &-\frac{\lambda_{\max}(A)+\iota+\vartheta+h_3}{\lambda_2(L)})\lambda_i^2(L)\\ &-(a+c-\frac{h_2^2}{4}\lambda_N(L))\lambda_i^2(L)\\ &+(\lambda_{\max}(A)+\iota+\vartheta+h_3)\lambda_i(L)\Big]v_i^2\\ &+2\sum_{i\in\mathcal{V}}\sum_{j>i}p_i^T[(a+c-\frac{h_2^2}{4}\lambda_N(L)\\ &-\frac{\lambda_{\max}(A)+\iota+\vartheta+h_3}{\lambda_2(L)})L^2-((a+c-\frac{h_2^2}{4}\lambda_N(L))L\\ &-(\lambda_{\max}(A)+\iota+\vartheta+h_3)I_N)L]p_jr_ir_j\\ =\ &\sum_{i=2}^N p_i^Tp_i[-\frac{1}{\lambda_2(L)}\lambda_i(L)+1](\lambda_{\max}(A)\\ &+\iota+\vartheta+h_3)\lambda_i(L)r_i^2\\ \le\ &0.\end{aligned}\tag{45}$$

Therefore, (44) is true. Thus, we find

$$\begin{aligned}\mathcal{C} &= \mathbb{E}\{\frac{1}{N}\int_0^\infty x^T(t)(L\Theta L\otimes I_n)x(t)dt\}\\ &\le -\mathbb{E}\{\frac{\hat{\alpha}\lambda_2(L)}{N[(a+c-\frac{h_2^2}{4}\lambda_N(L))\lambda_2(L)-\mathfrak{X}]}[\int_0^\infty\mathscr{L}Vdt]\}\\ &= \mathbb{E}\{\frac{\hat{\alpha}\lambda_2(L)}{N[(a+c-\frac{h_2^2}{4}\lambda_N(L))\lambda_2(L)-\mathfrak{X}]}[V_0-V_\infty]\}\\ &= \mathbb{E}\{\frac{\hat{\alpha}\lambda_2(L)}{N[(a+c-\frac{h_2^2}{4}\lambda_N(L))\lambda_2(L)-\mathfrak{X}]}[q_0\\ &+\sum_{i\in\mathcal{V}}\frac{1}{2\check{\alpha}}(2a\epsilon_{i,\infty}-\epsilon_{i,\infty}^2)]\}\\ &\le \mathbb{E}\{\frac{\hat{\alpha}\lambda_2(L)}{N[(a+c-\frac{h_2^2}{4}\lambda_N(L))\lambda_2(L)-\mathfrak{X}]}\\ &\times[q_0+\frac{aN}{\check{\alpha}}\mathcal{C}-\frac{N}{2\check{\alpha}}\mathcal{C}^2]\},\end{aligned}\tag{46}$$

where $\mathfrak{X}=\lambda_{\max}(A)+\iota+\vartheta+h_3, V_0=V(0)$, $V_\infty=\lim_{t\to\infty}V(t)$ and $q_0=\frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\|e_{ij}(0)\|^2$. Let $\tilde{\alpha}=\frac{\hat{\alpha}}{\check{\alpha}}$. By solving the last inequality in (46), an upper bound of $\mathcal{C}$

can be obtained

$$\mathcal{C} \leq \bar{\mathcal{C}} = \mathbb{E}\{\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \frac{2q_0\hat{\alpha}}{N}}\}, \tag{47}$$

where

$$\begin{aligned}\mathfrak{F} &= \frac{4\lambda_{\max}(A) + 4\iota + 4\vartheta + 4h_3 + \lambda_2(L)(h_2^2\lambda_N(L) - 4c)}{4\lambda_2(L)} \\ &\quad + a(\tilde{\alpha} - 1).\end{aligned} \tag{48}$$

According to the inequality $\sqrt{a^2+b^2} \leq a+b$, where a and $b \in \mathbb{R}$ are nonnegative real numbers, we have

$$\begin{aligned}\bar{\mathcal{C}} &\leq \hat{\mathcal{C}} = \mathbb{E}\Big\{\mathfrak{F} + \Big|\mathfrak{F}\Big| + \sqrt{\frac{2q_0\hat{\alpha}}{N}}\Big\} \\ &= \begin{cases} \mathbb{E}\Big\{2\mathfrak{F} + \sqrt{\frac{2q_0\hat{\alpha}}{N}}\Big\}, & \text{if } \mathfrak{F} \geq 0, \\ \mathbb{E}\sqrt{\frac{2q_0\hat{\alpha}}{N}}, & \text{else.}\end{cases}\end{aligned} \tag{49}$$

In the following, we are in a position to estimate an upper bound for $\mathcal{S}$. Denote $\mathfrak{D} = (d_{ij})$ with $d_{ij} = -\frac{1}{N}$ if $i \neq j$ and $d_{ii} = 1 - \frac{1}{N} (\forall i = 1, 2, ..., N)$ and $\mathfrak{W} = \frac{1}{m-1}\mathfrak{D}^T\mathfrak{D}$. Thus, $\mathcal{S}$ can be calculated as follows:

$$\mathcal{S} = \mathbb{E}\int_0^\infty x^T(t)(\mathfrak{W} \otimes I_n)x(t)dt. \tag{50}$$

The following inequality is true according to [21]:

$$\mathfrak{W} \leq \frac{1}{(N-1)\lambda_2^2(L)}L^2.$$

From (7), (42) and (50), it can be checked that

$$\begin{aligned}\mathcal{S} &\leq \mathbb{E}\Big\{\frac{1}{(N-1)\lambda_2^2(L)}\int_0^\infty x^T(t)(L\Theta L \otimes I_n)x(t)dt\Big\} \\ &\leq \mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\breve{\alpha}}\mathcal{C}\Big\} \\ &\leq \mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\breve{\alpha}}\Big[\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \frac{2q_0\hat{\alpha}}{N}}\Big]\Big\}.\end{aligned} \tag{51}$$

One can further have

$$\bar{\mathcal{S}} \leq \mathbb{E}\Bigg\{\frac{N}{(N-1)\lambda_2^2(L)\breve{\alpha}}\Big[\mathfrak{F} + |\mathfrak{F}| + \sqrt{\frac{2q_0\hat{\alpha}}{N}}\Big]\Bigg\}. \tag{52}$$

Hence, one has from (52):

$$\hat{\mathcal{S}} = \begin{cases} \mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\breve{\alpha}}\Big[2\mathfrak{F} + \sqrt{\frac{2q_0\hat{\alpha}}{N}}\Big]\Big\}, \\ \text{if } \mathfrak{F} \geq 0, \\ \mathbb{E}\Big\{\frac{N}{(N-1)\lambda_2^2(L)\breve{\alpha}}\sqrt{\frac{2q_0\hat{\alpha}}{N}}\Big\}, \text{else.}\end{cases} \tag{53}$$

This completes the proof. ■

E. Itô's formula

Itô's formula is given in [39] as Theorem 6.4, which is given as follows:

Theorem 6.4 in [39]. Let $x(t)$ be a d-dimensional Itô process on $t \geq 0$ with the stochastic differential

$$dx(t) = f(t)dt + g(t)dv(t). \tag{54}$$

Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R})$, where $C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R})$ denotes the family of all real-valued functions $V(x,t)$ defined on $\mathbb{R}^d \times \mathbb{R}^+$ which are continuously twice differentiable in $x \in \mathbb{R}^d$ and one differentiable in $\mathbb{R}^+$. Then $V(x(t),t)$ is again an Itô process with the stochastic differential given by

$$\begin{aligned}dV(x(t),t) &= [V_t(x(t),t) + V_x(x(t),t)f(t) \\ &\quad + \frac{1}{2}\text{trace}(g^T(t)V_{xx}(x(t),t)g(t))]dt \\ &\quad + V_x(x(t),t)g(t)dv(t) \\ &= \mathscr{L}V(t) + V_x(x(t),t)g(t)dv(t),\end{aligned} \tag{55}$$

where

$$\begin{aligned}&\mathscr{L}V(t) = V_t(x(t),t) + V_x(x(t),t)f(t) \\ &\qquad + \frac{1}{2}\text{trace}(g^T(t)V_{xx}(x(t),t)g(t)), \\ &V_t = \frac{\partial V}{\partial t}, V_x = (\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_d}), \\ &V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{d\times d}.\end{aligned} \tag{56}$$

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