Global generalized synchronization in networks of different time-delay systems

2013 EPL 103 50010

(http://iopscience.iop.org/0295-5075/103/5/50010)

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

September 2013

EPL, **103** (2013) 50010 www.epljournal.org
doi: 10.1209/0295-5075/103/50010

Global generalized synchronization in networks of different time-delay systems

D. V. SENTHILKUMAR[1], R. SURESH[2], M. LAKSHMANAN[2] and J. KURTHS[1,3,4]

[1] *Potsdam Institute for Climate Impact Research - 14473 Potsdam, Germany, EU*
[2] *Centre for Nonlinear Dynamics, School of Physics, Bharathidasan University - Tiruchirapalli 620 024, India*
[3] *Institute of Physics, Humboldt University - 12489 Berlin, Germany, EU*
[4] *Institute for Complex Systems and Mathematical Biology, University of Aberdeen - Aberdeen AB24 3UE, UK, EU*

received 1 April 2013; accepted in final form 4 September 2013
published online 30 September 2013

PACS 05.45.Xt – Synchronization; coupled oscillators
PACS 05.45.Pq – Numerical simulations of chaotic systems
PACS 05.45.-a – Nonlinear dynamics and chaos

Abstract – We show that global generalized synchronization (GS) exists in structurally different time-delay systems, even with different orders, with quite different fractal (Kaplan-Yorke) dimensions, which emerges via partial GS in symmetrically coupled regular networks. We find that there exists a smooth transformation in such systems, which maps them to a common GS manifold as corroborated by their maximal transverse Lyapunov exponent. In addition, an analytical stability condition using the Krasvoskii-Lyapunov theory is deduced. This phenomenon of GS in strongly distinct systems opens a new way for an effective control of pathological synchronous activity by means of extremely small perturbations to appropriate variables in the synchronization manifold.

Synchronization is a ubiquitous nonlinear phenomenon serving as a platform for information processing in diverse natural and man-made systems [1,2]. It has been investigated mainly in identical systems and in systems with parameter mismatch, with rare exceptions of distinctly (structurally) nonidentical systems [1,2]. However, in reality very often synchronization emerges in distinctly nonidentical systems such as respiratory arrhythmia between cardiovascular and respiratory systems [3], visual and motor systems [4], paced maternal breathing on fetus [5], different populations of species [6,7], in epidemics [8], in climatology [9], and many more. Considering the coherent coordination of living systems involving multiple organs such as brain, heart, lungs, limbs, etc., or machines consisting of distinct parts, cooperative evolution of distinct and often time-delayed systems is essential and challenging.

Among various kinds of synchronization admitted by coupled nonlinear dynamical systems [1,2], the intricate phenomenon of generalized synchronization (GS) refers to a (static) functional relationship between interacting systems [10–25]. While the phenomenon of GS has been well understood in unidirectionally coupled systems [10–13], it remains still in its infancy in bidirectionally coupled systems and in particular there exist only very limited results on GS, even in systems with parameter mismatches [14–23], and particularly in structurally different (nonidentical) systems with different fractal dimensions [24]. Thus, in general, the notion of GS in mutually coupled systems needs much in-depth investigation and in particular in distinctly different systems with different fractal dimensions involving time delay. Indeed, recent investigations have revealed that GS is essentially more likely to occur in complex networks (even with identical nodes) [15] due to the large heterogeneity (degree distribution) of many natural networks [21].

It is important to recall that the above-mentioned studies [3–9] have demonstrated only phase synchronization (PS) among such distinctly different complex systems, while the natural choice of GS in them has been largely neglected, except for the important study of Zheng *et al.* [14] on low-dimensional systems without delay and without a substantial difference in their fractal dimension. Further, depending on the relation between PS and GS[1] in such systems, which remains unclear, our understanding of their evolutionary mechanism, dynamical and functional

[1]The facts that PS (GS) emerges first for a low (high) degree of parameter mismatch and that they both occur simultaneously for a critical range of mismatch, have been clearly shown in low-dimensional systems, see, for example, [25].

behavior may need to be reinvestigated. Controlling the pathological synchronous activity and inducing coherent coordination in paralysed systems may be effectively done upon understanding the emergence of a common GS manifold in such systems. Furthermore, in a more general scenario of networks of distinctly nonidentical time-delay systems with different fractal dimensions, it remains unclear whether there exists a transformation (map) that maps the individual systems to a common GS manifold despite disparity in their degree of complexity.

In line with the above discussions, in this letter we will provide a substantial extension of the formulation of GS to mutually coupled distinctly different delay systems[2], which also holds for systems without delay. Based on our generalized formulation, we will specifically demonstrate the existence of *global GS* via *partial GS* in symmetrically coupled networks, which even consist of distinctly different time-delay systems (Mackey-Glass [26], piecewise linear [27], threshold nonlinearity [28] and Ikeda [29]). It is important to note that such a phenomenon also occurs among delay systems of different orders, namely, Ikeda and a second-order Hopfield neural network [30] as well as Mackey-Glass and a third-order plankton model [31] with multiple delays. It is surprising that there exists a common GS manifold even in an ensemble of distinctly different time-delay systems to achieve *global GS* in four different network (array, ring, star and all-to-all) configurations, *i.e.*, there exists a function (smooth map) for each system, even with different fractal dimensions, in a network which maps them to the common GS manifold. We calculate the maximal transverse Lyapunov exponent (MTLE) to evaluate the asymptotic stability of the complete synchronization (CS) manifold of each of the main systems with their corresponding auxiliary systems, which, in turn, asserts the stability of the GS manifold between the main systems. Further, we will also estimate cross-correlation (CC) and correlation of the probability of recurrence (CPR) [32] to establish relations between GS and PS.

Kocarev and Parlitz [10] formulated that GS in drive $\boldsymbol{x}$ and response $\boldsymbol{y}$ configuration occurs only if the response is asymptotically stable, *i.e.* $\forall\ \boldsymbol{y_i}(0), \boldsymbol{x}(0)$ in the basin of the synchronization manifold $\lim_{t\to\infty} ||\boldsymbol{y}(t,\boldsymbol{x}(0),\boldsymbol{y}_1(0)) - \boldsymbol{y}(t,\boldsymbol{x}(0),\boldsymbol{y}_2(0))|| = 0$, a mathematical formulation of the concept of auxiliary-system approach [11]. Now we give a substantial extension of this formulation to *mutually* coupled different dynamical systems with delay represented as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}_\tau,\boldsymbol{u}), \qquad \dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y},\boldsymbol{y}_\tau,\boldsymbol{v}), \qquad \boldsymbol{f} \neq \boldsymbol{g}, \qquad (1)$$

[2]The main reason to consider time-delay systems in this work is due to the ubiquity of time-delay systems as several real systems in ecology, epidemics, physiology, economics, engineering and control systems, etc. [2] inevitably require intrinsic delay (different from connection delay between different units) for a complete description. Even with a single time-delay system, one has the flexibility of choosing systems with different fractal dimensions just by adjusting their intrinsic delay, which is quite attractive from a modelling point of view.

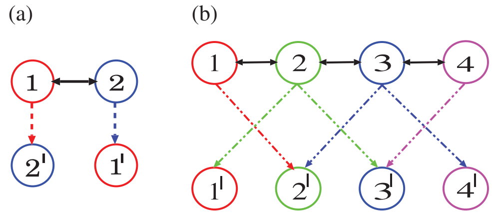

Fig. 1: (Color online) Schematic diagram of the auxiliary-system approach for mutually coupled time-delay systems for the case of a linear array for $N = 2$ (a) and $N = 4$ (b).

where $\boldsymbol{x},\boldsymbol{x}_\tau \in \boldsymbol{R}^n, \boldsymbol{y},\boldsymbol{y}_\tau \in \boldsymbol{R}^m$, $\tau \in \boldsymbol{R}$ and $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{R}^k, k \leq m,n$. $u_i = -v_i = h_i(\boldsymbol{x}(t,\boldsymbol{x}_0),\boldsymbol{y}(t,\boldsymbol{y}_0))$ correspond to the driving signals. System (1) is in GS if there exists a transformation $\boldsymbol{H}\colon (\boldsymbol{R}^n,\boldsymbol{R}^m) \to \ \subset \boldsymbol{R}^n \times \boldsymbol{R}^m$. That is, there may exist a set of transformations H that maps a given $\boldsymbol{x},\boldsymbol{x}_\tau \in \boldsymbol{R}^n$ and $\boldsymbol{y},\boldsymbol{y}_\tau \in \boldsymbol{R}^m$ to different subspaces of $\boldsymbol{R}^n \times \boldsymbol{R}^m$. This implies that the synchronization manifold $M = \{(\boldsymbol{x},\boldsymbol{y})\colon \boldsymbol{H}(\boldsymbol{x},\boldsymbol{y}) = 0\}$ is such that $\forall\ \boldsymbol{x}(\hat{\tau}), \boldsymbol{y}(\hat{\tau}), \hat{\tau} \in [-\tau, 0]$, which lies within the subset of the basin of attraction $B = B_{\boldsymbol{x}} \times B_{\boldsymbol{y}} \subset \boldsymbol{R}^n \times \boldsymbol{R}^m$, approaches $M \subset B$, so that M is an attracting manifold. Here $B_{\boldsymbol{x}}$ and $B_{\boldsymbol{y}}$ are the basins of attractions of systems x and y, respectively.

Thus, GS exists in system (1) only if both coupled systems are asymptotically stable such that $\forall\ (\boldsymbol{x}_i(\hat{\tau}), \boldsymbol{y}_i(\hat{\tau}))$, $\hat{\tau} \in [-\tau, 0] \subset B$, $i = 1,2$, $\lim_{t\to\infty} ||\boldsymbol{y}(t,\boldsymbol{x}_1(\hat{\tau}),\boldsymbol{y}_1(\hat{\tau})) - \boldsymbol{y}(t,\boldsymbol{x}_1(\hat{\tau}),\boldsymbol{y}_2(\hat{\tau}))|| = 0$ and $\lim_{t\to\infty} ||\boldsymbol{x}(t,\boldsymbol{x}_1(\hat{\tau}),\boldsymbol{y}_1(\hat{\tau})) - \boldsymbol{x}(t,\boldsymbol{x}_2(\hat{\tau}),\boldsymbol{y}_1(\hat{\tau}))|| = 0$. This is a mathematical formulation of the auxiliary-system approach to system (1), whose schematic diagrams with the main and auxiliary systems for the case of a linear array are depicted in figs. 1(a) and (b), respectively, which are similar in approach to that of [14] concerned with low-dimensional systems without delay. It is to be noted that the asymptotic stability of synchronization of the auxiliary systems and their corresponding original systems holds good only when there is no subharmonic entrainment of unstable periodic orbits [33]. Therefore, trajectories of (1) starting from B asymptotically reach M defined by the transformation function $\boldsymbol{H}(\boldsymbol{x},\boldsymbol{y})$, which can be smooth if the systems (1) uniformly converge (otherwise nonsmooth), *i.e.* their local Lyapunov exponents (LLEs) are always negative, to M [34].

Now, we will demonstrate the existence of GS in symmetrically coupled arbitrary network of distinctly nonidentical time-delay systems with different fractal dimensions using the above formalism. The dynamics of the i-th node in the network is represented as

$$\dot{\boldsymbol{x}}_i = -\alpha_i \boldsymbol{x}_i(t) + \beta_i \boldsymbol{f}_i(\boldsymbol{x}_i(t-\tau_i)) - \varepsilon \sum_{j=1}^{N} G_{ij}\boldsymbol{x}_j, \qquad (2)$$

where $i = 1,\ldots,N$, N is the number of nodes in the network, α_i and β_i are constant parameters, τ_i are the delay times, $\boldsymbol{f}_i$ is the nonlinear vector function of i-th node and

G is a Laplacian matrix, determining the topology of the arbitrary network. To determine the asymptotic stability of each of the nodes in the network, we define an idential (auxiliary) network (starting from different initial conditions) with node i represented as (see fig. 1)

$$\dot{\boldsymbol{x}}'_i = -\alpha_i \boldsymbol{x}'_i(t) + \beta_i \boldsymbol{f}'_i(\boldsymbol{x}'_i(t-\tau_i)) - \varepsilon \sum_{j=1}^{N} G_{ij}(\delta_{ij}(-\boldsymbol{x}_j) + \boldsymbol{x}_j). \quad (3)$$

First, we consider $N = 2$ mutually coupled distinctly nonidentical time-delay systems with the nonlinear function

$$f_1(x) = \frac{x_1(t-\tau_1)}{(1 + (x_1(t-\tau_1)^{10}))}, \quad (4)$$

of the Mackey-Glass (MG) [28] system and the piecewise linear (PWL) function

$$f_2(x) = \begin{cases} 0, & x \le -4/3, \\ -1.5x - 2, & -4/3 < x \le -0.8, \\ x, & -0.8 < x \le 0.8, \\ -1.5x + 2, & -0.8 < x \le 4/3, \\ 0, & x > 4/3 \end{cases} \quad (5)$$

of the PWL time-delay system [27], respectively. The parameters of the MG systems are chosen as $\alpha_1 = 0.5$, $\beta_1 = 1.0$, and $\tau_1 = 8.5$ and for the PWL systems we choose $\alpha_2 = 1.0$, $\beta_2 = 1.2$ and $\tau_2 = 10.0$, exhibiting hyperchaotic attractors with two ($D_{KY} = 2.957$) and three ($D_{KY} = 4.414$) positive Lyapunov exponents (LEs) [2], respectively.

At first sight, one may think that mutually interacting systems reach a common synchronization manifold simultaneously for a certain critical value of ε. But because of the distinctly different coupled systems with different fractal dimensions, one of the systems first reaches a GS manifold for a lower value of $\varepsilon_c^{(1)}$, while the other system remains in a desynchronized state, which we refer to as a *partial GS*. With further increase of ε the other system also converges at a different critical $\varepsilon_c^{(2)}$ to the common GS manifold achieving a *global GS*. Further, one may also expect that systems with lower dynamical complexity will converge to the GS manifold first and then the higher one in the order of their degree of complexity (here in terms of the number of positive LEs). On the contrary, the PWL system with three positive LEs reaches the GS manifold first at $\varepsilon_c^{(2)} \approx 0.26$ as indicated by the changes in the sign of the $\lambda^{(2)}_{MTLE}$ between 2 and 2′ in fig. 2(a) and then the MG system with only two positive LEs smoothly converges to the GS manifold at further larger $\varepsilon_c^{(1)} \approx 0.5$, where $\lambda^{(1)}_{MTLE} < 0$ as shown in fig. 2(a) confirming the emergence of a *global GS* in distinctly nonidentical time-delay systems.

The emergence and the transition from partial to global GS is also characterized by the correlation coefficient

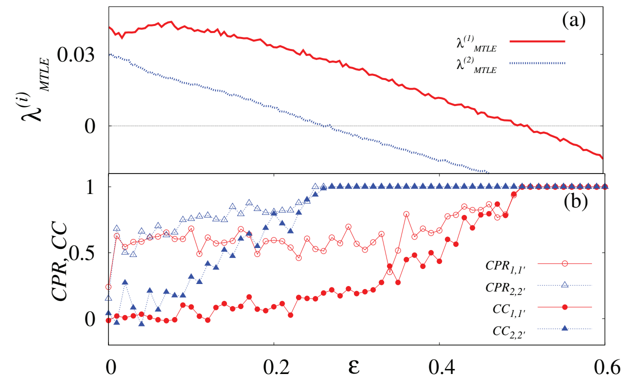

Fig. 2: (Color online) MTLEs (a) and CC, CPR (b) of the main and auxiliary systems of coupled MG-PWL systems.

(CC)[3]. If the two systems are in the CS state the $CC = 1$, otherwise $CC < 1$. Further, the existence of PS in highly non–phase-coherent hyperchaotic attractors of time-delay systems is characterized by the value of the correlation of probability of recurrence (CPR ≈ 1) [27,32].

In the absence of coupling ($\varepsilon = 0.0$), $CC_{1,1'}$ and $CC_{2,2'}$ and $CPR_{1,1'}$ and $CPR_{2,2'}$ (fig. 2(b)) are nearly zero and both $\lambda^{(1)}_{MTLE}$ and $\lambda^{(2)}_{MTLE} > 0$, indicating the absence of CS (GS) between the main and auxiliary systems. If we increase the coupling strength, $CC_{2,2'}$ and $CPR_{2,2'}$ start to increase towards unity and at $\varepsilon_c^{(1)} \approx 0.26$, $CC_{2,2'} = 1$ ($CPR_{2,2'} = 1$), where $\lambda^{(2)}_{MTLE} < 0$, which confirms the onset of GS (PS) in the PWL system while the MG system continues to remain in a desynchronized state ($CC_{2,2'} \approx 0.2$ and $\lambda^{(1)}_{MTLE} > 0$). Further, if we increase the coupling strength to $\varepsilon_c^{(2)} \approx 0.5$, a global GS occurs where both $CC_{1,1'}$ and $CC_{2,2'}$ become unity and while both $\lambda^{(1)}_{MTLE}$ and $\lambda^{(2)}_{MTLE} < 0$. It is also to be noted that both $CPR_{1,1'}$ and $CPR_{2,2'}$ are also in agreement with their CC confirming the existence of GS and PS together.

We have also analytically investigated the existence of partial and global GS using the Krasovskii-Lyapunov theory. For this purpose, we consider the difference in the state variables of the main and auxiliary systems ($\Delta_i = x_i - x'_i$). For small values of Δ_i, the evolution equation for the CS manifold for the Mackey-Glass time-delay systems ($\Delta_1 = x_1 - x'_1$) can be written as

$$\dot{\Delta}_1 = -(\alpha_1 + \varepsilon)\Delta_1 + \beta_1[f'_1(x_{\tau_1})]\Delta_{1\tau_1}, \quad (6)$$

where $x_{\tau_1} = x(t-\tau_1)$ and $\Delta_{1\tau_1} = \Delta_1(t-\tau_1)$. The CS manifold is locally attracting if the origin of $\dot{\Delta}_1$ is stable. Then, a continuous positive definite Lyapunov functional can be defined in the form

$$V(t) = \frac{1}{2}\Delta_1^2 + \mu \int_{-\tau}^{0} \Delta_1^2(t+\theta)\mathrm{d}\theta, \qquad V(0) = 0, \quad (7)$$

[3] $C_{i,i'} = \frac{\langle (x_i(t) - \langle x_i(t)\rangle)(x'_i(t) - \langle x'_i(t)\rangle)\rangle}{\sqrt{\langle (x_i(t) - \langle x_i(t)\rangle)^2\rangle \langle (x'_i(t) - \langle x'_i(t)\rangle)^2\rangle}}$, where the $\langle \ldots \rangle$ brackets indicate time average.

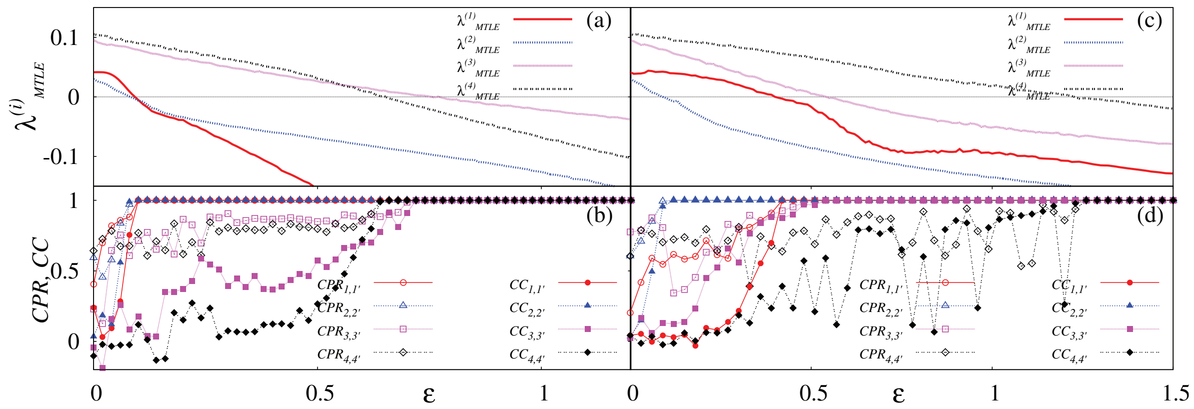

Fig. 3: (Color online) MTLEs ((a) and (c)) and CC, CPR ((b) and (d)) of the main and auxiliary systems of four distinctly nonidentical time-delay systems with ring and array configurations, respectively, as a function of ε.

where μ is an arbitrary positive parameter ($\mu > 0$). The Lyapunov function $V(t)$ approaches zero as $\Delta_1 \to 0$. The derivative of $V(t)$ along the trajectory of the CS manifold should be negative for the stability of the CS manifold $\Delta_1 = 0$. This requirement results in a sufficiency condition for the asymptotic stability as

$$(\alpha_1 + \varepsilon) > |\beta_1 f_1'(x_{\tau_1})|. \tag{8}$$

Note that the evolution equation of the synchronization manifold and hence the stability condition depends on the $f_1'(x_\tau)$, which, in turn, depends on the nonlinearity of the individual systems. From the form of the nonlinear function $f(x)$ for the MG system, the stability condition becomes

$$(\alpha_1 + \varepsilon) > \left|\beta_1 \left(\frac{(1 + x_{\tau_1}^c) - c x_{\tau_1}^c}{(1 + x_{\tau_1}^c)^2}\right)\right|. \tag{9}$$

It is not possible to find the exact value of the nonlinear function $f_1'(x_{\tau_1})$. However, in accordance with the Lyapunov-Razumikin function, a special class of the Krasovskii-Lyapunov theory, one can find the value of $f_1'(x_{max})$ from the maximal value of $x(t)$ and arrive at a sufficient condition for CS (GS).

From the hyperchaotic attractor of the Mackey-Glass system for the above chosen parameter values one can find numerically the maximum value of $x_{max} \approx 1.24$. So the stability condition becomes $\varepsilon > |\beta_1 f_1'(x_{max\ \tau_1})| - \alpha_1 \approx 0.33$ (which is a sufficiency condition for global GS). From fig. 2, the threshold value of the coupling strength to attain GS in the Mackey-Glass systems is $\varepsilon_c^{(2)} \approx 0.5$ (> 0.33) for which the stability condition is satisfied. With a similar procedure, the stability condition for the PWL systems become $(\alpha_2 + \varepsilon) > |\beta_2 f_2'(x_{\tau_2})|$. Now from the form of the piecewise linear function $f_2(x)$, we have

$$f_2'(x_{\tau_2}) = \begin{cases} 1.5, & 0.8 \le |x| \le \frac{4}{3}, \\ 1.0, & |x| < 0.8. \end{cases} \tag{10}$$

Consequently, the stability condition becomes $(\alpha_2 + \varepsilon) > |\beta_2|$ (see footnote [4]), for the asymptotic CS state $\Delta_2 = 0$. From fig. 2, we find that CS between the piecewise linear systems (x_2 and x_2') occur for the coupling strength $\varepsilon_c^{(1)} > 0.27$, which satisfies the sufficient stability condition for partial GS $\varepsilon > |\beta_2| - \alpha_2 = 0.2$ ($0.27 > 0.2$).

Next, we demonstrate the existence of GS in four (only in $N = 4$ for clear visibility of figures depicting synchronization transitions) mutually coupled, distinctly nonidentical time-delay systems in a ring, an array, a global and a star configuration. In addition to the above two time-delay systems, now we will consider the third system as a threshold piecewise linear time-delay (TPWL) system [28] with the nonlinear function

$$f_3(x) = AF^* - Bx. \tag{11}$$

Here

$$F^* = \begin{cases} -x^*, & x < -x^*, \\ x, & -x^* \le x \le x^*, \\ x^*, & x > x^*. \end{cases} \tag{12}$$

The system parameters are $\alpha_3 = 1.0$, $\beta_3 = 1.2$, $\tau_3 = 7.0$, $A = 5.2$, $B = 3.5$ and $x^* = 0.7$ and the system exhibits a hyperchaotic attractor with four positive LEs ($D_{KY} = 8.211$). As a fourth system, we take the Ikeda time-delay system [29] with the nonlinear function $f_4(x) = \sin(x(t-\tau_4))$. This system exhibits a hyperchaotic attractor with five positive LEs ($D_{KY} = 10.116$) [2] for $\alpha_4 = 1.0$, $\beta_4 = 5.0$ and $\tau_4 = 7.0$. The schematic diagram of four mutually coupled main systems and their auxiliary ones in a ring configuration is depicted in fig. 1(b). CC, CPR and MTLE for a mutually coupled ring of the above four distinctly nonidentical systems are shown in figs. 3(a) and (b). $\lambda_{MTLE}^{(i)} > 0$ and low values of $CC_{i,i'}$ and $CPR_{i,i'}$

[4] As the dynamics are confined to the inner regime of the nonlinear function [2].

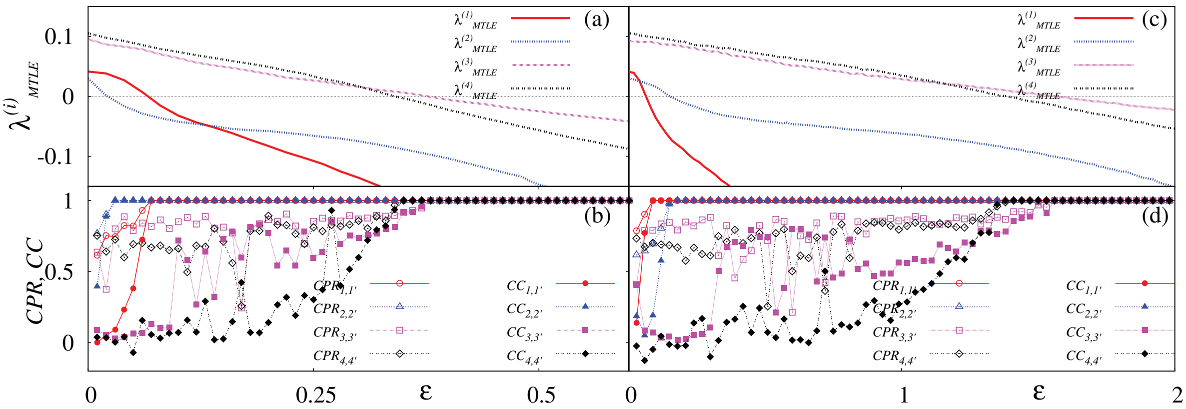

Fig. 4: (Color online) MTLEs ((a) and (c)) and CC, CPR ((b) and (d)) of the main and auxiliary systems of four distinctly nonidentical time-delay systems with global and star configurations, respectively, as a function of ε.

for $\varepsilon = 0$ indicate that the main and their auxiliary systems evolve independently. Increasing ε results in decreasing $\lambda^{(i)}_{MTLE}$ and increasing $CC_{i,i'}$ and $CPR_{i,i'}$. The PWL systems with 3 positive LEs become synchronized first at $\varepsilon_c^{(2)} = 0.088$, as evidenced by $\lambda^{(2)}_{MTLE} < 0$, indicating the onset partial GS, and then the MTLE of MG with 2 positive LEs becomes negative ($\lambda^{(1)}_{MTLE} < 0$) at $\varepsilon_c^{(1)} \approx 0.092$, while the other two systems are not yet synchronized. Further, the increase in ε leads to CS of Ikeda systems with 5 positive LEs at $\varepsilon_c^{(4)} \approx 0.65$ with $\lambda^{(4)}_{MTLE} < 0$ and finally the TPWL system becomes synchronized at $\varepsilon_c^{(3)} \approx 0.75$, confirming the existence of a global GS between the four mutually coupled distinctly nonidentical time-delay systems in a ring configuration. In addition, all $CC_{i,i'}$ and $CPR_{i,i'}$ reach unity (fig. 3(b)) exactly at the threshold values $\varepsilon_c^{(i)}$ of $\lambda^{(i)}_{MTLE}$ corroborating the simultaneous existence of GS and PS.

Next, we illustrate the transition from partial GS to global GS in a linear array of $N = 4$ (MG, PWL, TPWL and Ikeda) systems. MTLE, CC and CPR of all the four main systems and their corresponding auxiliary ones are shown in figs. 3(c) and (d). Without coupling, all the systems evolve independently as evidenced by the $\lambda^{(i)}_{MTLE} > 0$ and low values of $CC_{i,i'}$ and $CPR_{i,i'}$. Again, all the four systems reach their CS (GS) manifolds for the threshold values $\varepsilon_c^{(i)} \approx 0.09, 0.04, 0.54, 1.2$, respectively, as indicated by the transition of their $\lambda^{(i)}_{MTLE}$ below zero. Further, GS and PS also occur together as indicated by $CC_{i,i'}$ and $CPR_{i,i'}$ (fig. 3(d)) of the systems in the array.

Maximal TLEs and CC and CPR of all the four systems with global and star configurations are depicted in figs. 4(a), (b) and (c), (d), respectively. As in the case of ring configurations, $\lambda^{(2)}_{MTLE}$ transits first from positive to negative values elucidating CS between PWL and its auxiliary system thereby indicating the onset of partial GS. This is followed by the MG systems, then by Ikeda systems and finally the TPWL systems for appropriate $\varepsilon_c^{(i)}$ confirming the existence of GS in four distinctly nonidentical time-delay systems with different fractal dimensions in both global and star configurations.

Further, we have also confirmed that there exists a transition from a partial to global GS in other permutations on the order of systems between MG, PWL, TPWL, Ikeda systems in array and star configurations. Following a similar stability analysis as above for the four coupled time-delay systems with their auxiliary systems for CS, one can also arrive at a sufficiency stability condition for GS in all the above four configurations. All the above results have also been confirmed in a larger ensemble ($N = 7, 10$) of time-delay systems with distinct fractal dimensions in all the four configurations.

Finally, we have also confirmed the existence of the above transitions in time-delay systems with different orders, namely i) in a system consisting of a mutually coupled Ikeda time-delay system (which is a scalar first-order time-delay system) and a Hopfield neural network [2,30] (which is a second-order time-delay system), and ii) in a system of a mutually coupled Mackey-Glass time-delay system (which is a scalar first-order time-delay system) and a plankton model [31] (which corresponds to a third-order system with multiple delays). Complete details on these results will be presented in a forthcoming paper.

In summary, we have found that there exist smooth (as evidenced by the smooth convergence of MTLE) transformation functions even for distinctly nonidentical time-delay systems with different fractal dimensions and different orders in symmetrically coupled networks (ring, array, all-to-all and star), which map each of them to a common synchronization (GS) manifold. We have also shown that the asymptotic stability of each of the systems in the network guarantees the existence of GS as confirmed by their MTLEs and analytical stability conditions. Further, we have confirmed the existence of GS using the mutual false nearest-neighbour method in all the cases (the details will be published separately). In addition, we found that GS always coexists with PS or vice versa in these systems using CC and CPR and hence our understanding of the evolutionary mechanism including the dynamical and functional behavior of systems with different fractal dimensions [3–9] should be revisited and improved. For instance, GS leading to pathological disorders such as epilepsy, Parkinson's disease, paralysis, etc., may be effectively controlled with extremely small external perturbations to appropriate variables (to electrodes) without harming the subject. In this connection, investigations on desynchronization mechanisms and conditions in ensembles of such systems will also be crucial. In a recent work, GS among remote (identical) systems in simple network motifs with a chain of delay-coupled relay elements mediating them to maintaining total propagation delay time is demonstrated using the auxiliary-system approach [35]. Here, the issue of indirect connections and synchronization among remote elements in networks is addressed. But we have considered different time-delay systems with different number of positive Lyapunov exponents in regular networks and explore local and global GS using the auxiliary-system approach. The concept of *equivalence* and *predictability*, which is well understood in unidirectionally coupled nonidentical systems [10], remains an open problem for bidirectionally coupled systems. Further, one can extend the present work to complex networks and also include heterogenity by weighted couplings, etc., to generalize even more.

* * *

The work of RS and ML has been supported by the Department of Science and Technology (DST), Government of India sponsored IRHPA research project. ML has also been supported by a DST Ramanna project and a DAE Raja Ramanna Fellowship. ML also acknowledges the support by the Alexander von Humboldt Foundation to visit PIK, where the work was completed. DVS and JK acknowledge the support from EU under project No. 240763 PHOCUS(FP7-ICT-2009-C).

REFERENCES

[1] Pikovsky A. S., Rosenblum M. G. and Kurths J., *Synchronization - A Unified Approach to Nonlinear Science* (Cambridge University Press, Cambridge) 2001.
[2] Lakshmanan M. and Senthilkumar D. V., *Dynamics of Nonlinear Time-Delay Systems* (Springer-Verlag, Berlin) 2011.
[3] Schäfer C., Rosenblum M. G., Kurths J. and Abel H. H., *Nature (London)*, **293** (1998) 239.
[4] Farmer S. F., *J. Physiology*, **509** (1998) 3; Sebe J. Y., van Berderode J. F., Berger A. J. and Abel H. H., *J. Neurophysiol.*, **96** (2006) 391.
[5] Van Leeuwen P., Geue D., Thiel M., Cysarz D., Lange S., Romano M. C., Wessel N., Kurths J. and Grönemeyer D. H., *Proc. Natl. Acad. Sci. U.S.A.*, **106** (2009) 13661.
[6] Blasius B., Huppert A. and Stone L., *Nature (London)*, **399** (1999) 354.
[7] Amritkar R. E. and Rangarajan G., *Phys. Rev. Lett.*, **96** (2006) 258102.
[8] Grenfell B. T., Brornstad O. N. and Kappey J., *Nature (London)*, **414** (2000) 716; Earn D. T. D., Rohani P. and Grenfell B. T., *Proc. R. Soc. London, Ser. B*, **265** (1998) 1471.
[9] Stein K., Timmermann A. and Schneifer N., *Phys. Rev. Lett.*, **107** (2011) 128501; Maraun D. and Kurths J., *Geophys. Res. Lett.*, **32** (2005) L15709.
[10] Kocarev L. and Parlitz U., *Phys. Rev. Lett.*, **76** (1996) 1816.
[11] Rulkov N. F., Sushchik M. M., Tsimring L. S. and Abarbanel H. D. I., *Phys. Rev. E*, **51** (1995) 980; Abarbanel H. D. I., Rulkov N. F. and Sushchik M. M., *Phys. Rev. E*, **53** (1996) 4528.
[12] Brown R., *Phys. Rev. Lett.*, **81** (1998) 4835.
[13] Pyragas K., *Phys. Rev. E*, **54** (1996) R4508.
[14] Zheng Z., Wang X. and Cross M. C., *Phys. Rev. E*, **65** (2002) 056211.
[15] Hung Y. C., Huang Y. T., Ho M. C. and Hu C. K., *Phys. Rev. E*, **77** (2008) 016202.
[16] Guan S., Wang X., Gong X., Li K. and Lai C.-H., *Chaos*, **19** (2009) 013130.
[17] Chen J., Lu J.-a., Wu X. and Zheng W. X., *Chaos*, **19** (2009) 043119.
[18] Hu A., Xu Z. and Guo L., *Chaos*, **20** (2010) 013112.
[19] Guan S., Gong X., Li K., Liu Z. and Lai C.-H., *New J. Phys.*, **12** (2010) 073045.
[20] Shang Y., Chen M. and Kurths J., *Phys. Rev. E*, **80** (2009) 027201.
[21] Moskalenko O. I., Koronovskii A. A., Hramov A. E. and Boccaletti S., *Phys. Rev. E*, **86** (2012) 036216.
[22] Shahverdiev E. M. and Shore K. A., *IET Optoelectron.*, **3** (2009) 274.
[23] Acharyya S. and Amritkar R. E., *Eur. Phys. J. ST*, **222** (2013) 939.
[24] Boccaletti S., Valladares D. L., Kurths J., Maza D. and Mancini H., *Phys. Rev. E*, **61** (2000) 3712.
[25] Zheng Z. and Hu G., *Phys. Rev. E*, **62** (2000) 7882.
[26] Mackey M. C. and Glass L., *Science*, **197** (1977) 287.
[27] Senthilkumar D. V., Lakshmanan M., and Kurths J., *Phys. Rev. E*, **74** (2006) 035205(R); *Chaos*, **18** (2007) 023118.
[28] Senthilkumar D. V., Srinivasan K., Murali K., Lakshmanan M. and Kurths J., *Phys. Rev. E*, **82** (2010) 065201(R).
[29] Ikeda K., Daido H. and Akimoto O., *Phys. Rev. Lett.*, **45** (1980) 709.
[30] Hopfield J. J., *Proc. Natl. Acad. Sci. U.S.A.*, **79** (1982) 2554; Meng J. and Wang X., *Chaos*, **17** (2007) 023113.
[31] Abraham E. R., *Nature*, **391** (1998) 577; Gakkhar S. and Singh A., *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010) 3607.
[32] Marwan N., Romano M. C., Thiel M. and Kurths J., *Phys. Rep.*, **438** (2007) 237.
[33] Parlitz U., Junge L. and Kocarev L., *Phys. Rev. Lett.*, **79** (1997) 3158.
[34] Pecora L. M., Carroll T. L., Johnson G. A. and Mar D. J., *Chaos*, **7** (1997) 520.
[35] Soriano M. C. , Van der Sande G., Fischer I. and Mirasso C. R., *Phys. Rev. Lett.*, **108** (2012) 134101.