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Review

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Dedicated to Prof. Tatsien Li on the occasion of his 75th birthday.

Chaos synchronization in fractional differential systems

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This paper presents a brief overview of recent developments in chaos synchronization in coupled fractional differential systems, where the original viewpoints are retained. In addition to complete synchronization, several other extended concepts of synchronization, such as projective synchronization, hybrid projective synchronization, function projective synchronization, generalized synchronization and generalized projective synchronization in fractional differential systems, are reviewed.

1. Introduction

Fractional calculus was formulated in 1695, shortly after the development of classical calculus. The earliest systematic studies were attributed to Liouville, Riemann, Leibniz, etc. [1,2]. An outline of the simple history of fractional calculus can be found in Machado *et al.* [3].

For a long time, fractional calculus was regarded as a pure mathematical realm without real applications. But, in recent decades, this has changed. It was found that fractional calculus is useful, even powerful, for modelling viscoelasticity [4], electromagnetic waves [5], boundary layer effects in ducts [6], quantum evolution of complex systems [7], distributed-order dynamical systems [8] and others. That is, the fractional differential systems are more suitable to describe physical phenomena that have memory and genetic characteristics.

On the other hand, it is known that chaos is ubiquitous in most nonlinear systems. Owing to the various backgrounds of scientific communities, there exist several non-equivalent mathematical definitions of chaos [9]. However, the criterion that the positivity of the largest Lyapunov exponent implies chaos is generally accepted.

Similar to a nonlinear differential system, a nonlinear fractional differential system may also have complex dynamics, such as chaos and bifurcation. Since 1999, the oscillatory behaviours in fractional dynamical systems have attracted considerable attention [10–12]. These studies are useful in the design and implementation of fractional-order oscillators. Such fractional dynamics were observed mainly through numerical simulations. Detecting the mathematical machinery of fractional dynamics takes a long time. To date, chaotic motions have been found in fractional systems, for example in the fractional versions of the Chua circuit [13], Duffing system [14], Lorenz system [15], Chen system [16,17], Rössler system [18], Arneodo system [19] and the Lü system [20,21]. More recently, Li *et al.* [22] have studied the definition of Lyapunov exponents of fractional differential systems, which are often used to detect chaos in fractional differential systems. On the other hand, both frequency-domain methods and time-domain methods are used for computing the fractional differential systems [16,23,24]. It was noted that 'using the frequency-domain approximation methods can conceal chaotic behaviour for a chaotic fractional-order system or display chaos for a non-chaotic one' [23]. In comparison, the time-domain approximation method provides more effective numerical simulations to recognize chaos in fractional differential systems [16,23].

The evolution of a chaotic system sensitively depends on its initial conditions or parameter values, in that two identical systems starting from slightly different initial conditions or parameter values may separate exponentially in time. For quite a long time, there was doubt about the possibility of synchronization between two chaotic systems owing to such sensitive dependence. But, it turned out that chaos synchronization was not only possible but also actually not difficult. Starting with Pecora & Carroll [25], chaos synchronization has been extensively and intensively studied [26]. The simplest form of synchronization in coupled chaotic systems is identical synchronization, also referred to as complete synchronization (CS) [26,27]. Yet, there are other complex forms of chaos synchronization, depending on the coupling configurations and systems, such as projective synchronization (PS), hybrid projective synchronization (HPS), function projective synchronization (FPS), generalized synchronization (GS), generalized projective synchronization (GPS) and phase synchronization.

In recent years, synchronization of fractional chaotic systems has started to attract increasing attention because of its potential applications in secure communication and control processing [28]. It was found that some fractional differential systems with order less than 1 may behave chaotically in some way similar to their integer-order counterparts, in which chaos can also be synchronized. Synchronization of two coupled dynamical systems is essentially the stability of the zero solution of their error dynamical system. Therefore, some (stable and unstable) limit sets can be synchronized, as emphasized by Li & Deng [29]. For example, the fractional Brusselator with an effective dimension less than 1 has a limit cycle, first observed by Wang & Li [30], and the synchronization of this limit cycle in a coupled fractional Brusselators can be easily achieved [31].

Some remarks are given in order. On the one hand, as mentioned earlier, synchronization of chaotic systems is actually analysed by examining the stability of the zero solution of their error dynamical system, using for example Lyapunov functions. However, it is difficult or even impossible to explicitly construct Lyapunov functions for fractional differential systems. That is, not all analytical methods for synchronization of classical chaotic systems can be directly extended to the fractional-order setting. On the other hand, the lowest system dimension for the existence of chaos in autonomous ordinary differential equations is 3. Yet, chaos can exist in less than 3 efficient dimensions in fractional-order systems; for example, the smallest efficient dimension for the fractional Lorenz system to have a chaotic attractor is 1.07 [32], in which the appearance of the chaotic attractor depends on the efficient dimension displayed. Moreover, the randomness of a chaotic motion in a fractional chaotic system

3

can effectively hide information so as to significantly increase the security level in secure communication thanks to more system parameters, including the intrinsic parameter(s) and the order parameter(s).

It is worth mentioning that some new phenomena, such as riddled basins of attraction [33,34], attractor bubbling [35] and on–off intermittency [36–38], have been found in the studies of chaos synchronization problems in recent years. Here, riddled basins of attraction is the term used to indicate that every point in an attractor's basin has pieces of another attractor's basin arbitrarily nearby [39]. This discovery has attracted great interest from scientists and has inspired them to enter the field of chaotic synchronization [40–48]. It is now known that the transition of synchronized states is due to random or chaotic perturbations in some parameters of the system [34]. All this is observed or investigated in the realm of ordinary differential systems. Such phenomena appearing in classical systems possibly arise in fractional systems as well, but no reports in this respect are available in today's literature. Because this is a general review article, such intriguing problems are not discussed.

The present review has collected most key references on chaos synchronization of fractional differential systems, where the viewpoints of the original contributors are retained. The remainder of the article is organized as follows. In §2, some basic concepts of chaos synchronization of fractional differential systems are introduced. Section 3 reviews the developments in chaos synchronization of coupled fractional-order chaotic systems. The last section concludes this paper.

2. Some basic concepts

Let R, R_+ and Z_+ be the set of real numbers, the set of positive real numbers and the set of positive integer numbers, respectively.

Among several definitions for the fractional derivative, the Caputo derivative and the Riemann–Liouville derivative are most familiar. Engineers like to use the former, whereas physicists and mathematicians often choose the latter. In this paper, the involved fractional derivatives mean the Caputo derivative or the Riemann–Liouville derivative. These two fractional derivatives are not equivalent and have their respective applications [49–52].

Definition 2.1. The αth order *Caputo derivative* of a function $f(t)$ is defined by

$$_CD_{0,t}^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau, \tag{2.1}$$

where $m-1 < \alpha \le m \in Z_+$ and $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The αth order *Riemann–Liouville derivative* of a function $f(t)$ is defined by

$$_{RL}D_{0,t}^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\, d\tau, \tag{2.2}$$

where $m-1 \le \alpha < m \in Z_+$.

Among various kinds of synchronization, CS of two coupled fractional differential systems is the same as that of two coupled conventional differential systems, which are introduced in the appendices A–D. In this article, the fractional partially linear system is used to define CS, PS and HPS.

Definition 2.3. A *fractional partially linear system* is a set of fractional differential equations where the state vector can be decomposed into two parts, (u,z), in which the equation for z is nonlinear in u while that for the fractional derivative of the vector u is linear in z through a matrix M, which depends only on z, in the form of

$$\left.\begin{aligned}\frac{d^\alpha u}{dt^\alpha} &= M(z) \cdot u \\ \frac{d^\alpha z}{dt^\alpha} &= f(u,z),\end{aligned}\right\} \tag{2.3}$$

and

where α is the fractional order and d^α/dt^α denotes either $_CD_{0,t}^\alpha$ or $_{RL}D_{0,t}^\alpha$.

Now, some basic definitions about synchronization are given.

Consider two copies of a partially linear system, which are coupled through the variable z in the following manner:

$$
\left.
\begin{aligned}
\frac{d^{\alpha} u_m}{dt^{\alpha}} &= M(z) \cdot u_m, \\
\frac{d^{\alpha} z}{dt^{\alpha}} &= f(u_m, z) \\
\frac{d^{\alpha} u_s}{dt^{\alpha}} &= M(z) \cdot u_s,
\end{aligned}
\right\}
\tag{2.4}
$$

and

where α is the fractional order, and $u_m \in R^n$ and $u_s \in R^n$ are the state vectors of the drive and response systems, respectively.

Definition 2.4. The two coupled systems in (2.4) are said to reach CS if

$$
\lim_{t \to \infty} \| u_m - u_s \| = 0,
\tag{2.5}
$$

where $\| \cdot \|$ denotes a norm (usually, the Euclidean norm) of a vector.

Here, CS is defined through the fractional partially linear system (2.4) just for simplicity and convenience. CS has other coupled forms; see appendices A–D for more details.

Definition 2.5. The two coupled systems (2.4) are said to reach PS if, for the initial conditions, there is a constant β such that

$$
\lim_{t \to \infty} \| u_m - \beta u_s \| = 0.
\tag{2.6}
$$

Definition 2.6. The two coupled systems (2.4) are said to reach HPS, if there exist n constants h_i $(1 \le i \le n)$ such that

$$
\lim_{t \to \infty} \| u_m - H u_s \| = 0,
\tag{2.7}
$$

where $H = \mathrm{diag}(h_1, h_2, \ldots, h_n)$ is called the scaling matrix and $h_1, h_2, \ldots, h_n$ are the scaling factors.

Remark 2.7. Definitions 2.3 and 2.6 are generalized from the integer-order partially linear systems defined by Mainieri & Rehacek [53] and the HPS defined by Hu *et al.* [54], respectively.

Now, consider the following coupled drive and response systems:

$$
\left.
\begin{aligned}
\frac{d^{q_m} x}{dt^{q_m}} &= f(x) \\
\frac{d^{q_s} y}{dt^{q_s}} &= g(y) + u(x, y),
\end{aligned}
\right\}
\tag{2.8}
$$

and

where q_m and q_s are fractional orders satisfying $0 < q_m \le 1$ and $0 < q_s \le 1$, respectively, $f, g : R^n \to R^n$ are two continuous nonlinear vector functions, and $u(x, y) : R^n \times R^n \to R^n$ is a controller to be designed.

Definition 2.8. The two coupled systems (2.8) are said to reach FPS if there exists a controller $u(x, y)$ such that

$$
\lim_{t \to \infty} \| y - K(x) x \| = 0,
\tag{2.9}
$$

where $K(x) = \mathrm{diag}(k_1(x), k_2(x), \ldots, k_n(x))$ with $k_i(x)$ being continuous functions, $i = 1, 2, \ldots, n$.

Next, considering the following two unidirectionally coupled fractional systems:

$$
\left.
\begin{aligned}
\frac{d^p x}{dt^p} &= f(x) \\
\frac{d^q y}{dt^q} &= g(y, u) = g(y, h(x)),
\end{aligned}
\right\}
\tag{2.10}
$$

and

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n$, $y = (y_1, y_2, \ldots, y_m)^{\mathrm{T}} \in R^m$,
$d^p x / dt^p = (d^{p_1} x_1 / dt^{p_1}, \ d^{p_2} x_2 / dt^{p_2}, \ldots, d^{p_n} x_n / dt^{p_n})^{\mathrm{T}}$,

$\mathrm{d}^q y/\mathrm{d}t^q = (\mathrm{d}^{q_1} y_1/\mathrm{d}t^{q_1}, \mathrm{d}^{q_2} y_2/\mathrm{d}t^{q_2}, \ldots, \mathrm{d}^{q_m} y_m/\mathrm{d}t^{q_m})^{\mathrm{T}}, p_i, q_j \in R_+, p = (p_1, \ldots, p_n),$
$q = (q_1, \ldots, q_m), f : R^n \to R^n, g : R^m \times R^k \to R^m$
and $u(t) = (u_1(t), u_2(t), \ldots, u_k(t))^{\mathrm{T}}$ with $u_j(t) = h_j(x(t, x_0))$.

Definition 2.9. The two coupled systems (2.10) are said to reach GS if there exist a transformation $H : R^n \to R^m$, a manifold $M = \{(x, y) : y = H(x)\}$, and a subset $B = B_x \times B_y \subset R^n \times R^m$ with $M \subset B$, such that, with any initial conditions in B, one has

$$(x, y) \to M \quad (t \to \infty). \tag{2.11}$$

Furthermore, consider the following two coupled fractional systems:

$$\left. \begin{aligned} \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} &= f(x) \\ \frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} &= g(y, h(x, y)), \end{aligned} \right\} \tag{2.12}$$

and

where α is the fractional order, $x \in R^n, y \in R^n, f : R^n \to R^n, h : R^n \times R^n \to R^n, g : R^n \times R^n \to R^n$ and $g(x, 0) \equiv f(x)$.

Definition 2.10. The two coupled systems (2.12) are said to reach GPS if there exists a constant $\sigma \in R - \{0\}$ such that

$$\lim_{t \to \infty} \|x - \sigma y\| = 0. \tag{2.13}$$

Note that definitions 2.5–2.10 (whose original viewpoints are retained) have some relations but their synchronizations appear in the fractional differential systems with different couplings.

3. Synchronization of fractional chaotic systems

In this section, typical methods for various synchronizations of two coupled fractional chaotic systems are reviewed and discussed.

(a) Complete synchronization

CS can be achieved by means of different coupling schemes. In general, CS can roughly be divided into two categories: unidirectional coupling (drive–response coupling) configuration and bidirectional configuration. In a unidirectional coupling configuration, the evolution of one of the coupled systems is not influenced by the other via coupling. On the contrary, in a bidirectional coupling configuration both systems mutually influence each other [55]. CS is the simplest setting in synchronization of chaotic systems and is easy to apply in practice.

In the following, numerical and analytical methods for CS of the fractional differential systems are introduced.

(i) Numerical methods

There are two popular numerical methods for computing the chaotic attractors of fractional systems and their synchronization diagrams. One is the frequency-domain method and the other is the time-domain method. The former is mainly used to approximate the transfer function $1/s^\alpha$. The latter is used to directly approximate the temporal fractional derivatives. In the study by Li *et al.* [56], the frequency-domain technique was used to numerically analyse CS of two identical fractional chaotic systems via a one-way coupling configuration (A 1) (see appendix A), with $k = c\Gamma$, where $c > 0$ is the coupling strength and $\Gamma \in R^{n \times n}$ is a constant 0–1 matrix linking the coupling variables. CS of many other fractional chaotic systems via one-way coupling was studied numerically. For example, CS via one-way coupling of two electronic fractional chaotic oscillators in a canonical structure was numerically studied by Gao & Yu [57], who pointed out that the synchronization rate of a fractional chaotic oscillator was slower than its integer-order counterpart. The one-way coupling technique was also applied to numerically study CS of

chaotic fractional Lü systems [21] and of the chaotic fractional Ikeda systems with delays [58]. In the study by Ge & Jhuang [59], CS of a fractional rotational mechanical system with a centrifugal governor was studied for both autonomous and non-autonomous cases. It was shown that the rotational mechanical system, with its total order less than or more than the number of state variables, exhibited chaos. In addition, it was pointed out that practical chaos synchronization of different fractional systems needs a large coupling strength.

In the study by Tavazoei & Haeri [23], however, it was pointed out that the time-domain method is more reliable than the frequency-domain method in detecting chaotic attractors of fractional differential systems. One of the most used time-domain methods is the predictor–corrector algorithm [60]. The time-domain method is more flexiable than the frequency-domain method, since approximating the transfer function $1/s^\alpha$ is not so convenient if the fractional derivative order α has a large number of digits after the decimal point.

CS of the Chua, Rössler and Chen systems with different fractional orders was investigated numerically by using the predictor–corrector algorithm in the time domain. By selecting proper parameters, numerical results illustrated that synchronization of the fractional Chua, Rössler and Chen systems is slower than that of their respective integer-order systems, where the different fractional orders lie in (0,1).

In addition to the one-way coupling configuration, a control technique was also applied to synchronizing the fractional chaotic systems. For example, the synchronizations of two identical generalized van der Pol systems could be achieved, which was called 'chaos excited chaos synchronization' [61]. Chaos synchronization of fractional modified Duffing systems was also studied, and was called 'parameter excited chaos synchronization' [62]. Moreover, the active sliding mode controller [63] and adaptive proportional–integral–derivative controller [64] were applied to the synchronization of fractional chaotic systems.

(ii) Laplace transform method

The Laplace transform theory was applied by Deng & Li [17] to theoretically study CS of fractional Lü systems by one-way and Pecora–Carroll (PC) coupling configurations (see appendix B). And then the Laplace transform theory was used to theoretically study CS of the Chua systems [65], the unified chaotic systems [66] and the fractional neuron network systems with time-varying delays [67]. In the study by Li & Deng [68], the Laplace transform method was applied to investigating CS of the fractional Lorenz systems (x, y, z) in the PC coupling configuration, where (x, z) were driven by y. For coupled fractional Lorenz systems, CS can also be achieved if the driving signal is selected as x [69], i.e. CS of fractional Lorenz systems can be realized using driving signal x or y, which is in accordance with the case of integer-order Lorenz systems [25].

Now, the Laplace transform method for synchronization is illustrated by the following examples.

Example 3.1. Consider two identical Chua circuits in a one-way coupling form [65], in which the drive system is described by

$$\left.\begin{aligned}
{}_C D_{0,t}^{q_1} x_m(t) &= p_1(y_m - x_m - f(x_m)), \\
{}_C D_{0,t}^{q_2} y_m(t) &= x_m - y_m + z_m \\
{}_C D_{0,t}^{q_3} z_m(t) &= -p_2 y_m,
\end{aligned}\right\} \tag{3.1}$$

and

and the response system by

$$\left.\begin{aligned}
{}_C D_{0,t}^{q_1} x_s(t) &= p_1(y_s - x_s - f(x_s)) + k(x_s - x_m), \\
{}_C D_{0,t}^{q_2} y_s(t) &= x_s - y_s + z_s \\
{}_C D_{0,t}^{q_3} z_s(t) &= -p_2 y_s,
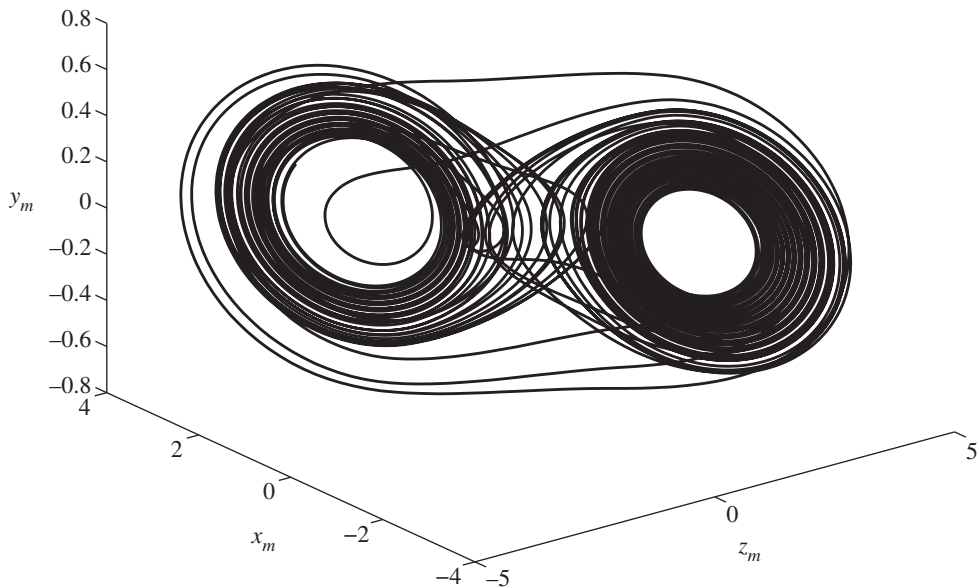\end{aligned}\right\} \tag{3.2}$$

and

Figure 1. The fractional Chua circuit in R^3. The diagram shows that the fractional Chua system can also exhibit chaotic behaviour, where $p_1 = 10$, $p_2 = 14.87$, $a = -1.27$, $b = -0.68$, $q_1 = 0.92$, $q_2 = 0.92$, $q_3 = 0.98$. The time step length is 0.02, the first 100 points are removed [65].

where the fractional orders satisfy $0 < q_1, q_2, q_3 \leq 1$, k is the coupling strength, p_1 and p_2 are positive constants, $f(x) = bx + \frac{1}{2}(a - b)(|x + 1| - |x - 1|)$ with $a < b < 0$.

The error dynamical system between systems (3.1) and (3.2) is

$$\left.\begin{aligned}
{}_cD_{0,t}^{q_1}e_1(t) &= -p_1 e_1 + p_1 e_2 - p_1(f(x_s) - f(x_m)) + ke_1, \\
{}_cD_{0,t}^{q_2}e_2(t) &= e_1 - e_2 + e_3 \\
\text{and} \qquad {}_cD_{0,t}^{q_3}e_3(t) &= -p_2 e_2,
\end{aligned}\right\} \tag{3.3}$$

where the error variables $e_1 = x_s - x_m$, $e_2 = y_s - y_m$, $e_3 = z_s - z_m$. Denoting $E_i(s) = \mathcal{L}\{e_i(t)\}$, $i = 1, 2, 3$, and applying the Laplace transform to both sides of (3.3), one obtains

$$\left.\begin{aligned}
s^{q_1}E_1(s) - s^{q_1-1}e_1(0) &= -p_1 E_1(s) + p_1 \mathcal{L}(f(x_s) - f(x_m)) + kE_1(s), \\
s^{q_2}E_2(s) - s^{q_2-1}e_2(0) &= E_1(s) - E_2(s) + E_3(s) \\
\text{and} \qquad s^{q_3}E_3(s) - s^{q_3-1}e_3(0) &= -p_2 E_2(s).
\end{aligned}\right\} \tag{3.4}$$

With the assumption $|E_3(s)| \leq N \in R_+$ and applying the final-value theorem of the Laplace transform [70], one obtains

$$\lim_{t \to \infty} e_i(t) = \lim_{s \to 0} sE_i(s) = 0, \quad i = 1, 2, 3, \tag{3.5}$$

which implies that CS between systems (3.1) and (3.2) is realized. If $q_1 = q_2 = q_3 = 1$, system (3.1) is the usual Chua system. When the intrinsic parameters are chosen as $p_1 = 10$, $p_2 = 14.87$, $a = -1.27$, $b = -0.68$, the usual Chua system has a strange attractor. Similarly, with the same intrinsic parameter values and the order parameters chosen as $q_1 = 0.92$, $q_2 = 0.92$, $q_3 = 0.98$,
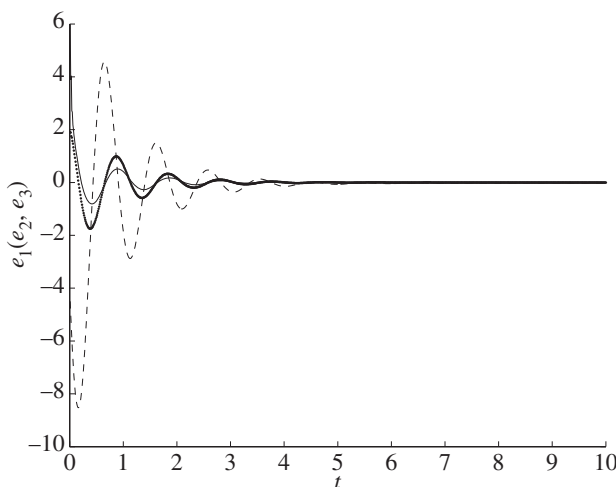
Figure 2. The evolution diagram of the synchronization errors between (3.1) and (3.2), which shows that the fractional Chua circuits (3.1) and (3.2) are asymptotically synchronized. Solid line shows $e_1(t) = x_s - x_m$; dashed line shows $e_2(t) = y_s - y_m$; and dotted line shows $e_3(t) = z_s - z_m$. Here, $p_1 = 10$, $p_2 = 14.87$, $a = -1.27$, $b = -0.68$, $q_1 = 0.92$, $q_2 = 0.92$, $q_3 = 0.98$, $k = 16$ [65].

a chaotic attractor is produced in the uncoupled fractional Chua circuit (3.1) (figure 1). With these chosen parameters and $k = 16$, the numerical simulation of CS between systems (3.1) and (3.2) is illustrated in figure 2.

From figure 2, one can see that the fractional Chua circuit (3.1) and its slave system (3.2) with one-way coupling can also reach CS with the same parameter values as the integer-order forms of (3.1) and (3.2) by choosing a suitable coupling parameter k.

Remark 3.2. It follows from the above example that the fractional orders chosen are close to 1 in the numerical simulations. In our opinion, according to the conclusion $\lim_{\alpha \to 1^-} cD_{0,t}^\alpha x(t) = x^{(1)}(t)$, the fractional system can produce a chaotic attractor similar to its integer-order counterpart with the same parameters.

In the following, this issue is further discussed. For a fractional differential system with a derivative order α lying in $(0,1)$, the smaller the α is taken, the less likely this fractional differential system is to display chaotic behaviour. The reason is possibly that, as α gets smaller and smaller, the stable region becomes larger and larger. For simplicity, take the chaotic fractional Chua circuit [65] as an example. When $q_1 = q_2 = q_3 = 0.95$, other parameters are the same as those in example 3.1. Figure 3 shows the phase portrait. It can be seen that system (3.1) is stable. Then, with $q_1 = q_2 = q_3 = 0.96$, the system generates a limit cycle, as shown in figure 4. As $q_1 = q_2 = q_3$ becomes bigger, chaos appears (figure 5) where $q_1 = q_2 = q_3 = 0.965$. When $q_1 = q_2 = q_3 = 0.97$ and 0.99, chaotic attractors are found again, and the phase portraits are shown in figures 6 and 7, respectively. With the increase of $q_1 = q_2 = q_3$, the chaotic attractors are more and more similar to those of the ordinary Chua system. Moreover, $q_1 = q_2 = q_3 = 0.96$ is the critical value of transition from stable equilibrium dynamics over self-sustained oscillations to chaos in the fractional Chua system (3.1), which is also demonstrated by a one-dimensional bifurcation diagram in figure 8.

Remark 3.3. From example 3.1, the stability analysis of CS between (3.1) and (3.2) discusses the stability of the zero solution of the error dynamic system of systems (3.1) and (3.2). Here, the Laplace transform is used. By fixing the parameter values as those in example 3.1 and approximately computing them from the predictor–corrector approach [71], one can find that the set of initial conditions leading to synchronization between systems (3.1) and (3.2) is not arbitrary.
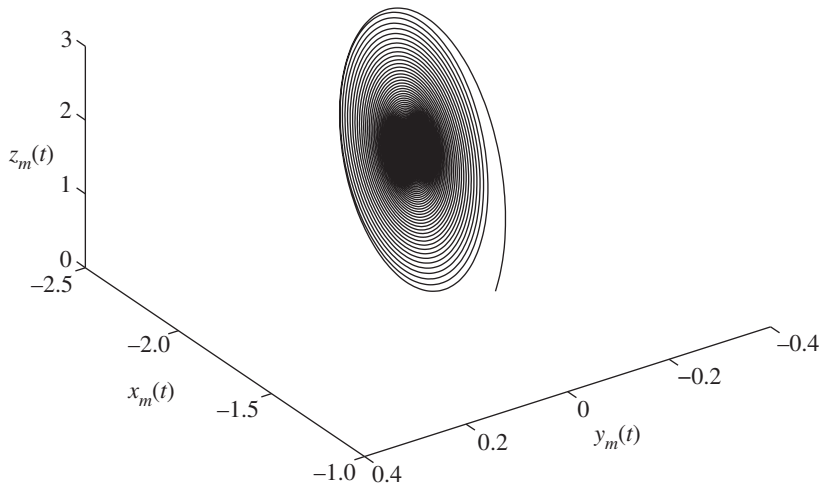
Figure 3. The phase portrait of the fractional Chua system with $q_1 = q_2 = q_3 = 0.95$, a stable point.
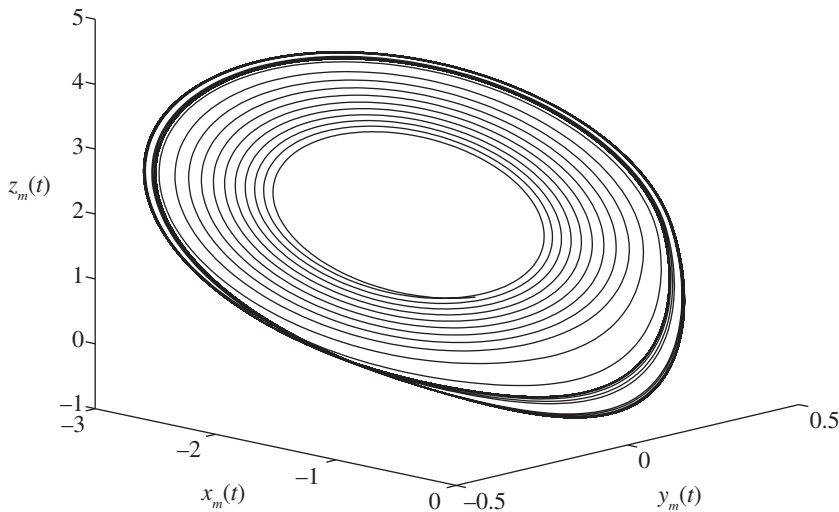

Figure 4. The phase portrait of the fractional Chua system with $q_1 = q_2 = q_3 = 0.96$, a stable limit cycle.

Given the drive initial conditions $(x_m(0), y_m(0), z_m(0)) = (0.1, -0.2, 0.1)$, the set of response initial conditions leading to synchronization between systems (3.1) and (3.2) lies in

$$W = \{(x_s(0), y_s(0), z_s(0)) \mid x_s(0) \geq 1.6 \text{ or } x_s(0) \leq -1.3; y_s(0) \geq 0.4$$

$$\text{or } y_s(0) \leq -0.8; \ z_s(0) \geq 1.6 \text{ or } z_s(0) \leq -1.4\},$$

which can be approximately located by numerical calculation.

In the study by Zhu *et al.* [72], the Laplace transform method was also applied to investigating CS of the following fractional Chua systems with the coupled matrix (k_1, k_2, k_3), where the drive system is given by

$$\left.\begin{aligned}
{}_C D_{0,t}^{q_1} x_m(t) &= p_1(y_m - x_m - f(x_m)), \\
{}_C D_{0,t}^{q_2} y_m(t) &= x_m - y_m + z_m \\
{}_C D_{0,t}^{q_3} z_m(t) &= -p_2 y_m - p_3 z_m,
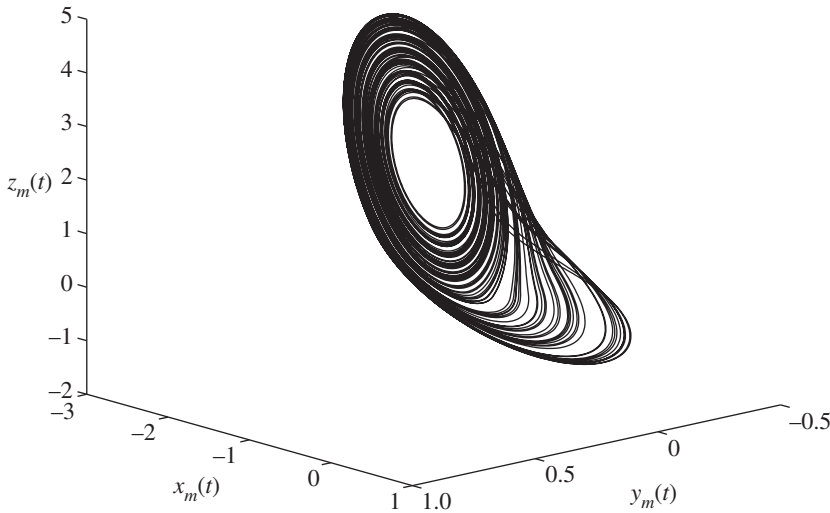\end{aligned}\right\} \tag{3.6}$$

and

Figure 5. The phase portrait of the fractional Chua system with $q_1 = q_2 = q_3 = 0.965$, a chaotic attractor.
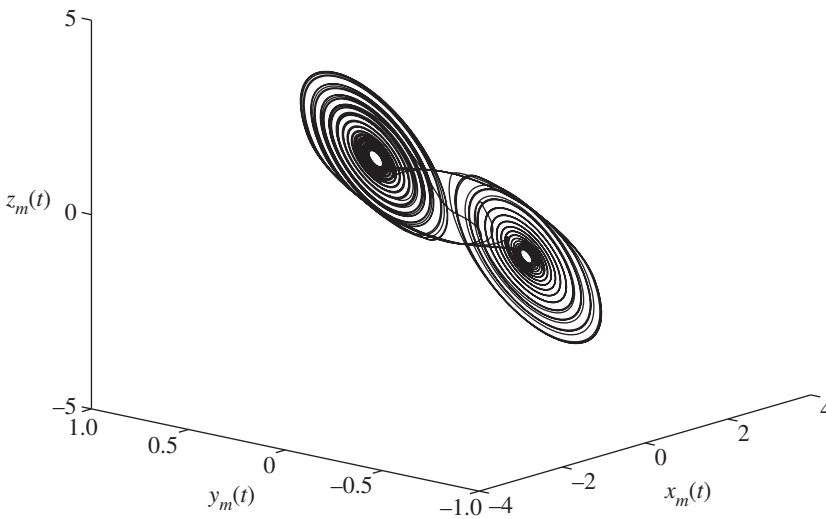

Figure 6. The phase portrait of the fractional Chua system with $q_1 = q_2 = q_3 = 0.97$, a chaotic attractor.

and the response system by

$$
\left.
\begin{aligned}
{}_C D_{0,t}^{q_1} x_s(t) &= p_1(y_s - x_s - f(x_s)) - k_1(x_s - x_m), \\
{}_C D_{0,t}^{q_2} y_s(t) &= x_s - y_s + z_s - k_2(y_s - y_m) \\
{}_C D_{0,t}^{q_3} z_s(t) &= -p_2 y_s - p_3 z_s - k_3(z_s - z_m),
\end{aligned}
\right\}
\tag{3.7}
$$

and

in which $f(x)$ is the same as that in example 3.1. Taking $p_1 = 10.725$, $p_2 = 10.593$, $p_3 = 0.268$, $a = -1.1726$, $b = -0.7872$, $q_1 = 0.93$, $q_2 = 0.99$, $q_3 = 0.92$, the fractional Chua system (3.6) also has a chaotic attractor. And, for systems (3.6) and (3.7), the synchronization thresholds were determined by using bifurcation graphs. Set the coupled matrix (k_1, k_2, k_3) to be $(k, 0, 0)$. Then, the transition diagrams can be obtained as shown in figure 9.
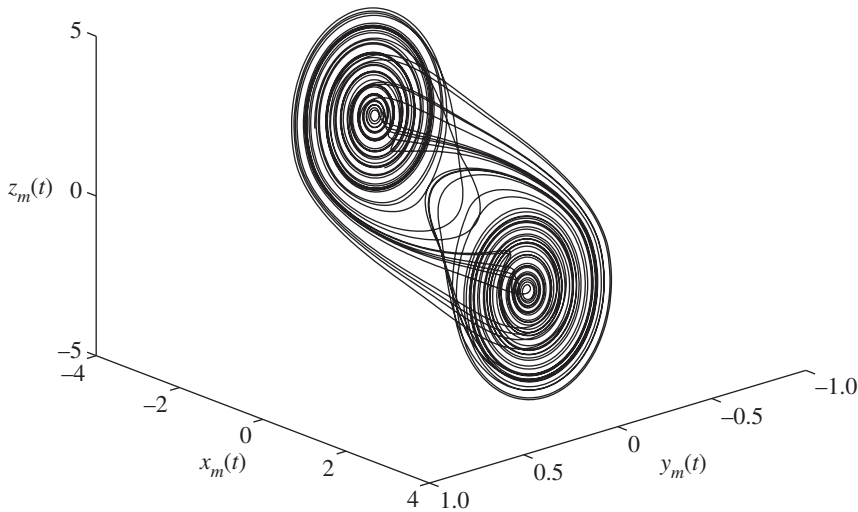
Figure 7. The phase portrait of the fractional Chua system with $q_1 = q_2 = q_3 = 0.99$, a chaotic attractor.
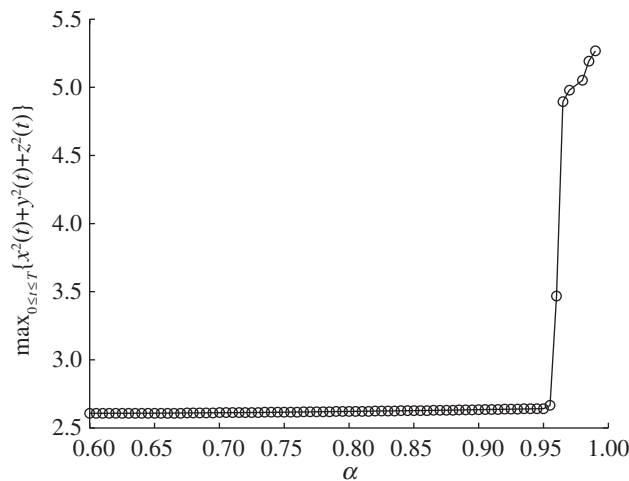

Figure 8. The transition diagram demonstrating the transition from stable equilibrium dynamics over self-sustained oscillations to chaos as the fractal dimension increases in the fractional Chua system (3.1). Here, $T = 100$, $q_1 = q_2 = q_3 = \alpha$.

From figure 9, it can be seen that the coupled system (3.6) and (3.7) with the coupled matrix $(k, 0, 0)$ is synchronized when the parameter k is greater than 4. Similarly, set the coupled matrix (k_1, k_2, k_3) to be $(k, k, 0)$ and $(k, k, 0)$ in system (3.7), respectively. Then, the synchronization can be realized when the parameter k is greater than approximately 1.0 and 0.5, respectively. Thus, it can be seen that the synchronization rate of the coupled matrix (k, k, k) is the fastest one [72].

Example 3.4. Consider a PC drive–response configuration with the drive system given by the fractional Lü system (with three state variables denoted by the subscript m) and the response system given by its subsystem containing the (x, z) variables [17].

The drive system is described by

$$
\left.\begin{aligned}
{}_C D_{0,t}^{q_1} x_m(t) &= a(y_m - x_m), \\
{}_C D_{0,t}^{q_2} y_m(t) &= -x_m z_m + c y_m \\
{}_C D_{0,t}^{q_3} z_m(t) &= x_m y_m - b z_m,
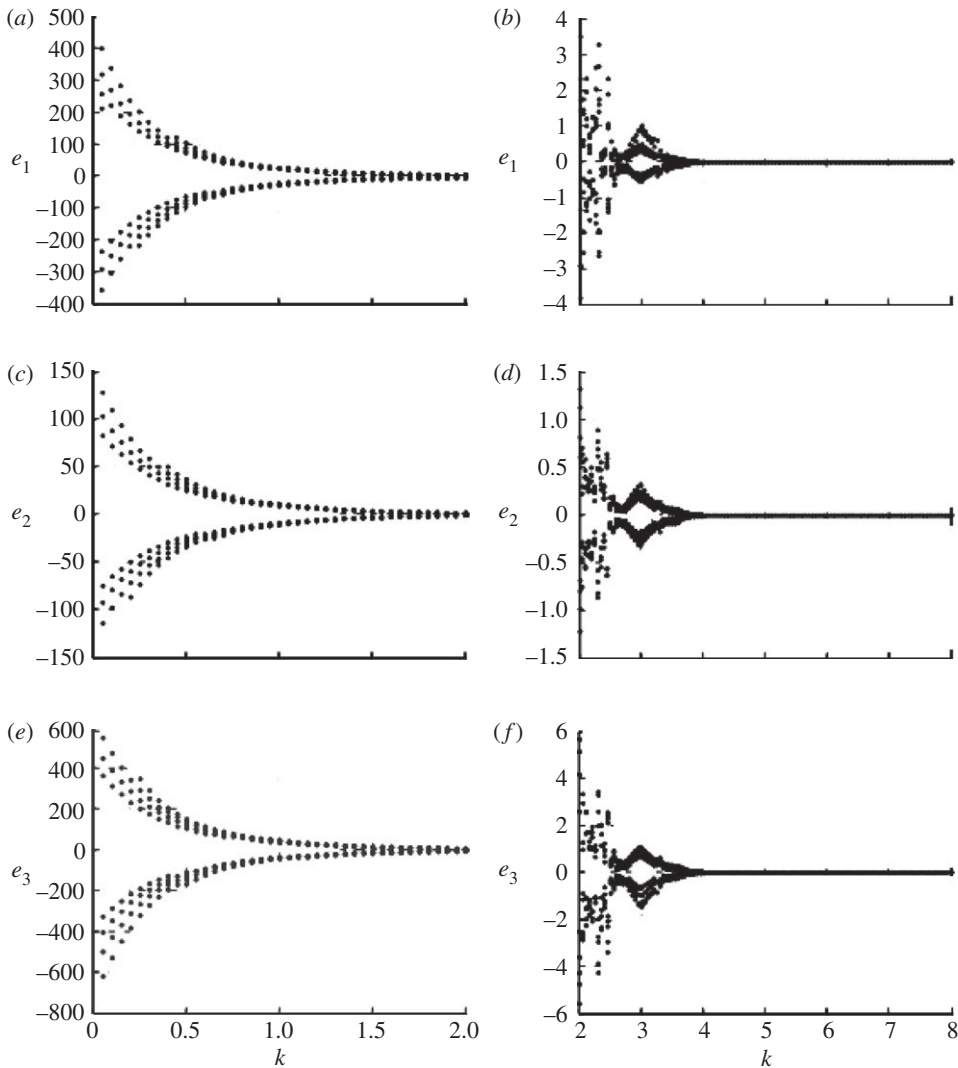\end{aligned}\right\}
\tag{3.8}
$$

and

Figure 9. The error diagrams of the synchronization configuration of the Chua systems (3.6) and (3.7) [72]. (a) e_1 versus k, (b) e_1 versus k, (c) e_2 versus k, (d) e_2 versus k, (e) e_3 versus k, (f) e_3 versus k. Here, $e_1 = x_s - x_m$, $e_2 = y_s - y_m$, $e_3 = z_s - z_m$.

and the response system by

$$\left. \begin{aligned} {}_C D_{0,t}^{q_1} x_s(t) &= a(y_m - x_s) \\ {}_C D_{0,t}^{q_3} z_s(t) &= x_s y_m - b z_s, \end{aligned} \right\} \tag{3.9}$$

and

where $0 < q_1, q_2, q_3 \le 1$, the response subsystem's variables are denoted by subscript s, and the chaotic signal y_m is used to drive the response subsystem.

Subtracting system (3.9) from system (3.8) leads to the following error dynamical system:

$$\left. \begin{aligned} {}_C D_{0,t}^{q_1} e_1(t) &= -a e_1 \\ {}_C D_{0,t}^{q_3} e_3(t) &= y_m e_1 - b e_3, \end{aligned} \right\} \tag{3.10}$$

and

where $e_1 = x_s - x_m$ and $e_3 = z_s - z_m$. Then, applying the Laplace transform to (3.10) as in example 3.1, one can achieve CS of systems (3.8) and (3.9) in the y-drive configuration. This result is illustrated by Deng & Li [17], with $(a, b, c) = (36, 3, 20)$ and $q_1 = 0.985$, $q_2 = 0.99$, $q_3 = 0.98$.

When $a = 36$, $b = 3$, $c = 20$, the usual Lü system, i.e. $q_1 = q_2 = q_3 = 1$, has a chaotic attractor. Its counterpart also behaves chaotically. Systems (3.8) and (3.9) can be asymptotically synchronized through a PC drive–response configuration. The diagram of the synchronization errors is provided in Deng & Li [17].

Remark 3.5. The analysis method in example 3.4 is almost the same as that in example 3.1.

Example 3.6. Consider applying the Laplace transform method to the fractional Chua circuit [65] via the active–passive decomposition (APD) configuration (see appendix C),

$$\left.\begin{array}{l} {}_CD_{0,t}^{q_1}x(t) = p_1(y - x - s(t)), \\[2mm] {}_CD_{0,t}^{q_2}y(t) = x - y + z \\[2mm] {}_CD_{0,t}^{q_3}z(t) = -p_2y, \end{array}\right\} \tag{3.11}$$

and

driven by signal $s(t) = f(x) = bx + \frac{1}{2}(a - b)(|x + 1| - |x - 1|)$ with $a < b < 0$, where q_i $(i = 1, 2, 3)$ are positive constants in $(0, 1]$.

By the final-value theorem of the Laplace transform, CS between the response system and its replica is implemented. When the coupling configuration is changed to the APD one, the coupled fractional Chua systems can be asymptotically synchronized with the parameter values $p_1 = 10$, $p_2 = 14.87$, $a = -1.27$, $b = -0.68$, $q_1 = 0.92$, $q_2 = 0.92$, $q_3 = 0.98$.

It is worth noting that the PC scheme for synchronization is a special case of the more general APD method. The freedom to choose the driving signal makes the APD scheme flexible in applications. For this reason, the APD scheme is usually combined with the simple one-way method to study CS by using the Laplace transform [20,67,68].

Example 3.7. Consider applying the Laplace transform method to studying synchronization of the fractional Duffing systems by using a combination of the APD method and the one-way coupling method [68]. The drive system is

$$\left.\begin{array}{l} {}_CD_{0,t}^{q_1}x_m(t) = y_m \\[2mm] {}_CD_{0,t}^{q_2}y_m(t) = -\frac{1}{25}y_m + \frac{1}{5}x_m - \frac{8}{15}s(t) + \frac{2}{5}\cos(0.2t), \end{array}\right\} \tag{3.12}$$

and

and the response system is

$$\left.\begin{array}{l} {}_CD_{0,t}^{q_1}x_s(t) = y_s + u(x_s - x_m) \\[2mm] {}_CD_{0,t}^{q_2}y_s(t) = -\frac{1}{25}y_s + \frac{1}{5}x_s - \frac{8}{15}s(t) + \frac{2}{5}\cos(0.2t), \end{array}\right\} \tag{3.13}$$

and

where $0 < q_1, q_2 \le 1$, u is a control parameter, and $s(t) = x_m^3$ is regarded as the driving signal.

If $u = 0$, then this drive–response configuration corresponds to the APD method. If $s(t) = x_m^3$ in the drive system and $s(t) = x_s^3$ in the response system, then it corresponds to the one-way coupling method. Applying the Laplace transform to the corresponding final-value theorem, the CS state can be realized as long as $u \ne -5$. By comparing the diagrams of the synchronization errors, it is found that this synchronization method is more effective for the Duffing system, since reaching synchronization takes longer than using only the APD scheme [68].

Apart from the aforementioned unidirectional coupling configuration, there is a more effective bidirectional coupling method (see appendix D) for CS of fractional chaotic systems. By applying the bidirectional coupling scheme to a pair of coupled fractional Rössler systems [68],

$$\left.\begin{array}{l} {}_CD_{0,t}^{q_1}x_m = -y_m - z_m + c_1(x_s - x_m) \\[2mm] {}_CD_{0,t}^{q_2}y_m = x_m + ay_m + c_2(y_s - y_m) \\[2mm] {}_CD_{0,t}^{q_3}z_m = 0.2 + z_m(x_m - 10) + c_3(z_s - z_m) \end{array}\right\} \text{drive} \tag{3.14}$$
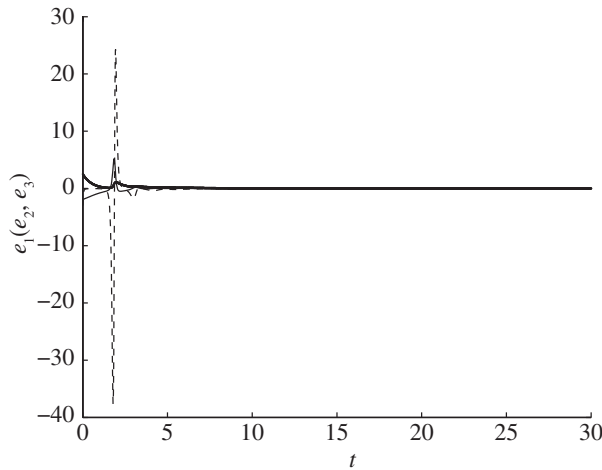
Figure 10. Synchronization error evolution of the drive–response systems (3.14) and (3.15) with the bidirectional coupling method, where the phase curves of synchronization errors show that the synchronized chaotic state is realized, where $c_1 = 0.8$, $c_2 = c_3 = 0.6$ and $q_1 = q_2 = q_3 = 0.9$. Here, solid line shows $e_1(t) = x_s - x_m$; dotted line shows $e_2(t) = y_s - y_m$; and dashed line shows $e_3(t) = z_s - z_m$ [68].

and

$$
\left.
\begin{aligned}
_cD_{0,t}^{q_1}x_s &= -y_s - z_s + c_1(x_m - x_s) \\
_cD_{0,t}^{q_2}y_s &= x_s + ay_s + c_2(y_m - y_s) \\
_cD_{0,t}^{q_3}z_s &= 0.2 + z_s(x_s - 10) + c_3(z_m - z_s),
\end{aligned}
\right\} \text{response} \tag{3.15}
$$

one has the following error dynamical system:

$$
\left.
\begin{aligned}
_cD_{0,t}^{q_1}e_1 &= -e_2 - e_3 - 2c_1 \cdot e_1, \\
_cD_{0,t}^{q_2}e_2 &= e_1 + ae_2 - 2c_2 \cdot e_2 \\
_cD_{0,t}^{q_3}e_3 &= z_se_1 + x_me_3 - 10e_3 - 2c_3 \cdot e_3,
\end{aligned}
\right\} \tag{3.16}
$$

and

where $0 < q_1, q_2, q_3 \le 1$, $e_1 = x_s - x_m$, $e_2 = y_s - y_m$ and $e_3 = z_s - z_m$. By using the Laplace transform and the final-value theorem, CS between systems (3.14) and (3.15) can be achieved under some prior assumptions. Select $a = 0.4$ and $q_1 = q_2 = q_3 = 0.9$, so as to produce chaotic dynamics in the uncoupled fractional Rössler system. With these parameters and $c_1 = 0.8$, $c_2 = c_3 = 0.6$, all the synchronization errors e_i ($i = 1, 2, 3$) soon converge to zero. The synchronization error evolution of the bidirectional coupling method is shown in figure 10.

(iii) Stability analysis

In this section, the stability theory of fractional systems is applied to studying CS of fractional chaotic systems with various kinds of couplings. It is well known that the stability region of the fractional case is greater than the stability region of the corresponding integer-order case if the fractional order lies in $(0, 1)$. Based on this fact, CS of fractional modified autonomous Van der Pol–Duffing (MAVPD) circuits was studied by a one-way coupling scheme as follows [73].

The drive system is

$$
\frac{d^\alpha x_1}{dt^\alpha} = -v(x_1^3 - \mu x_1 - y_1), \quad \frac{d^\alpha y_1}{dt^\alpha} = x_1 - \gamma y_1 - z_1 \quad \text{and} \quad \frac{d^\alpha z_1}{dt^\alpha} = \beta y_1, \tag{3.17}
$$

and the response system is

$$
\left.\begin{aligned}
\frac{d^\alpha x_2}{dt^\alpha} &= -\nu(x_2^3 - \mu x_2 - y_2) - k_1(x_2 - x_1), \\
\frac{d^\alpha y_2}{dt^\alpha} &= x_2 - \gamma y_2 - z_2 - k_2(y_2 - y_1) \\
\frac{d^\alpha z_2}{dt^\alpha} &= \beta y_2 - k_3(z_2 - z_1).
\end{aligned}\right\}
\tag{3.18}
$$

and

When $\alpha = 1$, the two coupled integer-order MAVPD systems can be asymptotically synchronized, if the feedback control gains k_1, k_2 and k_3 satisfy the following inequalities [74]:

$$
k_1 > \tfrac{1}{2}(2\nu(\mu - k_{x_1,x_2}) + |\nu + 1|), \quad k_2 > \tfrac{1}{2}(-2\gamma + |\nu + 1| + |\beta - 1|) \quad \text{and} \quad k_3 > \tfrac{1}{2}(|\beta - 1|), \tag{3.19}
$$

where $k_{x_1,x_2} = x_2^2 + x_1 x_2 + x_2^2 \geq 0$. Furthermore, for $\alpha \in (0, 1]$, CS of the coupled fractional MAVPD systems (3.17) and (3.18) can be achieved if k_i $(i = 1, 2, 3)$ satisfy the conditions (3.19). This can be verified (see fig. 6 in [73]) by selecting the parameter values $\beta = 200, \mu = 0.1$, $\nu = 100, \gamma = 1.6, \alpha = 0.98$ and the feedback control gains $k_1 = 280, k_2 = 250, k_3 = 100$, which satisfy the inequalities (3.19).

In addition, one can apply the stability theory to studying CS of fractional chaotic systems by one-way coupling [75]. Especially, based on the stability theory of delayed fractional differential systems, CS of delayed fractional chaotic systems by one-way coupling was investigated by Deng et al. [76], who simulated CS of the coupled Duffing oscillators.

Next, the stability theory of fractional differential systems is employed to investigate CS of fractional chaotic systems with the PC drive–response configuration. Consider the PC drive–response configuration with the drive system given by the fractional Chen system (with subscript m)

$$
\left.\begin{aligned}
{}_C D_{0,t}^{\alpha_1} x_m &= a(y_m - x_m), \\
{}_C D_{0,t}^{\alpha_2} y_m &= (c - a)x_m - x_m z_m + c y_m \\
{}_C D_{0,t}^{\alpha_3} z_m &= x_m y_m - b z_m,
\end{aligned}\right\}
\tag{3.20}
$$

and

and the response system chosen as the subsystem of (x, z) [77]

$$
\left.\begin{aligned}
{}_C D_{0,t}^{\alpha_1} x_s &= a(y_m - x_s) \\
{}_C D_{0,t}^{\alpha_3} z_s &= x_s y_m - b z_s.
\end{aligned}\right\}
\tag{3.21}
$$

and

For the error dynamical system of systems (3.20) and (3.21), by applying the stability theorem of multi-rational-order fractional differential systems [76], CS is achieved for the parameters $(a, b, c) = (35, 3, 28)$, $(\alpha_1, \alpha_2, \alpha_3) = (0.9, 0.95, 0.95)$.

For the fractional Lorenz system, several PC drive–response configurations were studied with the drive system given by the same order fractional Lorenz system and the response system given by its subsystems containing one state variable and two state variables [78]. The stability theorem of fractional differential systems was applied to discuss all possible drive–response subsystems, which can divide the Lorenz system. With the drive system containing one state variable, only two choices can induce CS, which agrees with the integer-order Lorenz system case. Yet, all possible choices can induce the appearance of CS when the drive system contains two state variables (table 1).

The APD scheme is usually combined with the one-way scheme to study CS of fractional chaotic systems by using the stability theorem of fractional differential systems [77,78]. In addition, the APD configuration combined with a linear or nonlinear controller is also commonly used to study synchronization of fractional chaotic systems [19,79,80].

In addition, the bidirectional coupling method can be used to achieve CS of two different fractional chaotic systems. From a system point of view, the bidirectional coupling method for identical chaotic systems can be regarded as a special case. In the study by

Table 1. The arguments of Jacobian eigenvalues of the fractional Lorenz system for different PC coupling configurations [78].

system	drive	response	argument
Lorenz system	(y, z)	x	π
$\alpha = 0.993, b = \frac{8}{3}$	(x, z)	y	π
$\sigma = 10, \gamma = 28$	(x, y)	z	π

Wu & Lu [81], the bidirectional coupling scheme of two different fractional systems was applied to two fractional networks with identical topological structures and different topological structures.

Compared with the unidirectional coupling configuration, this bidirectional coupling method is more effective in completely synchronizing the dynamical variables of coupled fractional chaotic systems because of the additional dissipation introduced. However, their synchronization manifold no longer follows the state variables of the uncoupled fractional chaotic systems.

Remark 3.8. Chaos synchronization of coupled systems has potential applications in secure communication. Hence, various kinds of techniques have been developed with the aim of hiding information in chaotic attractors. Compared with the classical chaotic systems, the chaotic fractional different systems possibly provide more efficient secure communication since the derivative order(s) can be regarded as parameter(s) in addition to the intrinsic parameter(s). Apart from the above popular coupling schemes, there are several unidirectional coupling methods for constructing drive–response configurations, such as the linear control method [82,83] and the nonlinear control method [73,81,83–86]. In fact, the procedure for analysing CS of two identical (different) fractional chaotic systems is the same as that which uses linear or nonlinear controllers.

(b) Projective synchronization

In the study by Mainieri & Rehacek [53], a new phenomenon was observed in coupled partially linear chaotic systems, called PS. PS is a dynamical behaviour in which the responses of two identical systems synchronize up to a constant scaling factor. This PS phenomenon was then studied widely in coupled integer-order chaotic systems. For fractional chaotic systems, PS was achieved to synchronize up to a scaling factor, i.e. the two variable vectors become proportional to each other [87].

As an example, consider PS of coupled fractional Chen systems [87], which is partially linear with $u = (x, y)$ and

$$M(z) = \begin{pmatrix} -a & a \\ c - a - z & c \end{pmatrix}.$$

Specifically, the coupled fractional partially linear chaotic systems are

$$
\left.
\begin{aligned}
_{\mathrm{RL}}D_{0,t}^{\alpha} x_m &= a(y_m - x_m), \\
_{\mathrm{RL}}D_{0,t}^{\alpha} y_m &= (c - a)x_m - x_m z + c y_m, \\
_{\mathrm{RL}}D_{0,t}^{\alpha} z &= x_m y_m - bz, \\
_{\mathrm{RL}}D_{0,t}^{\alpha} x_s &= a(y_s - x_s) \\
_{\mathrm{RL}}D_{0,t}^{\alpha} y_s &= (c - a)x_s - x_s z + c y_s.
\end{aligned}
\right\}
\tag{3.22}
$$

and

Taking the fractional order $\alpha = 0.9$ and selecting the parameters $(a, b, c) = (35, 3, 28)$, the fractional Chen system is chaotic. Numerical results for PS of the coupled fractional Chen systems are shown in figure 11 [87], where the projections of the master and the slave system onto the x–y plane are
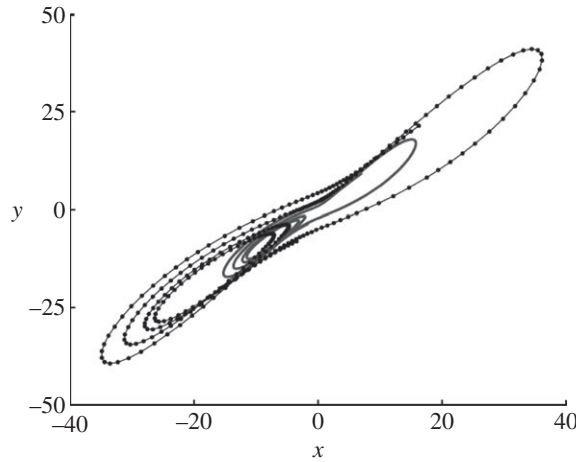
Figure 11. The projection diagram of the drive and response systems (3.22), which outlines the projections of the master system (thick line) and slave system (thin line with dots) onto the x–y plane. The structure of the two attractors indicates the PS of the coupled fractional Chen systems. Here, $\alpha = 0.9$ and $(a, b, c) = (35, 3, 28)$ [87].

provided. It follows from figure 8 that the two attractors corresponding to the master and slave systems are the same in structure except for size, which demonstrates that the coupled fractional Chen systems realize PS.

PS was further extended to general nonlinear systems by using a controller to the response system. In the study by Shao *et al.* [88], based on the stability theory of fractional systems, PS of coupled fractional chaotic Rössler systems was investigated. Also, PS of a new fractional chaotic system was investigated by Wu & Wang [89] through designing a suitable nonlinear controller, based on the stability of the fractional systems.

Finally, it is worth noting that a linear separation method was proposed by Wang & He [90] to achieve PS of coupled fractional unified systems according to the proportionality of the PS states. This linear separation method deals with a fractional chaotic system of the form

$$_C D_{0,t}^q x(t) = f(x(t)), \tag{3.23}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, $f : R^n \to R^n$ is a continuous vector function, $0 < q \leq 1$. Assume that the function $f(x(t))$ can be decomposed as $f(x(t)) = \hat{A}x(t) + \hat{h}(x(t))$, where $\hat{A}x(t)$ is the linear part and $\hat{h}(x(t))$ is the nonlinear part of $f(x(t))$. Then, $\hat{A}x(t)$ is suitably decomposed as $\hat{A}x(t) = Ax(t) + \bar{A}x(t)$, where A is a full-rank constant matrix and all of its eigenvalues have negative real parts. Let $h(x(t)) = \bar{A}x(t) + \hat{h}(x(t))$. Then, system (3.23) can be rewritten as

$$_C D_{0,t}^q x(t) = Ax(t) + h(x(t)). \tag{3.24}$$

Now, construct a new system as follows:

$$_C D_{0,t}^q y(t) = Ay(t) + \frac{h(x(t))}{\alpha}, \tag{3.25}$$

where $y(t) \in R^n$ is the state vector of system (3.25), and α is a desired scaling factor. Then, the error dynamical system between systems (3.23) and (3.25) is obtained as

$$_C D_{0,t}^q e(t) = Ae(t), \tag{3.26}$$

where $e(t) = x(t) - \alpha y(t)$. It was shown [90] that the state vectors $x(t)$ and $y(t)$ of the fractional systems (3.24) and (3.25) will reach PS, since all eigenvalues of the matrix A have negative real parts.

(c) Hybrid projective synchronization

Recently, the concept of PS has been generalized to HPS [54]. Similarly, this type of synchronization was studied by Chang & Chen [91], through considering two general coupled fractional chaotic systems,

$$_{c}D_{0,t}^{q}x(t) = f(x) \text{ (drive system)} \tag{3.27}$$

and

$$_{c}D_{0,t}^{q}y(t) = g(y) + U(x,y) \text{ (response system)}, \tag{3.28}$$

where $x \in R^n$, $f,g : R^n \to R^n$ are continuous vector functions, $y \in R^n$, and $U : R^n \times R^n \to R^n$ is a controller. Then, HPS means that there exists a constant matrix $H = \text{diag}(h_1, h_2, \ldots, h_n) \in R^{n \times n}$ such that $\lim_{t \to \infty} \|x - Hy\| = 0$.

Remark 3.9. In fact, HPS means that different state variables can synchronize up to some different scaling factors, where the scaling factors can be arbitrarily designed for different state variables by designing a suitable controller. Clearly, in secure communications, this feature could be used to enhance security.

Meanwhile, based on the stability of fractional differential systems, HPS of commensurate and incommensurate fractional Chen–Lee chaotic systems was studied by Chang & Chen [91], by designing a nonlinear controller. Using a specific state variable and its time derivatives, a new HPS scheme was also presented and applied to three-dimensional fractional unified chaotic systems by Chang & Chen [91].

The HPS idea was also extended to the fractional chaotic systems in different dimensions by Wang *et al.* [92] by considering the m-dimensional system (3.27) and the n-dimensional system (3.28), i.e. $x \in R^m$, $f : R^m \to R^m$ is a continuous vector function, $y \in R^n$, $g : R^n \to R^n$ is a continuous vector function and $U : R^m \times R^n \to R^n$ is a controller. Decompose the fractional drive–response systems (3.27) and (3.28) as

$$_{c}D_{0,t}^{q}x(t) = Ax + F(x) \tag{3.29}$$

and

$$_{c}D_{0,t}^{q}y(t) = By + G(y) + U(x,y), \tag{3.30}$$

where $A \in R^{m \times m}$, $B \in R^{n \times n}$ are the linear parts, and $F : R^m \to R^m$ and $G : R^n \to R^n$ are the nonlinear parts. Then, HPS means that there exists a real matrix $C \in R^{n \times m}$ such that $\lim_{t \to \infty} \|y - Cx\| = 0$. Based on the stability theory of fractional linear systems, the effectiveness of the above proposed scheme for HPS between systems (3.27) and (3.28) was shown by Wang *et al.* [92], including two cases: reduced-order synchronization with $m > n$ and increased-order synchronization with $m < n$.

(d) Function projective synchronization

Among all kinds of chaos synchronizations, PS has been most extensively studied in recent years because it can obtain faster communication speeds with its proportional feature. Apart from HPS, PS has also been extended to a more general synchronization setting, i.e. FPS. FPS means that the drive and response systems can be synchronized up to a scaling function, not just a constant. Clearly, the unpredictability of the scaling function in FPS can further enhance the security of communication.

In the study by Zhou & Zhu [93], a detailed account of FPS of fractional chaotic systems was considered. More generally, in Zhou & Cao [94], FPS between fractional chaotic systems and integer-order chaotic systems was discussed, based on the stability theory of linear fractional systems. Moreover, this FPS scheme was applied to synchronizing the integer-order Lorenz chaotic system and the fractional Chen chaotic system.

Remark 3.10. From the generalized definitions of PS for fractional chaotic systems, it is obvious that FPS covers all the others. In other words, if $K(x) = \alpha I, \alpha \in R$, the FPS problem reduces to PS, where I is the identity matrix with proper dimension. In particular, if $\alpha = 1$, FPS reduces to CS. Moreover, if $K(x) = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$, FPS becomes HPS.

(e) Generalized synchronization

GS means that the states of two coupled systems satisfy a functional relation or asymptotically satisfy a functional relation as time goes to infinity. GS has many potential applications in secure communications, chemical reactions and modelling brain activity, etc.

The definition of GS was extrapolated from classical systems to fractional systems.

Theorem 3.11. *GS is achieved in systems* (2.10) *with the Caputo derivative if and only if, for all* $(x_0, y_0) \in B$, *the response system* $d^q y / dt^q = g(y, u) = g(y, h(x))$ *is asymptotically stable about zero, i.e.* $\forall y_{10}, y_{20} \in B_y$, $\lim_{t \to \infty} \|y(t, x_0, y_{10} - y(t, x_0, y_{20}))\| = 0$, *where f, g and h are continuous functions in systems* (2.10).

In the study by Deng [95], three methods for achieving GS of fractional systems were discussed from the so-called auxiliary system approach [96], where theorem 3.11 above was specifically used to realize GS.

From then on, several GS schemes for some special types of coupled fractional chaotic systems were developed based on the stability theory of fractional systems. In the study by Zhou *et al.* [97], the following fractional chaotic system was considered:

$$
\left.
\begin{aligned}
\frac{d^q x_1}{dt^q} &= \sum_{i=1}^{3} a_i x_i, \\
\frac{d^q x_2}{dt^q} &= \sum_{i=1}^{3} b_i x_i \\
\frac{d^q x_3}{dt^q} &= f_3(x_1, x_2, x_3),
\end{aligned}
\right\}
\tag{3.31}
$$

and

where a_i, b_i $(i = 1, 2, 3)$ are real numbers satisfying $a_3 \neq 0$, $b_3 \neq 0$ or $a_3 = 0$, $a_2 \neq 0$, $b_3 \neq 0$ or $b_3 = 0$, $b_1 \neq 0$, $a_3 \neq 0$, $0 < q \leq 1$ is the fractional order, $x = (x_1, x_2, x_3)^T$ are state variables, and $f_3(x_1, x_2, x_3)$ is a nonlinear function. The following response system, which is different from system (3.31), was used:

$$
\left.
\begin{aligned}
\frac{d^q x_1'}{dt^q} &= \sum_{i=1}^{3} a_i' x_i', \\
\frac{d^q x_2'}{dt^q} &= \sum_{i=1}^{3} b_i' x_i' \\
\frac{d^q x_3'}{dt^q} &= \phi + g_3(x_1', x_2', x_3'),
\end{aligned}
\right\}
\tag{3.32}
$$

and

where a_i', b_i' $(i = 1, 2, 3)$ are real numbers satisfying $a_3' \neq 0$, $b_3' \neq 0$ or $a_3' = 0$, $a_2' \neq 0$, $b_3' \neq 0$ or $b_3' = 0$, $b_1' \neq 0$, $a_3' \neq 0$, $0 < q \leq 1$ is the fractional order, and ϕ is a scalar controller. When the controller ϕ satisfies some suitable conditions [97], GS between systems (3.31) and (3.32) can be realized.

Based on the stability theory of linear fractional differential systems, another GS method for the fractional chaotic systems was presented by Zhang *et al.* [98], as follows.

Theorem 3.12. *Let* $A, M, K \in R^{n \times n}$, *where M is an invertible matrix. If every eigenvalue* λ *of the matrix* $(MAM^{-1} + K)$ *satisfies* $|\arg(\lambda)| > \alpha \pi / 2$, *then the following coupled systems* (3.33) *and* (3.34) *can reach GS via a linear transform* $y = Mx$:

$$
\frac{d^\alpha x}{dt^\alpha} = Ax + \varphi(x)
\tag{3.33}
$$

and

$$
\frac{d^\alpha y}{dt^\alpha} = MAM^{-1}y + M\varphi(x) + Ky - KMx,
\tag{3.34}
$$

where $0 < \alpha \leq 1$.

Remark 3.13. In theorem 3.12, the GS scheme is easy to understand, because it is constructed by the linear transform $y = Mx$. In this case, the GS problem is converted to the stability problem of a fractional system.

(f) Generalized projective synchronization

Recall that PS means that the drive and response state vectors synchronize up to a scaling factor. Chen & Sun [99] studied PS for a general class of chaotic systems including non-partially linear systems, which is known as GPS.

In the study by Peng & Jiang [100], GPS of fractional chaotic systems was introduced (see definition 2.10). GPS may be achieved through properly adjusting the controller. Moreover, the transmitted synchronizing signal used to drive the fractional response system can be in a scalar form [100] or a nonlinear vector form [101]. Based on a partially linear decomposition and the stability theory of integer-order systems, a GPS scheme for coupled fractional Rössler systems was developed by Shao [102]. Especially, the Laplace transform method was applied to discussing GPS of fractional Chen hyperchaotic systems by Wu & Lu [103], and GPS of time-delay fractional chaotic systems was investigated by Zhou *et al.* [67] by using the stability theory of linear fractional systems with time delays.

In the study by Zhou *et al.* [104], a GPS scheme was constructed, which has different scaling factors, as follows: the fractional chaotic drive system is

$$\frac{\mathrm{d}^q x}{\mathrm{d} t^q} = Ax + F(x), \tag{3.35}$$

for which the following fractional response system is constructed:

$$\frac{\mathrm{d}^q y}{\mathrm{d} t^q} = C^{-1}[ACy + F(Cy) + (K - DF(x))(Cy - x)], \tag{3.36}$$

where $0 < q \le 1$, $x, y \in R^m$, $DF(x) \in R^{m \times m}$ is the Jacobian matrix of $F(x)$; $A \in R^{m \times m}$, $C \in R^{m \times m}$ is a real invertible matrix, and $K \in R^{m \times m}$ is a real matrix to be determined. Moreover, based on the stability theory of fractional differential systems, it was proved that there exists a matrix K such that $\lim_{t \to \infty} \|Cy - x\| = 0$.

Remark 3.14. Taking into account uncertainties, robust synchronization of two perturbed fractional Chen systems was studied by Asheghan *et al.* [105]. And anticipated synchronization was also investigated by Zhou & Zhu [106].

Normally speaking, compared with other synchronizations, phase synchronization in classical systems most approximately reflects the real world so attracts much more attention, although there is no suitable analytical method available. For the fractional case, phase synchronization of coupled fractional differential systems in memory processes is a challenging topic that needs far more attention in future research.

4. Conclusions and comments

This paper presents an overview of chaos synchronization of coupled fractional differential systems. A list of coupling schemes is presented, including one-way coupling, PC coupling, APD coupling, bidirectional coupling and other unidirectional coupling configurations. Also, several extended concepts of synchronization are introduced, namely, PS, HPS, FPS, GS and GPS. Corresponding to different kinds of synchronization schemes, various analysis methods are presented and discussed.

It should be mentioned that this review covers most contributions in the area. Although great efforts have been made to prepare a comprehensive review, it is literally impossible to be complete. Nevertheless, it is the authors' hope that the present review can serve as a good starting point for future advanced research work in studying chaos synchronization of fractional differential systems. On the other hand, similar to the classical dynamics, some interesting but

somewhat knotty problems have not been investigated in this review article, such as the attractive basin of the fractional attractor, the transverse stability of the synchronized state, the so-called weak stability in which any neighbourhood to the synchronized state is 'riddled' with a dense set of initial conditions that produce divergence, the effect of a mismatch between the two oscillators that cannot be compensated, etc. The new dynamical phenomena in coupled fractional systems, together with the dynamical ones observed in the classical systems, should attract attention and be further explored. This paper is just a review article where the existing studies are collected and commented upon. The authors do hope such topics will stimulate future research interests.

Appendices

In general, CS can be roughly divided into two categories: unidirectional coupling (drive–response coupling) configuration and bidirectional coupling configuration.

Appendix A. One-way coupling configuration

Consider two identical fractional chaotic systems: $d^\alpha x/dt^\alpha = f(x)$ and $d^\alpha y/dt^\alpha = f(y)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ ($\alpha_i > 0$, $i = 1, 2, \ldots, n$), $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T$ with $f : R^n \rightarrow R^n$ defining a vector field, $d^\alpha x/dt^\alpha = (d^{\alpha_1} x_1/dt^{\alpha_1}, d^{\alpha_2} x_2/dt^{\alpha_2}, \ldots, d^{\alpha_n} x_n/dt^{\alpha_n})^T$.

The one-way coupling technique employs a coupling term $k(x - y)$ in the second equation, which leads to a coupled system as follows [17]:

$$\left.\begin{aligned}\frac{d^\alpha x}{dt^\alpha} &= f(x) \\ \frac{d^\alpha y}{dt^\alpha} &= f(y) + k(x - y),\end{aligned}\right\} \tag{A 1}$$

and

where the diagonal matrix $k = \text{diag}(k_1, \ldots, k_n)$ controls the strength of the feedback into the coupling system, $k_i \geq 0$, $i = 1, 2, \ldots, n$. This type of coupling does not change the solution to the autonomous uncoupled system $d^\alpha x/dt^\alpha = f(x)$, and CS is realized by designing the coupling strength such that $\|x - y\| \rightarrow 0$ as $t \rightarrow \infty$.

Appendix B. Pecora–Carroll configuration

This scheme was proposed by Pecora & Carroll [25]. Consider an autonomous n-dimensional chaotic dynamical system of fractional differential equations [17]

$$\frac{d^q u}{dt^q} = F(u), \tag{B 1}$$

where $q = (q_1, q_2, \ldots, q_n)^T$, $u = (u_1, u_2, \ldots, u_n)^T$, with $F = (f_1, f_2, \ldots, f_n)^T : R^n \rightarrow R^n$ defining a vector field. The PC scheme decomposes the fractional dynamical system (B 1) into two subsystems,

$$\frac{d^\alpha v}{dt^\alpha} = g(v, w) \tag{B 2}$$

and

$$\frac{d^\beta w}{dt^\beta} = h(v, w), \tag{B 3}$$

where $\quad u = (v, w)^T, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T, \quad \beta = (\beta_{m+1}, \beta_{m+2}, \ldots, \beta_n)^T, \quad v = (v_1, v_2, \ldots, v_m)^T,$ $w = (w_{m+1}, w_{m+2}, \ldots, w_n)^T$. The first subsystem (B 2) defines the drive system, whereas the second one (B 3) represents the response system, whose evolution is guided by the drive trajectory through the driving signal v.

Now, create a new subsystem w' with the same chaotic driving signal $v = (v_1, v_2, \ldots, v_m)^T$, which is the replica of subsystem w, as follows:

$$\frac{d^\beta w'}{dt^\beta} = h(v, w'). \tag{B 4}$$

In this situation, CS means that the trajectories of the response system w will converge to the trajectories of the replica w' and they will remain together with each other. That is, CS implies $w \to w'$ as $t \to \infty$.

Appendix C. Active–passive decomposition configuration

The APD method for achieving identical chaotic synchronization systems was proposed by Boccaletti *et al.* [55] and Kocarev & Parlitz [107] and is more general than the PC configuration. The APD method starts from a chaotic autonomous system,

$$\frac{d^q z}{dt^q} = F(z), \tag{C 1}$$

and rewrites it as a non-autonomous system,

$$\frac{d^q x}{dt^q} = f(x, s(t)), \tag{C 2}$$

where $q = (q_1, q_2, \ldots, q_n)^T, f : R^n \to R^n$, and $s(t)$ is the driving signal: $s(t) = h(x)$ or $d^q s/dt^q = h(x, s)$. Let

$$\frac{d^q y}{dt^q} = f(y, s(t)) \tag{C 3}$$

be a copy of the non-autonomous system that is driven by the same signal $s(t)$. The CS error dynamical state between the system (C 2) and system (C 3) will approach zero asymptotically as $t \to \infty$.

Appendix D. Bidirectional coupling configuration

In the study by Li *et al.* [68] and that by Li & Deng [65], a bidirectional coupling scheme was introduced between two identical fractional chaotic systems. It amounts to introducing additional dissipation in the dynamics,

$$\left.\begin{aligned}
\frac{d^q X}{dt^q} &= f(X) + \hat{C} \cdot (Y - X)^T \\
\frac{d^q Y}{dt^q} &= f(Y) + \hat{C} \cdot (X - Y)^T,
\end{aligned}\right\} \tag{D 1}$$

and

where $q = (q_1, q_2, \ldots, q_n)^T$, X and Y represent the n-dimensional state vectors of the chaotic systems, f is a vector field from R^n to R^n, and $\hat{C}$ is an $n \times n$ matrix whose elements rule the dissipative coupling.

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