Pinning noise-induced stochastic resonance

Yang Tang,[1,2,*] Huijun Gao,[3] Wei Zou,[1,2,4] and Jürgen Kurths[1,2,5]

[1] *Institute of Physics, Humboldt University Berlin, Berlin D-12489, Germany*
[2] *Potsdam Institute for Climate Impact Research, Telegraphenberg, Potsdam D-14415, Germany*
[3] *Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150080, China*
[4] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China*
[5] *Institute for Complex systems and Mathematical Biology, University of Aberdeen, Aberdeen AB24 3FX, UK*

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This paper proposes the concept of *pinning* noise and then investigates the phenomenon of stochastic resonance (SR) of coupled complex systems driven by pinning noise, where the noise has an α-stable distribution. Two kinds of pinning noise are taken into account: partial noise and switching noise. In particular, we establish a connection between switching noise and global noise when Gaussian noise is considered. It is shown that switching noise can not only achieve a stronger resonance effect, but it is also more robust to induce the resonance effect than partial noise.

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I. INTRODUCTION

Noise-induced phenomena have been a major focus in the field of nonlinear physics and statistical physics [1–5], where noise usually has a disordering impact but under certain conditions can be employed to induce order in nonlinear systems under certain conditions, mainly to mention, stochastic resonance (SR) [1], coherence resonance (CR) [6], enhanced stability [7–9] and resonant activation phenomena [10]. SR is a noise-induced phenomenon of signal amplification, which has been experimentally observed in several physical and biological systems [2, 11–16]. The output of the system may be amplified in the presence of noise at a proper level and a weak external periodic signal [17–20]. In addition to Gaussian white noise, different types of noises are used to drive the systems to exhibit noise-induced phenomena such as colored noise and α-stable noise [21, 22]. Due to generality and wide observations in social, financial and biological systems [23–25], noise having an α-stable distribution has gained increasing research attention [20, 22, 26, 27].

It is to be noted that the majority of past studies about SR have been hitherto confined to the case that noise is injected in each unit and all the time. Such an assumption on noise in coupled networks is related to various applications but it is also convenient for both theoretical study and numerical simulation. However, it is natural to ask what will happen in coupled networks driven by *pinning* noise, where only some units in the networks are driven by noise while the others are not. Some kinds of recently reported phenomena can be regarded as special cases of pinning mechanisms such as partial time-delay coupling [28] and switching nonlinearities [29, 30], which can be widely observed in technological and natural systems [29–32]. The pinning noise could happen in nature and one can use it for engineering applications like other pinning mechanisms [33, 34]. The idea of the pinning noise-induced phenomenon in this paper is quite similar to pinning control problem [33, 35]. The major difference between pinning control and pinning noise-induced phenomena is that usually in pinning control problem, the more nodes are allowed to control, the better the control performance will be yielded [33, 34]. Different from pinning control, there exists an intermediate value of percent of units with noise to exhibit the best response under a certain intensity of noise.

Our intention of this contribution is to elucidate effects of partial and switching noises on SR in coupled complex systems, where the noise has an α-stable distribution. Partial noise indicates that the units suffering from noise are fixed along the time evolution, while the switching noise means that the units injected by noise depend on the probabilities of Bernoulli variables. One may intuitively conjecture that switching noise can only have a “partial” weak effect on the overall signal amplification like partial noise. Nevertheless, via an analytical and numerical approach, we present in this paper an intriguing result: switching noise facilitates SR more than partial noise by achieving a stronger resonance behavior. The interplays between the percent of the units subjected to noise p (or the probability of the units subjected to noise p), the stability parameter α and the scale parameter γ are demonstrated to show resonance effects to an external stimulus.

The rest of this paper is structured as follows. Section II introduces the methods and results. We investigate the problem of SR in an array of globally coupled bistable systems under pinning noise, where pinning noise includes partial and switching noises satisfying a Lévy α-stable distribution. Finally, the paper concludes with a summary in Section III.

II. MODEL AND PRELIMINARIES

In this section, we present some preliminaries of Lévy α-stable distribution noises, Bernoulli variables and bistable systems.

First, we consider the following ensemble of coupled

*Corresponding author. tangtany@gmail.com

bistable systems with pinning noise:

$$\dot{x}_i = x_i - x_i^3 + \frac{g}{d_i}\sum_{j=1}^{N} c_{ij}(x_j - x_i) + F\sin(\Psi t) + \xi_i(t)\eta_i, i = 1, 2, ..., N, \quad (1)$$

where $x_i(t)$ is the state of the ith unit at time t. The system is subjected to an external periodic forcing with amplitude F and frequency $\Psi = \frac{2\pi}{T}$. $g > 0$ is the global coupling strength; d_i is the degree of unit i; c_{ij} denotes the coupling strength between the ith and jth unit, which is defined as $c_{ij} = c_{ji} = 1$ if two units are coupled, otherwise $c_{ij} = c_{ji} = 0$; η_i is the noise satisfying an α-stable distribution [36]. $\xi_i(t)$ denote constants for partial noise or Bernoulli variables for switching noise, respectively. Note that our results can not only be extended to neural systems like FitzHugh-Nagumo systems, but also be shown under a different scale N.

For partial noise, $\xi_i(t)$ are constants to characterize the following cases: $i \in \mathcal{M} \subseteq \mathcal{V} = \{1, ..., N\}$ if $\xi_i(t) = 1$; otherwise, $i \notin \mathcal{M}$, if $\xi_i(t) = 0$. $\mathcal{M}$ is the set of units subjected to noise. Since the systems are coupled, without loss of generality, we assume that the first l units are subjected to noise. That is, $\xi_i(t) = 1, i = 1, ..., l$ and $\xi_i(t) = 0, i = l + 1, ..., N$. $l = \lfloor N * p \rfloor$ denotes the element number of finite set $\mathcal{M}$ composed of the units to be injected by noise. $\lfloor N * p \rfloor$ is the minimum integer close to $N * p$, where p is the percent of the units suffering from noise. If $p = 1$, $\mathcal{M} = \mathcal{V}$ holds. The pinning noise here turns into global noise and the problem considered here reduces the one considered in [20]. If $p = 0$, $\mathcal{M} = \varnothing$ and there is no noise in coupled systems.

For switching noise, $\xi_i(t)$ are Bernoulli stochastic variables, which are used to define the following events: $i \in \mathcal{N}(t) \subseteq \mathcal{V} = \{1, ..., N\}$ if $\xi_i(t) = 1$; otherwise, $i \notin \mathcal{N}(t)$. $\mathcal{N}(t)$ is the time-varying set of units driven by noise. $\mathbb{E}\{\xi_i(t)\} = p$, $P\{\xi_i(t) = 1\} = p$ and $\mathbb{D}\{\xi_i(t)\} = p(1 - p)$, where $\mathbb{E}\{.\}$ is the expectation operator, $P\{.\}$ is the probability of one event and $\mathbb{D}\{.\}$ is the variance operator.

The location and relative stability of the fixed points of an isolated ith unit are perturbed by the noise η_i. We assume that the noise η_i follow a Lévy α-stable distribution whose characteristic function is given as follows [36]:

$$\psi(t, \alpha, \beta, \gamma, \delta) = \exp\left[it\delta - |\gamma t|^{\alpha}(1 - i\beta \mathrm{sgn}(t)\tan\frac{\pi\alpha}{2})\right], \quad \text{for} \quad \alpha \neq 1, \quad (2)$$

and

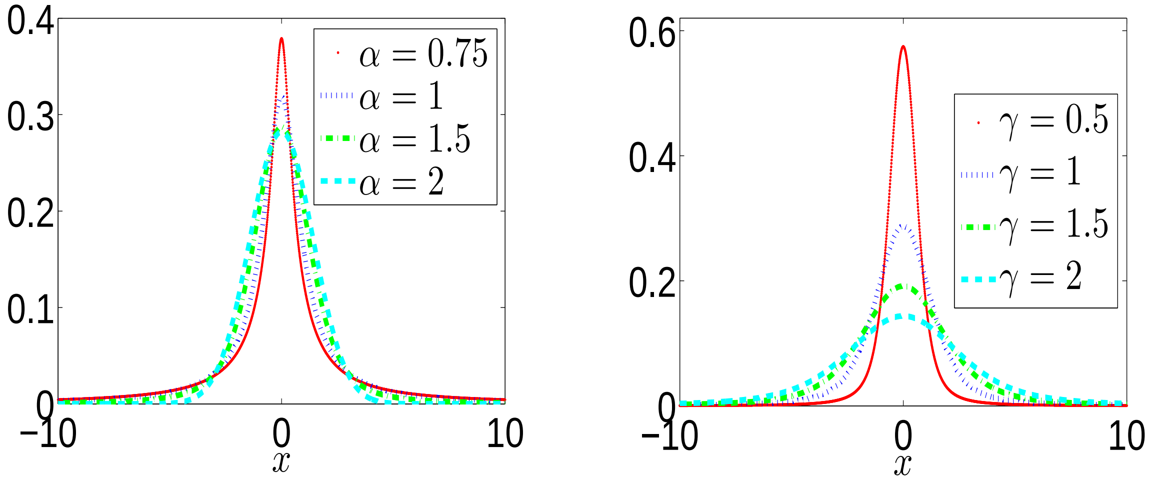
$$\psi(t, \alpha, \beta, \gamma, \delta) = \exp\left[it\delta - \gamma|t|(1 + i\beta\frac{2}{\pi}\mathrm{sgn}(t)\log|t|)\right], \quad \text{for} \quad \alpha = 1, \quad (3)$$

where $\alpha \in (0, 2]$ stands for a stability parameter; $\beta \in [-1, 1]$ represents the skewness parameter for measuring asymmetry; $\gamma \in (0, \infty)$ and $\delta \in (-\infty, \infty)$ are a scale parameter and a location parameter, respectively. When $\beta = 0$, the distribution is symmetric around δ and is called as a (Lévy) symmetric α-stable distribution. γ is used to measure the width of the distribution and α indicates the exponent or index of the distribution and specifies the asymptotic behavior of the distribution when $\alpha < 2$. The normal distribution, the Cauchy distribution, and the Lévy distribution are special cases of stable distributions [20, 36]. When $\alpha = 2$, the distribution reduces to a Gaussian distribution with variance $\sigma^2 = 2\gamma^2$ and the skewness parameter β has no effect on the distribution. It is found in [20] that the stability parameter α can be employed to represent diversity, which is similar to the scale parameter γ. A typical probability density function of a Lévy α-stable distribution is depicted in Fig. 1.

To study the resonance effect of the periodically driven system, the spectral amplification factor $R = 4\frac{1}{F^2}|\langle e^{i\Psi t}X(t)\rangle|$ is used [4], where $X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is the mean-field value of the units. The spectral amplification factor R can well characterize the amount of information in the signal transmission with an external forcing. In what follows, we simply assume that $\beta = 0$ and $\delta = 0$, which neglects the effects of skewness and location. Moreover, as pointed out in [20, 23], we generate the noise η_i according to the Lévy α-stable distribution within a predefined bound $[-\epsilon_1, \epsilon_2]$ such that the generated random variables cannot be extremely large. This type of distribution is a truncated Lévy α-stable distribution with zero value outside the range [23]. This kind of setting is also used in characterizing random time delays with a normal distribution [39]. In addition, the Bernoulli stochastic variables $\xi_i(t)$ and noise η_i are mutually independent. If without specific mentioning, the parameters are fixed as $T = 100, F = 0.2, g = 1, \epsilon_1 = \epsilon_2 = \epsilon = 500$ and $N = 200$. For numerical simulation, we use the Euler-Maruyama method in [37]. If without specific mentioning, the parameters for numerical simulation are chosen as follows: the simulation time $W = 100$, the step size $h = 0.01$ and the running times $Q = 20$. The simulation of Lévy α-stable distribution noise follows the method in [23, 38]. Denote p_e as the parameter $p_e = \log(p)$, i. e., $p = 10^{p_e}$.

III. MAIN RESULTS

Figure 2 depicts the effect of the expectation (average) of Bernoulli variables p (i. e., using switching noise), the stability parameter α and the scale parameter γ on amplification factor R. From Fig. 2(a), we observe that there exists a peak when varying α for a fixed p_e. As p_e increases, a larger α is necessary to reach the best resonance effect of the external forcing. Moreover, a more pronounced resonance effect can be seen for a larger α, i. e., the peaks become higher for a larger α. We can elucidate the observed phenomenon as follows. The stable distribution approaches the normal distribution as α increases while the stable distribution will have a heavy tail when α decreases [36], as shown in Fig. 1. Therefore, when α is small, a small value of p_e is needed to ensure that large random variables do not occur frequently. With an appropriate value of p_e, the best response to an external forcing can be achieved. Although decreasing the stability parameter α can induce diversity to the unit, it will inevitably lead to large random variables which is possible to destroy the resonance behavior. To summarize, the peaks become higher for



(a) (b)

FIG. 1: (Color online) Probability density function of noise having a stable distribution: (a) varying α when $\beta = 0, \gamma = 1, \delta = 0$; (b) varying γ when $\alpha = 1.5, \beta = 0, \delta = 0$.

a larger α, since the diversity can be preserved by tuning p_e and overlarge random variables can be avoided.

A similar phenomenon can be observed from Fig. 2(b) by tuning γ. It can be seen that a smaller p_e is required to exhibit the best resonance effect when increasing γ. Increasing γ means that the width of the distribution turns wider, i. e., the diversity of the generated noise becomes richer. Therefore, a smaller p_e is necessary such that the diversity can be kept at a proper level to enhance the resonance behavior when γ increases. In addition, we find that when p_e turns smaller, a large set of γ can be allowed to show resonance behaviors, i. e., the smaller p_e will broaden the area of γ for showing resonance behaviors. From the above observations, the probability p, the stability parameter α and the scale parameter γ play constructive roles in enhancing resonance effects of an array of coupled complex systems.

In order to show the properties of switching noise, stochastic resonance with partial noise and switching noise are compared. In switching noise, $\mathbb{E}\{\sum_{i=1}^{N} \xi_i(t)\} = p * N$ holds, which means that there exists the same number of units (in expectation sense) subjected to noise as the case of partial noise.

Figure 3(a) depicts the comparison of partial noise and switching noise in globally coupled networks (GCNs) and nearest neighbor networks (NNNs). By employing switching noise in both GCNs and NNNs, as p increases, the amplification factor R first increases and the best resonance behavior can be reached. When p further increases, the amplification factor R decreases slowly, i. e., there exists a peak by varying p. In addition, GCNs can present more pronounced SR than NNNs in that NNNs cannot efficiently transmit the signal of noise through networks, when partial noise is applied. Figure 3(a) shows that partial noise has a "partial" weak effect on the signal amplification R. One can also observe that, with the same variance of noise, switching noise can deliver a more pronounced resonance effect than partial noise, even with a smaller p. Moreover, in either GCNs or NNNs, the best resonance effect of switching noise is closer to each other than that of partial noise, i. e., switching noise is more robust to induce resonance behaviors for different types of networks than partial noise. This can be explained as follows. In partial noise, the effects of noise have to be spread all over the network by connection between the units. Hence, the diversity will be weakened due to the transmission in the network. Nevertheless, in switching noise, all the units can be injected by noise frequently according to probability p. Therefore, the diversity of the units can be maintained to enhance the resonance behavior.

Next, we will show stochastic resonance of partial noise and switching noise for different network sizes of networks and simulation time W. We consider networks under $N = 50$, $N = 100$ and $N = 800$ with a global coupling structure. It can be observed from Fig. 4 that the results are robust under different scale, since the injection of noise in the nodes is according to the probability p. Figure 5 depicts the effects of simulation time W on SR. It is shown that the results are also robust for different W.

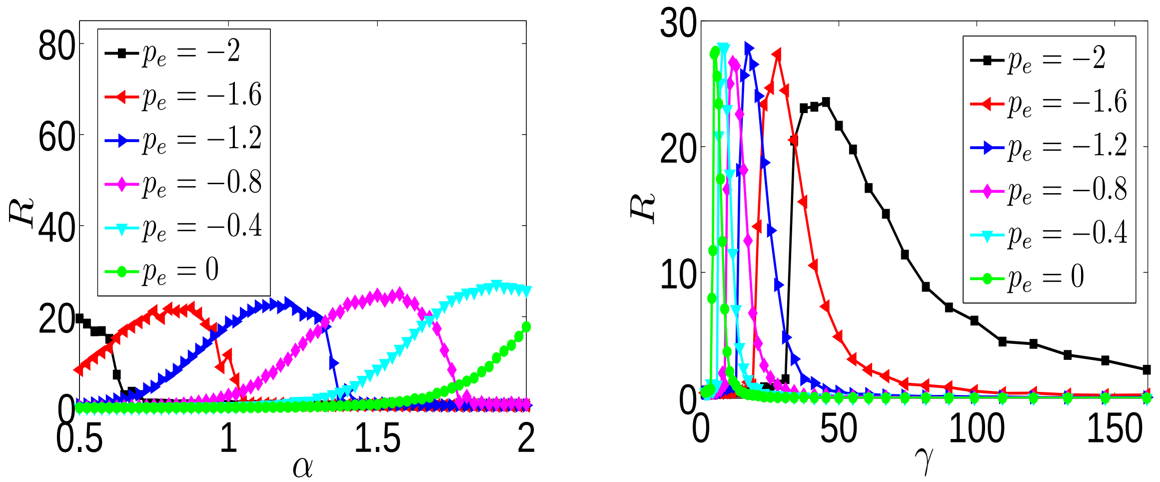
In the following, we will establish a connection between switching noise and usual global noise. Denote

$$\lambda(x) = \frac{x - \mathbb{E}(x)}{\mathbb{E}(x)}, h(x) = \frac{\mathbb{D}(x)}{[\mathbb{E}(x)]^2}.$$

Following the way of Ref. [40], if two variables x and y are independent, one has:

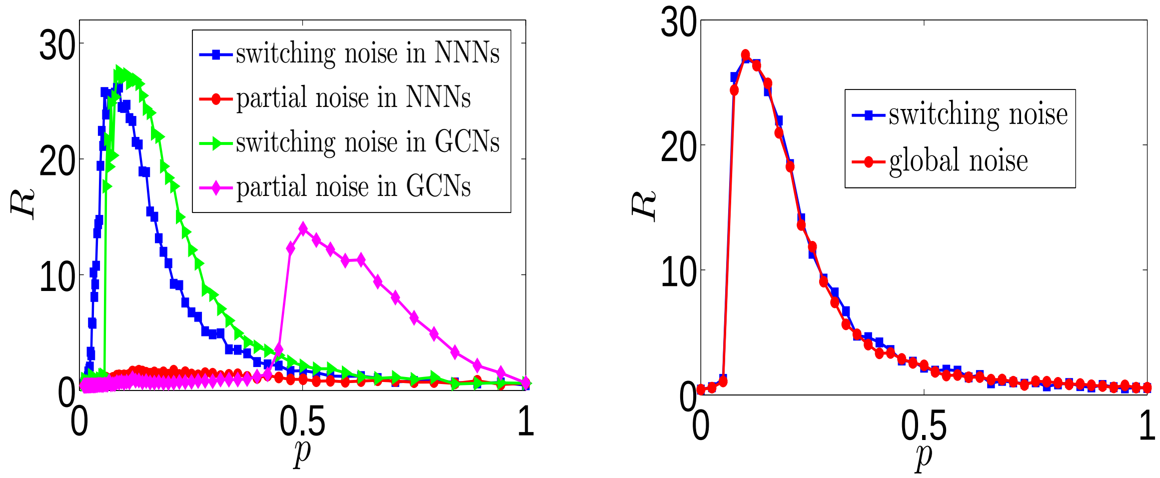
$$\begin{aligned} xy - \mathbb{E}(x)\mathbb{E}(y) &= \mathbb{E}(x)\mathbb{E}(y)[(\lambda(x)+1)(\lambda(y)+1) - 1] \\ &= \mathbb{E}(x)\mathbb{E}(y)[\lambda(x) + \lambda(y) \\ &\quad +\lambda(x)\lambda(y)]. \qquad (4) \end{aligned}$$

Note that $\mathbb{D}(x) = \mathbb{E}[(x)^2] - [\mathbb{E}(x)]^2$ and by considering (4),



(a) (b)

FIG. 2: (Color online) The spectral amplification factor R of globally coupled bistable systems is plotted as a function of α, γ and p_e, when switching noise is considered. (a) R as a function of α and p_e when $\beta = 0, \gamma = 5\sqrt{2}, \delta = 0$ and $W = 100$; (b) R as a function of γ and p_e when $\alpha = 2, \beta = 0, \delta = 0$ and $W = 100$.



(a) (b)

FIG. 3: (Color online) Properties of SR of coupled bistable systems with partial noise and switching noise: (a) comparison of SR of coupled bistable systems with partial noise and switching noise when $\alpha = 2, \beta = 0, \gamma = 10^{1.2}, \delta = 0$ and $W = 100$; (b) comparison of SR of globally coupled bistable systems with global noise and switching noise when $\alpha = 2, \beta = 0, \gamma = 10^{1.2}, \delta = 0, \check{\gamma} = \sqrt{p} * 10^{1.2}$ and $W = 100$.

we have

$$\begin{aligned}\mathbb{D}(xy) &= \mathbb{E}[xy - \mathbb{E}(xy)]^2 \\ &= [\mathbb{E}(x)\mathbb{E}(y)]^2[h(x) + h(y) + h(x)h(y)] \\ &= [\mathbb{E}(x)]^2\mathbb{D}(y) + [\mathbb{E}(y)]^2\mathbb{D}(x) + \mathbb{D}(x)\mathbb{D}(y). \quad (5)\end{aligned}$$

For switching noise with a normal distribution, i. e., noise η_i having a stable distribution with $\alpha = 2, \beta = 0, \delta = 0$ characterized by Bernoulli stochastic variables $\xi_i(t)$, we get from (5) that

$$\begin{aligned}\mathbb{D}(xy) &= p^2 * 2\gamma^2 + p(1-p) * 2\gamma^2 \\ &= 2p\gamma^2. \quad (6)\end{aligned}$$

Therefore, when $\alpha = 2, \beta = 0, \delta = 0$, one can write (1) into the one without Bernoulli stochastic variables as an equivalent version:

$$\begin{aligned}\dot{x}_i &= x_i - x_i^3 + \frac{g}{d_i}\sum_{i=1}^{N}(x_j - x_i) + F\sin(\Psi t) \\ &\quad + \varphi_i, i = 1, 2, ..., N, \quad (7)\end{aligned}$$

where $\mathbb{D}(\varphi_i) = 2p\gamma^2$. The scale parameter $\check{\gamma}$ is $\check{\gamma} = \sqrt{p}\gamma$.

In Fig. 3(b), we show the equivalence of systems (1) and (7). It can be seen that the values of the amplification factor R of (1) subjected to switching noise fit well with R of (7) driven by global noise. However, although we can develop (7)

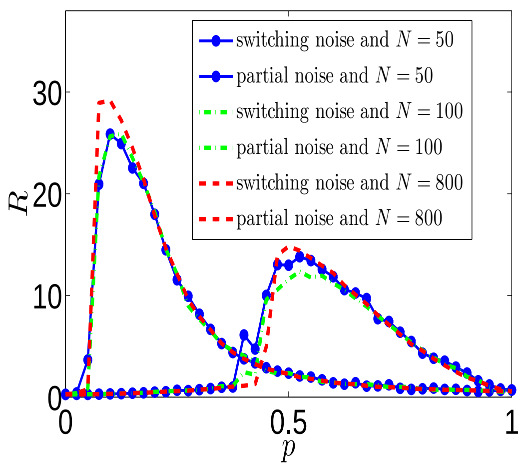


FIG. 4: (Color online) Results of SR of coupled bistable systems with partial noise and switching noise under different scale N, when $\alpha = 2, \beta = 0, \gamma = 10^{1.2}, \delta = 0$ and $W = 100$.

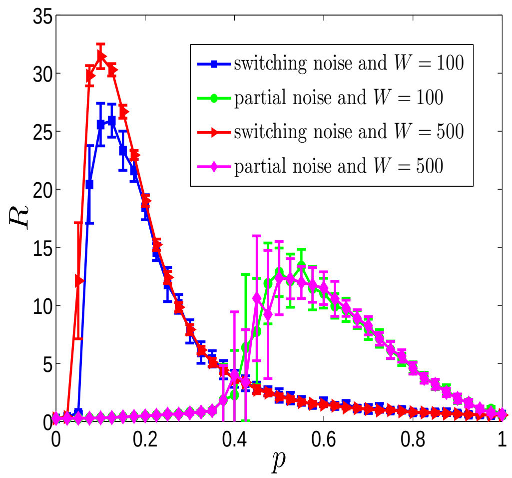


FIG. 5: (Color online) Results of SR of coupled bistable systems with partial noise and switching noise under different simulation time W, when $\alpha = 2, \beta = 0, \gamma = 10^{1.2}, \delta = 0$ and $N = 200$.

by selecting appropriate variance to amplify R like (1), there exists a major difference here. System (1) should be injected with noise by a large variance such that it can compensate the effect of random switching of noise. The noise acts like a "pinning controller" in [33], in which the noise is not necessary to work at each unit all the time and thus can reduce the implementation burden and the noise under random switching can also enhance the occurrence of resonance behaviors like global noise [20]. The maximum value of R reached by system (1) with switching noise is also close to that of (7), different from that of system (1) with partial noise.

In the following, to quantify the response of the coupled bistable systems to the external forcing, we utilize the approximate theory to conduct an analytical analysis [18]. $X(t)$ is the mean-field value of the units and thus (1) with global coupling can be rewritten as follows:

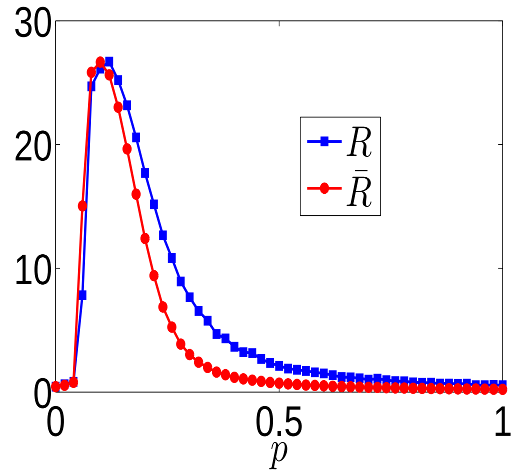


FIG. 6: (Color online) The approximating theory for the spectral amplification factor R when $\alpha = 2, \beta = 0, \gamma = 10^{1.2}, \delta = 0$, $W = 100$ and $N = 200$. $\bar{R}$ is an approximation of R.

$$\begin{aligned}\dot{x}_i &= \frac{gN}{N-1}X + \left(1 - \frac{gN}{N-1}\right)x_i - x_i^3 + \xi_i(t)\eta_i \\ &\quad + F\sin(\Psi t).\end{aligned} \tag{8}$$

By averaging (8), one yields

$$\dot{X} = X - \frac{1}{N}\sum_{i=1}^{N} x_i^3 + \frac{1}{N}\sum_{i=1}^{N}\xi_i(t)\eta_i + F\sin(\Psi t). \tag{9}$$

Let θ_i be $\theta_i = x_i - X$ and $\frac{1}{N}\sum_{i=1}^{N}\theta_i^2 = \mathcal{V}(t)$. It yields from (9) that

$$\dot{X} = X(1 - 3\mathcal{V}(t)) - X^3 + \sum_{i=1}^{N}\xi_i(t)\eta_i + F\sin(\Psi t). \tag{10}$$

Note that $\mathbb{E}\{\sum_{i=1}^{N}\xi_i(t)\eta_i\} = 0$, we obtain

$$\dot{X} = X(1 - 3\mathcal{V}(t)) - X^3 + F\sin(\Psi t). \tag{11}$$

The amplification factor obtained by (11) is denoted by $\bar{R}$. Figure 6 illustrates the approximating results. It can be seen that the method is in good agreement with the results by a direct simulation of the original system (1).

IV. CONCLUSION

To summarize, in this paper, we have investigated the problem of SR in coupled complex systems under pinning noise, in which the pinning noise satisfies a Lévy α-stable distribution. It is shown that the pinning noise in the forms of switching noise and partial noise make the system to deliver a resonance behavior in response to an external periodic stimulus. We demonstrate the crucial roles of the percent of the units subjected to noise p (or the probability of the units subjected to noise p), the stability parameter α and the scale parameter γ in showing SR. Adjusting p can efficiently control the width of γ to generate SR, where a smaller p will broaden the interval of γ to exhibit resonance behavior. The interplay of these three parameters induces a proper diversity to exhibit the best resonance effect.

We have found that the system with switching noise can not only lead to a more pronounced resonance effect, but it is also more robust to exhibit the resonance effect than partial noise in different types of networks, due to the existence of blinking mechanisms in switching noise. The relationship between (switching) pinning control and (switching) pinning noise is discussed. Different from global noise, the switching noise acts like a "pinning controller", i. e., the noise is not required to work at each unit all the time and thus is more flexible than global noise. We have also developed a mathematical relationship between the global noise-induced phenomenon and the switching noise-induced phenomenon.

The results about switching noise seem to have strong potential applications to control pathological activities in neuroscience, ecology and in engineering problems [41]. We expect that our findings could be of importance for providing valuable guidelines not only in bistable systems but also in neural and other excitable media [42]. The theoretical findings will also stimulate experimental works to verify SR with pinning noise in real physical systems such as nonlinear electronic circuits [14] and robots coordination. One direction for future research is to study SR with nonidentical units. Another one is to utilize switching noise to investigate other kinds of pinning noise-induced phenomena such as synchronization [35, 43–45] and coherence [6].

Acknowledgments

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[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A, **14**, L453 (1981).
[2] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[3] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, England, 2001).
[4] P. Jung and P. Hänggi, EPL, **8**, 505 (1989).
[5] C. Zhou, J. Kurths, and B. Hu, Phys. Rev. Lett. **87**, 098101 (2001).
[6] A. Pikovsky and J. Kurths, Phys. Rev. Lett., **78**, 775 (1997).
[7] R. N. Mantegna and B. Spagnolo, Phys. Rev. Lett. **76**, 563 (1996).
[8] N. Agudov and B. Spagnolo, Phys. Rev. E **64**, 035102(R)(2001).
[9] A. A. Dubkov, N. V. Agudov and B. Spagnolo, Phys. Rev. E **69**, 061103(2004).
[10] R. N. Mantegna and B. Spagnolo, Phys. Rev. Lett. **84**, 3025-3028(2000).
[11] B. Lindner, J. García-Ojalvo, A. Neiman, and L. Schimansky-Geier, Phys. Rep. **392**, 321 (2004).
[12] T. Wellens, V. Shatokhin, and A. Buchleitner, Rep. Prog. Phys. **67**, 45 (2004).
[13] P. Jung and F. Marchesoni, Chaos, **21**, 047516 (2011).
[14] A. Zaikin, J. Kurths, and L. Schimansky-Geier, Phys. Rev. Lett., **85**, 227 (2000).
[15] M. Ruskoni, A. Zaikin, N. Marwan, and J. Kurths, Phys. Rev. Lett. **100**, 128101 (2008).
[16] A. La Cognata, D. Valenti, A. A. Dubkov, and B. Spagnolo, Phys. Rev. E **82**, 011121(2010).
[17] A. Pikovsky, A. Zaikin, and M. Casa, Phys. Rev. Lett. **88**, 050601 (2002).
[18] C. Tessone, C. Mirasso, Raúl Toral, and J. Gunton, Phys. Rev. Lett., **97**, 194101 (2006).
[19] M. Perc, Phys. Rev. E, **78**, 036105 (2008).
[20] Y. Tang, W. Zou, J. Lu, and J. Kurths, Phys. Rev. E, **85**, 046207 (2012).
[21] C. Zhou, J. Kurths, I. Z. Kiss, and J. L. Hudson, Phys. Rev. Lett., **89**, 014101 (2002).
[22] B. Dybiec, E. Gudowska-Nowak, J. Stat. Mech. P05004 (2009).
[23] R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge, England, 2000).
[24] G. M. Viswanathan, M. E. da Luz, E. Raposo, and H. E. Stanley, *The Physics of Foraging: An Introduction to Random Searches and Biological Encounters* (Cambridge University Press, Cambridge, England, 2011).
[25] M. Perc, Phys. Rev. E **75**, 022101 (2007).
[26] B. Dybiec, Phys. Rev. E., **80**, 041111 (2009).
[27] L. Zhang, A. Song, and J. He, J. Phys. A, **42**, 475003(2009).
[28] W. Zou, M. Zhan, Phys. Rev. E, **80**, 065204(R) (2009).
[29] Y. Tang, H. Gao, W. Zou, and J. Kurths, IEEE Trans. On Cybernetics, **43**, 358-370(2013).
[30] Y. Tang, and W. K. Wong, IEEE Trans. On Neural Networks and Learning Systems, **24**, 435-447(2013).
[31] T. Preis, J. Schneider, and H. E. Stanley, Proc. Natl. Acad. Sci. USA, **108**, 7674-7678 (2011).
[32] X. Mao and C. Yuan, Stochastic Differential Equations with Markovian Switching. Imperial College Press, 2008.
[33] Y. Liu, J. Slotine, and A. Barabasi, Nature, **473**, 167(2011).
[34] Y. Tang, H. Gao, W. Zou and J. Kurths, PLoS ONE, **7**, e41375(2012).
[35] M. Frasca, A. Buscarino, A. Rizzo and L. Fortuna, Phys. Rev. Lett., **108**, 204102 (2012).
[36] J. Nolan, Stable Distributions-Models for Heavy Tailed Data (Birkhäuser, Boston, 2013), in progress, Chapter 1 online at academic2.american.edu/∼jpnolan.
[37] D. J. Higham, SIAM Review, **43**, 525 (2001).

[38] J. Nolan, http://academic2.american.edu/∼jpnolan/stable/stable.html.

[39] C. Masoller and A. C. Martí, Phys. Rev. Lett., **94**, 134102 (2005).

[40] L. A. Goodman, J. Am. Stat. Assoc., **55**, 708 (1960).

[41] W. Zou, Y. Tang, L. Li and J. Kurths, Phys. Rev. E, **85**, 046206 (2012).

[42] E. Ullner, A. Zaikin, J. García-Ojalvo, and J. Kurths, Phys. Rev. Lett., **91**, 180601(2003).

[43] Y. Tang, Z. Wang, H. Gao, S. Swift, and J. Kurths, IEEE/ACM Trans. On Computational Biology and Bioinformatics, **9**, 1569-1581(2012).

[44] Y. Tang, Z. Wang, W. K. Wong, J. Kurths and J. Fang, Chaos, **21**, 025114(2011).

[45] Y. Tang, J. Fang, Q. Miao, Int. J. of Neural Systems, **19** 43(2009).