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Equivalent system for a multiple-rational-order fractional differential system

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The equivalent system for a multiple-rational-order (MRO) fractional differential system is studied, where the fractional derivative is in the sense of Caputo or Riemann–Liouville. With the relationship between the Caputo derivative and the generalized fractional derivative, we can change the MRO fractional differential system with a Caputo derivative into a higher-dimensional system with the same Caputo derivative order lying in $(0, 1)$. The stability of the zero solution to the original system is studied through the analysis of its equivalent system. For the Riemann–Liouville case, we transform the MRO fractional differential system into a new one with the same order lying in $(0, 1)$, where the properties of the Riemann–Liouville derivative operator and the fractional integral operator are used. The corresponding stability is also studied. Finally, several numerical examples are provided to illustrate the derived results.

1. Introduction

In 1695, fractional calculus was born, with a question about the meaning of a derivative of the order one-half. Although fractional calculus is a mathematical topic with more than a 300 year old history, the development of fractional calculus was a bit slow at the early stage and was mainly focused on the pure mathematical field. The earliest systematic studies were made in the nineteenth century. With the development of fractional calculus theory, it has been found only in recent years that the behaviours of many systems can be described by using fractional differential systems, such as viscoelastic systems, dielectric polarization, electrode–electrolyte polarization, electromagnetic waves, power-law phenomena in fluid and complex networks, allometric scaling laws in biology and ecology, coloured noise, boundary-layer effects in ducts, quantitative finance, quantum evolution of complex systems, fractional kinetics, etc.

The extensive applications of fractional differential systems in various fields of science and engineering have greatly accelerated their advance in theoretical analysis and numerical calculation, especially in stability analysis, fractional dynamics and numerical computation, etc. Over the past few decades, since the work of Matignon [1], the stability analysis of fractional differential systems has become more and more interesting and important. Matignon's stability analysis is devoted to a linear fractional differential system with a Caputo derivative whose order lies in $(0, 1]$. Recently, Qian *et al.* [2] investigated the stability of fractional differential systems with Riemann–Liouville derivatives whose order belongs to $(0, 1)$. For nonlinear fractional differential systems, the stability analysis is much more difficult and only a few studies are available, including the continuous dependence of the solution on the initial conditions [3,4] and the stability in the sense of Lyapunov [5–7]. In [5], the definition of Mittag–Leffler stability was first defined, and the corresponding theoretical theorems were also derived. The generalized Mittag–Leffler stability was studied in [6]. All of the above literature deals with same-order fractional differential systems. On the other hand, the stability analysis of multiple-order fractional differential systems has been also discussed. For the multiple-rational-order (MRO) case with MRO in $(0, 1)$, we can refer to [8]. A survey on stability analysis of fractional differential systems has recently been presented, where multiple-order systems are also mentioned [9].

It is often inconvenient to study the MRO system directly. However, we change it into an equivalent system with the same derivative order. There are some studies in this respect. For more details, see [8,10–17] and references therein, where [8,16] mainly focused on the stability of solutions.

The rest of this paper is organized as follows. In §2, some definitions and properties are introduced. In §3, the equivalence and stability analysis of the MRO fractional differential system with a Caputo derivative are studied. The equivalence and stability analysis of the MRO fractional differential system with a Riemann–Liouville derivative, together with illustrative examples, are given in the following section. The conclusions are given in the final section.

2. Preliminaries and definitions

Let us denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}_+$ denote the set of positive real numbers and by $\mathbb{Z}_+$ denote the set of positive integer numbers.

In this section, we will recall the main definitions and properties of the relevant fractional derivative operators. Among several definitions of the fractional derivatives, the Caputo derivative and the Riemann–Liouville derivative are often used in applied mathematics and engineering [1,8,18,19]. Throughout this paper, we always assume the existence of the fractional integral and fractional derivatives of a given function, together the composite operations, as usual. Detailed discussions of such existence can be found in [18,20,21].

Definition 2.1. The αth-order Riemann–Liouville integral of function $x(t)$ is defined as follows:

$$D_{0,t}^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)\,\mathrm{d}\tau, \tag{2.1}$$

where $\alpha > 0$ and $\Gamma(\cdot)$ is the Euler Gamma function. In some situations, we use ${}_{\mathrm{RL}}D_{0,t}^{-\alpha}x(t)$ instead of $D_{0,t}^{-\alpha}x(t)$ for $\alpha > 0$.

Definition 2.2. The αth-order Riemann–Liouville derivative of function $x(t)$ is defined as follows:

$${}_{\mathrm{RL}}D_{0,t}^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_0^t (t-\tau)^{m-\alpha-1}x(\tau)\,\mathrm{d}\tau, \tag{2.2}$$

where $m-1 \le \alpha < m \in \mathbb{Z}_+$.

Definition 2.3. The αth-order Caputo derivative of function $x(t)$ is defined as follows:

$$_{\mathrm{C}}D^{\alpha}_{0,t}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}x^{(m)}(\tau)\,\mathrm{d}\tau, \tag{2.3}$$

where $m-1<\alpha\le m\in\mathbb{Z}_+$.

Unlike classical differentiation and integration, fractional differentiation and integration cannot commute. Neither the Caputo-derivative nor the Riemann–Liouville-derivative operator satisfies the semigroup property. In the following, we just list some properties of the fractional calculus where the calculations involved are meaningful [18,19,21,22].

Property 2.4. The fractional integral operator satisfies the semigroup property, i.e.

$$D^{-\alpha}_{0,t}(D^{-\beta}_{0,t}x(t)) = D^{-\alpha-\beta}_{0,t}x(t), \tag{2.4}$$

where $\alpha,\beta>0$.

Property 2.5. The compositions of Riemann–Liouville derivative operators $_{\mathrm{RL}}D^{\alpha}_{0,t}$ and $_{\mathrm{RL}}D^{\beta}_{0,t}$ are as follows:

$$_{\mathrm{RL}}D^{\alpha}_{0,t}(_{\mathrm{RL}}D^{\beta}_{0,t}x(t)) = {}_{\mathrm{RL}}D^{\alpha+\beta}_{0,t}x(t) - \sum_{j=1}^{m}[_{\mathrm{RL}}D^{\beta-j}_{0,t}x(t)]_{t=0}\frac{t^{-\alpha-j}}{\Gamma(1-\alpha-j)} \tag{2.5}$$

and

$$_{\mathrm{RL}}D^{\beta}_{0,t}(_{\mathrm{RL}}D^{\alpha}_{0,t}x(t)) = {}_{\mathrm{RL}}D^{\beta+\alpha}_{0,t}x(t) - \sum_{j=1}^{n}[_{\mathrm{RL}}D^{\alpha-j}_{0,t}x(t)]_{t=0}\frac{t^{-\beta-j}}{\Gamma(1-\beta-j)}, \tag{2.6}$$

where $n-1\le\alpha<n$, $m-1\le\beta<m$ and $m,n\in\mathbb{Z}_+$.

Property 2.6. The compositions of Riemann–Liouville derivative operator $_{\mathrm{RL}}D^{\alpha}_{0,t}$ and fractional integral operator $_{\mathrm{RL}}D^{-\beta}_{0,t}$ are as follows:

$$_{\mathrm{RL}}D^{\alpha}_{0,t}(_{\mathrm{RL}}D^{-\beta}_{0,t}x(t)) = {}_{\mathrm{RL}}D^{\alpha-\beta}_{0,t}x(t) \tag{2.7}$$

and

$$_{\mathrm{RL}}D^{-\beta}_{0,t}(_{\mathrm{RL}}D^{\alpha}_{0,t}x(t)) = {}_{\mathrm{RL}}D^{\alpha-\beta}_{0,t}x(t) - \sum_{j=1}^{n}[_{\mathrm{RL}}D^{\alpha-j}_{0,t}x(t)]_{t=0}\frac{t^{\beta-j}}{\Gamma(1+\beta-j)}, \tag{2.8}$$

where $n-1\le\alpha<n\in\mathbb{Z}_+$ and $\beta>0$.

Properties 2.4–2.6 can be found in [19]. From [23], we can also conclude the following result on the fractional integral.

Remark 2.7. If $x(t)\in C^0[0,T]$ for $T>0$ and $\alpha>0$, then

$$[_{\mathrm{RL}}D^{-\alpha}_{0,t}x(t)]_{t=0}=0,$$

i.e.

$$\lim_{t\to0}\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}x(\tau)\,\mathrm{d}\tau=0.$$

Definition 2.8. Υ_α, the convolution kernel of order $\alpha>0$ for the fractional integral, is defined as follows:

$$\Upsilon_\alpha(t)\triangleq\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}\in L^1_{\mathrm{loc}}(\mathbb{R}_+), \tag{2.9}$$

where

$$t_+^{\alpha-1}=\begin{cases}t^{\alpha-1}, & t>0,\\ 0, & t\le0.\end{cases}$$

Remark 2.9.

— According to definitions 2.1 and 2.8, the αth-order Riemann–Liouville integral of a continuous, even $L^1_{\mathrm{loc}}(\mathbb{R}_+)$, causal function $x(t)$ can be written as

$$D_{0,t}^{-\alpha}x(t) = Y_\alpha \star x. \tag{2.10}$$

— Convolution property: $Y_\alpha \star Y_\beta = Y_{\alpha+\beta}$ for $\alpha > 0$ and $\beta > 0$.

Definition 2.10. $Y_{-\alpha}$, the causal distribution or the generalized function in the sense of Schwartz [24,25], is defined as follows:

$$Y_{+\alpha} \star Y_{-\alpha} = \delta, \tag{2.11}$$

where δ is the Dirac distribution, which is the neutral element of convolution.

Definition 2.11. The generalized fractional derivative with order α of a casual function or distribution $x(t)$ (abstract fractional differential operator) is defined as

$${}_{\mathrm{G}}D_{0,t}^{\alpha}x(t) \triangleq Y_{-\alpha} \star x. \tag{2.12}$$

Remark 2.12.

— Convolution property: $Y_\alpha \star Y_\beta = Y_{\alpha+\beta}$ holds for any real numbers α, β.
— Sequential property: ${}_{\mathrm{G}}D_{0,t}^{\alpha} \cdot {}_{\mathrm{G}}D_{0,t}^{\beta} = {}_{\mathrm{G}}D_{0,t}^{\alpha+\beta}$ for any real numbers α, β.
— ${}_{\mathrm{G}}D_{0,t}^{\alpha}(D_{0,t}^{-\alpha}x(t)) = D_{0,t}^{-\alpha}({}_{\mathrm{G}}D_{0,t}^{\alpha}x(t)) = x(t)$, where $x(t)$ is usually a causal function or a distribution.
— For $n-1 < \alpha \le n \in \mathbb{Z}_+$, $x^{(n)}(t) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$,

$${}_{\mathrm{C}}D_{0,t}^{\alpha}x = {}_{\mathrm{G}}D_{0,t}^{\alpha}x - \sum_{k=0}^{n-1} x^{(k)}(0)Y_{1+k-\alpha}. \tag{2.13}$$

— For $n-1 < \alpha < n \in \mathbb{Z}_+$, $x(t) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$,

$${}_{\mathrm{RL}}D_{0,t}^{\alpha}x(t) = {}_{\mathrm{G}}D_{0,t}^{\alpha}x(t) - \delta(t)[{}_{\mathrm{RL}}D_{0,t}^{\alpha-1}x(t)]_{t=0} - \sum_{k=1}^{n-1} Y_{-k}[{}_{\mathrm{RL}}D_{0,t}^{\alpha-k-1}x(t)]_{t=0}. \tag{2.14}$$

The above properties in remark 2.12 can be found in [22]. The generalized fractional derivative is often used in abstract analysis, see [1,22] for more details.

3. Analysis of a multiple-rational-order fractional differential system with a Caputo derivative

In this section, we investigate the equivalent system with the same order of the following system of fractional differential equations:

$$\left.\begin{aligned} {}_{\mathrm{C}}D_{0,t}^{\alpha_1}x_{1_1}(t) &= f_1(\bar{x}, t),\\ {}_{\mathrm{C}}D_{0,t}^{\alpha_2}x_{2_1}(t) &= f_2(\bar{x}, t),\\ &\vdots\\ {}_{\mathrm{C}}D_{0,t}^{\alpha_n}x_{n_1}(t) &= f_n(\bar{x}, t), \end{aligned}\right\} \tag{3.1}$$

and with the initial-value conditions

$$x_{i_1}^{(k)}(0) = x_{i0}^{(k)} \quad (i = 1, 2, \ldots, n;\ k = 0, 1, \ldots, m_i - 1), \tag{3.2}$$

where the time variable $t \ge 0$, $\bar{x}(t) = [x_{1_1}(t), x_{2_1}(t), \ldots, x_{n_1}(t)]^{\mathrm{T}}$, the vector fields $f_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, are continuous. All α_i, $i = 1, 2, \ldots, n$, are rational numbers satisfying $m_i - 1 < \alpha_i < m_i \in \mathbb{Z}_+$. For all

α_i lying in $(0,1)$, the reader can refer to [8,12,13,15] for more information. We always assume that system (3.1) with the initial-value conditions (3.2) has a solution $\bar{x}(t) \in C([0,b]^n)$ for some $b > 0$.

(a) Equivalent system

In this subsection, we derive the equivalent system of system (3.1) together with the initial-value conditions (3.2).

It follows from system (3.1) that there exist $p_i, q_i \in \mathbb{Z}_+$ such that $\alpha_i = p_i/q_i$, where p_i and q_i are two co-prime numbers, $i = 1, 2, \ldots, n$. Let M be the lower common multiple of the denominators q_i, $i = 1, 2, \ldots, n$. Let us take $\gamma = 1/M$ and $N = M(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$, then one can obtain the following equivalence result.

Theorem 3.1. *System* (3.1) *with the initial-value conditions* (3.2) *is equivalent to the N-dimensional system of fractional differential equations with derivative order* γ,

$$\left.\begin{aligned}
{}_{C}D^{\gamma}_{0,t}x_{1_1}(t) &= x_{1_2}(t),\\
&\vdots\\
{}_{C}D^{\gamma}_{0,t}x_{1_M}(t) &= x_{1_{M+1}}(t),\\
&\vdots\\
{}_{C}D^{\gamma}_{0,t}x_{1_{(m_1-1)M}}(t) &= x_{1_{(m_1-1)M+1}}(t),\\
&\vdots\\
{}_{C}D^{\gamma}_{0,t}x_{1_{\alpha_1 M}}(t) &= f_1(\bar{x},t),\\
{}_{C}D^{\gamma}_{0,t}x_{2_1}(t) &= x_{2_2}(t),\\
&\vdots\\
{}_{C}D^{\gamma}_{0,t}x_{2_M}(t) &= x_{2_{M+1}}(t),\\
&\vdots\\
{}_{C}D^{\gamma}_{0,t}x_{2_{(m_2-1)M}}(t) &= x_{2_{(m_2-1)M+1}}(t),\\
&\vdots\\
{}_{C}D^{\gamma}_{0,t}x_{2_{\alpha_2 M}}(t) &= f_2(\bar{x},t),\\
&\vdots\\
{}_{C}D^{\gamma}_{0,t}x_{n_1}(t) &= x_{n_2}(t),\\
&\vdots\\
{}_{C}D^{\gamma}_{0,t}x_{n_M}(t) &= x_{n_{M+1}}(t),\\
&\vdots\\
{}_{C}D^{\gamma}_{0,t}x_{n_{(m_n-1)M}}(t) &= x_{n_{(m_n-1)M+1}}(t),\\
&\vdots\\
\text{and}\qquad {}_{C}D^{\gamma}_{0,t}x_{n_{\alpha_n M}}(t) &= f_n(\bar{x},t),
\end{aligned}\right\} \tag{3.3}$$

with the initial-value conditions

$$x_{i_j}(0) = \begin{cases} x_{i0}^{(k)}, & j = kM+1;\ k = 0, 1, \ldots, m_i - 1, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.4}$$

where $i = 1, 2, \ldots, n$, *that is,*

— *whenever* $[x_{1_1}(t), x_{1_2}(t), \ldots, x_{1_{\alpha_1 M}}(t), x_{2_1}(t), x_{2_2}(t), \ldots, x_{2_{\alpha_2 M}}(t), \ldots, x_{n_1}(t), x_{n_2}(t), \ldots, x_{n_{\alpha_n M}}(t)]^{\mathrm{T}}$ *is a solution to system (3.3) equipped with the initial-value conditions (3.4),* $[x_{1_1}(t), x_{2_1}(t), \ldots, x_{n_1}(t)]^{\mathrm{T}} \in C^{m_1}[0,b] \times C^{m_2}[0,b] \times \cdots \times C^{m_n}[0,b]$, *then* $[x_{1_1}(t), x_{2_1}(t), \ldots, x_{n_1}(t)]^{\mathrm{T}}$ *solves system (3.1) and satisfies its corresponding initial-value conditions (3.2);*

— *whenever* $[x_{1_1}(t), x_{2_1}(t), \ldots, x_{n_1}(t)]^{\mathrm{T}} \in C^{m_1}[0,b] \times C^{m_2}[0,b] \times \cdots \times C^{m_n}[0,b]$ *is a solution to system (3.1) equipped with the initial-value conditions (3.2), then* $[x_{1_1}(t), x_{1_2}(t), \ldots, x_{1_{\alpha_1 M}}(t), x_{2_1}(t), x_{2_2}(t), \ldots, x_{2_{\alpha_2 M}}(t), \ldots, x_{n_1}(t), x_{n_2}(t), \ldots, x_{n_{\alpha_n M}}(t)]^{\mathrm{T}} = [x_{1_1}(t), {}_{C}D_{0,t}^{\gamma} x_{1_1}(t), \ldots, {}_{C}D_{0,t}^{(\alpha_1 M-1)\gamma} x_{1_1}(t), x_{2_1}(t), {}_{C}D_{0,t}^{\gamma} x_{2_1}(t), \ldots, {}_{C}D_{0,t}^{(\alpha_2 M-1)\gamma} x_{2_1}(t), \ldots, x_{n_1}(t), {}_{C}D_{0,t}^{\gamma} x_{n_1}(t), \ldots, {}_{C}D_{0,t}^{(\alpha_n M-1)\gamma} x_{n_1}(t)]^{\mathrm{T}}$ *satisfies system (3.3) and its initial-value conditions (3.4).*

Proof. (1) Suppose that the vector $[x_{1_1}(t), x_{1_2}(t), \ldots, x_{1_{\alpha_1 M}}(t), x_{2_1}(t), x_{2_2}(t), \ldots, x_{2_{\alpha_2 M}}(t), \ldots, x_{n_1}(t), x_{n_2}(t), \ldots, x_{n_{\alpha_n M}}(t)]^{\mathrm{T}}$ is a solution to system (3.3) with the initial-value conditions (3.4), then the following relations hold:

$$\left.\begin{aligned} {}_{C}D_{0,t}^{\gamma} x_{1_{\alpha_1 M}}(t) &= f_1(\bar{x}, t), \\ {}_{C}D_{0,t}^{\gamma} x_{2_{\alpha_2 M}}(t) &= f_2(\bar{x}, t), \\ &\vdots \\ {}_{C}D_{0,t}^{\gamma} x_{n_{\alpha_n M}}(t) &= f_n(\bar{x}, t). \end{aligned}\right\} \tag{3.5}$$

and

From remark 2.12, system (3.3) and the initial-value conditions (3.4), we have

$$\begin{aligned} {}_{C}D_{0,t}^{\gamma} x_{1_{\alpha_1 M}}(t) &= \underbrace{{}_{C}D_{0,t}^{\gamma} \cdots {}_{C}D_{0,t}^{\gamma}(\underbrace{{}_{C}D_{0,t}^{\gamma} \cdots {}_{C}D_{0,t}^{\gamma}({}_{C}D_{0,t}^{\gamma}({}_{C}D_{0,t}^{\gamma}}_{M+1} x_{1_1}(t))))}_{\alpha_1 M} \\ &= {}_{C}D_{0,t}^{\gamma} \cdots {}_{C}D_{0,t}^{\gamma}({}_{C}D_{0,t}^{\gamma} \cdots {}_{C}D_{0,t}^{\gamma}({}_{C}D_{0,t}^{\gamma}({}_{G}D_{0,t}^{\gamma} x_{1_1}(t) - x_{1_1}(0) Y_{1-\gamma}))) \\ &= {}_{C}D_{0,t}^{\gamma} \cdots {}_{C}D_{0,t}^{\gamma}({}_{C}D_{0,t}^{\gamma} \cdots {}_{C}D_{0,t}^{\gamma}({}_{G}D_{0,t}^{2\gamma} x_{1_1}(t) - x_{1_1}(0) Y_{1-2\gamma} - x_{1_2}(0) Y_{1-\gamma})) \\ &= {}_{C}D_{0,t}^{\gamma} \cdots {}_{C}D_{0,t}^{\gamma}({}_{C}D_{0,t}^{\gamma} \cdots {}_{C}D_{0,t}^{\gamma}({}_{G}D_{0,t}^{2\gamma} x_{1_1}(t) - x_{1_1}(0) Y_{1-2\gamma})) \\ &\ \ \vdots \\ &= {}_{C}D_{0,t}^{\gamma} \cdots {}_{C}D_{0,t}^{\gamma}({}_{G}D_{0,t}^{(M+1)\gamma} x_{1_1}(t) - x_{1_1}(0) Y_{1-(M+1)\gamma} - x_{1_{M+1}}(0) Y_{1-\gamma}) \\ &\ \ \vdots \\ &= {}_{G}D_{0,t}^{\alpha_1 M\gamma} x_{1_1}(t) - x_{1_1}(0) Y_{1-\alpha_1 M\gamma} - x_{1_{M+1}}(0) Y_{1-(\alpha_1 M-M)\gamma} \\ &\quad - x_{1_{2M+1}}(0) Y_{1-(\alpha_1 M-2M)\gamma} - \cdots - x_{1_{(m_1-1)M+1}}(0) Y_{1-[\alpha_1 M-(m_1-1)M]\gamma} \\ &= {}_{G}D_{0,t}^{\alpha_1 M\gamma} x_{1_1}(t) - x_{1_1}(0) Y_{1-\alpha_1} - x_{1_1}^{(1)}(0) Y_{1-(\alpha_1-1)} - x_{1_1}^{(2)}(0) Y_{1-(\alpha_1-2)} \\ &\quad - \cdots - x_{1_1}^{(m_1-1)}(0) Y_{1-[\alpha_1-(m_1-1)]} \end{aligned}$$

$$= {}_{\mathrm{G}}D_{0,t}^{\alpha_1} x_{1_1}(t) - \sum_{k=0}^{m_1-1} x_{1_1}^{(k)}(0) Y_{1+k-\alpha_1}$$
$$= {}_{\mathrm{C}}D_{0,t}^{\alpha_1} x_{1_1}(t).$$

Similar to the above derivation, one can obtain

$${}_{\mathrm{C}}D_{0,t}^{\gamma} x_{2\alpha_2 M}(t) = {}_{\mathrm{C}}D_{0,t}^{\alpha_2} x_{2_1}(t),$$
$${}_{\mathrm{C}}D_{0,t}^{\gamma} x_{3\alpha_3 M}(t) = {}_{\mathrm{C}}D_{0,t}^{\alpha_3} x_{3_1}(t),$$
$$\vdots$$
and
$${}_{\mathrm{C}}D_{0,t}^{\gamma} x_{n_{\alpha_n M}}(t) = {}_{\mathrm{C}}D_{0,t}^{\alpha_n} x_{n_1}(t).$$

Also note that

$$x_{i_1}^{(k)}(0) = {}_{\mathrm{C}}D_{0,t}^{kM\gamma} x_{i_1}(0) = x_{i_{kM+1}}(0) = x_{i0}^{(k)}$$
$$(i = 1, 2, \ldots, n;\ k = 0, 1, \ldots, m_i - 1).$$

Therefore, the first part of this theorem is completed.

(2) Suppose that $[x_{1_1}(t), x_{2_1}(t), \ldots, x_{n_1}(t)]^{\mathrm{T}}$ is a solution to system (3.1) with the initial-value conditions (3.2). Then, we have

$${}_{\mathrm{C}}D_{0,t}^{\alpha_1} x_{1_1}(t) = f_1(\bar{x}, t),$$
$${}_{\mathrm{C}}D_{0,t}^{\alpha_2} x_{2_1}(t) = f_2(\bar{x}, t),$$
$$\vdots$$
and
$${}_{\mathrm{C}}D_{0,t}^{\alpha_n} x_{n_1}(t) = f_n(\bar{x}, t),$$

i.e.

$$\left.\begin{aligned} {}_{\mathrm{C}}D_{0,t}^{\alpha_1 M\gamma} x_{1_1}(t) &= f_1(\bar{x}, t), \\ {}_{\mathrm{C}}D_{0,t}^{\alpha_2 M\gamma} x_{2_1}(t) &= f_2(\bar{x}, t), \\ &\vdots \\ \text{and}\quad {}_{\mathrm{C}}D_{0,t}^{\alpha_n M\gamma} x_{n_1}(t) &= f_n(\bar{x}, t). \end{aligned}\right\} \tag{3.6}$$

Taking into account remark 2.12 yields

$${}_{\mathrm{C}}D_{0,t}^{\alpha_1 M\gamma} x_{1_1}(t) = f_1(\bar{x}, t)$$
$$\Rightarrow {}_{\mathrm{G}}D_{0,t}^{\alpha_1 M\gamma} x_{1_1}(t) - \sum_{k=0}^{m_1-1} x_{1_1}^{(k)}(0) Y_{1+k-\alpha_1} = f_1(\bar{x}, t)$$
$$\Rightarrow {}_{\mathrm{G}}D_{0,t}^{(\alpha_1 M-1)\gamma} ({}_{\mathrm{G}}D_{0,t}^{\gamma} x_{1_1}(t) - x_{1_1}(0) Y_{1-\gamma})$$
$$- \sum_{k=1}^{m_1-1} x_{1_1}^{(k)}(0) Y_{1+k-\alpha_1} = f_1(\bar{x}, t)$$
$$\Rightarrow {}_{\mathrm{C}}D_{0,t}^{\gamma} x_{1_1}(t) = D_{0,t}^{-(\alpha_1 M-1)\gamma} f_1(\bar{x}, t) + \sum_{k=1}^{m_1-1} x_{1_1}^{(k)}(0) Y_{1+k-\gamma}$$
$$\Rightarrow [{}_{\mathrm{C}}D_{0,t}^{\gamma} x_{1_1}(t)]_{t=0} = 0.$$

Similarly,

$$
\begin{aligned}
&{}_{\mathrm{C}}D_{0,t}^{\alpha_1 M\gamma} x_{1_1}(t) = f_1(\bar{x}, t)\\
&\quad \Rightarrow {}_{\mathrm{G}}D_{0,t}^{\alpha_1 M\gamma} x_{1_1}(t) - \sum_{k=0}^{m_1-1} x_{1_1}^{(k)}(0) Y_{1+k-\alpha_1} = f_1(\bar{x}, t)\\
&\quad \Rightarrow {}_{\mathrm{G}}D_{0,t}^{(\alpha_1 M-2)\gamma} ({}_{\mathrm{G}}D_{0,t}^{2\gamma} x_{1_1}(t) - x_{1_1}(0) Y_{1-2\gamma})\\
&\qquad\qquad - \sum_{k=1}^{m_1-1} x_{1_1}^{(k)}(0) Y_{1+k-\alpha_1} = f_1(\bar{x}, t)\\
&\quad \Rightarrow {}_{\mathrm{C}}D_{0,t}^{2\gamma} x_{1_1}(t) = D_{0,t}^{-(\alpha_1 M-2)\gamma} f_1(\bar{x}, t) + \sum_{k=1}^{m_1-1} x_{1_1}^{(k)}(0) Y_{1+k-2\gamma}\\
&\quad \Rightarrow [{}_{\mathrm{C}}D_{0,t}^{2\gamma} x_{1_1}(t)]_{t=0} = 0.
\end{aligned}
$$

It follows from the above reasoning that

$$
\begin{aligned}
&[{}_{\mathrm{C}}D_{0,t}^{3\gamma} x_{1_1}(t)]_{t=0} = 0, \ldots, [{}_{\mathrm{C}}D_{0,t}^{(M-1)\gamma} x_{1_1}(t)]_{t=0} = 0,\\
&[{}_{\mathrm{C}}D_{0,t}^{(M+1)\gamma} x_{1_1}(t)]_{t=0} = 0, \ldots, [{}_{\mathrm{C}}D_{0,t}^{(2M-1)\gamma} x_{1_1}(t)]_{t=0} = 0,\\
&\vdots\\
&[{}_{\mathrm{C}}D_{0,t}^{((m_1-1)M+1)\gamma} x_{1_1}(t)]_{t=0} = 0, \ldots, [{}_{\mathrm{C}}D_{0,t}^{(\alpha_1 M-1)\gamma} x_{1_1}(t)]_{t=0} = 0
\end{aligned}
$$

and

$$[{}_{\mathrm{C}}D_{0,t}^{kM\gamma} x_{1_1}(t)]_{t=0} = x_{1_1}^{(k)}(0) = x_{10}^{(k)}, \quad k = 0, 1, 2, \ldots, m_1 - 1.$$

Then, applying repeatedly remark 2.12 and the above initial-value conditions leads to

$$\underbrace{{}_{\mathrm{C}}D_{0,t}^{\gamma} \cdots ({}_{\mathrm{C}}D_{0,t}^{\gamma}({}_{\mathrm{C}}D_{0,t}^{\gamma}}_{k} x_{1_1}(t))) = {}_{\mathrm{C}}D_{0,t}^{k\gamma} x_{1_1}(t), \quad k = 1, 2, \ldots, \alpha_1 M.$$

Until now, we affirm that $[x_{1_1}(t), {}_{\mathrm{C}}D_{0,t}^{\gamma} x_{1_1}(t), \ldots, {}_{\mathrm{C}}D_{0,t}^{(\alpha_1 M-1)\gamma} x_{1_1}(t)]^{\mathrm{T}}$ solves

$$
\begin{aligned}
&{}_{\mathrm{C}}D_{0,t}^{\gamma} x_{1_1}(t) = x_{1_2}(t),\\
&{}_{\mathrm{C}}D_{0,t}^{\gamma} x_{1_2}(t) = x_{1_3}(t),\\
&\vdots\\
&{}_{\mathrm{C}}D_{0,t}^{\gamma} x_{1_M}(t) = x_{1_{M+1}}(t),\\
&\vdots\\
&{}_{\mathrm{C}}D_{0,t}^{\gamma} x_{1_{(m_1-1)M}}(t) = x_{1_{(m_1-1)M+1}}(t),\\
&\vdots
\end{aligned}
$$

and

$${}_{\mathrm{C}}D_{0,t}^{\gamma} x_{1_{\alpha_1 M}}(t) = f_1(\bar{x}, t),$$

and satisfies the corresponding part of the initial-value conditions (3.4).

Proceeding with the same procedure yields that the vector $[x_{1_1}(t), {}_{\mathrm{C}}D_{0,t}^{\gamma} x_{1_1}(t), \ldots, {}_{\mathrm{C}}D_{0,t}^{(\alpha_1 M-1)\gamma} x_{1_1}(t),$ $x_{2_1}(t), {}_{\mathrm{C}}D_{0,t}^{\gamma} x_{2_1}(t), \ldots, {}_{\mathrm{C}}D_{0,t}^{(\alpha_2 M-1)\gamma} x_{2_1}(t), \ldots, x_{n_1}(t), {}_{\mathrm{C}}D_{0,t}^{\gamma} x_{n_1}(t), \ldots, {}_{\mathrm{C}}D_{0,t}^{(\alpha_n M-1)\gamma} x_{n_1}(t)]^{\mathrm{T}}$ solves system (3.3) and satisfies the initial-value conditions (3.4).

The proof is now completed. ■

Now, we study the equivalent system with the same order of the following MRO fractional differential equation:

$$ {}_{\mathrm{C}}D_{0,t}^{\alpha_n}x(t) + b_1 {}_{\mathrm{C}}D_{0,t}^{\alpha_{n-1}}x(t) + \cdots + b_{n-1} {}_{\mathrm{C}}D_{0,t}^{\alpha_1}x(t) + b_n = f(x,t), \quad t>0, \tag{3.7} $$

with the initial-value conditions

$$ x^{(k)}(0) = x_0^{(k)} \quad (k=0,1,\ldots,m_n-1), \tag{3.8} $$

where $x(t)\in\mathbb{R}$, function $f\in\mathbb{R}$ is continuous and $b_i, i=1,2,\ldots,n$, are constant numbers. The orders $\alpha_i, i=1,2,\ldots,n$, are rational numbers such that $m_i-1<\alpha_i<m_i, m_i\in\mathbb{Z}_+$ and $\alpha_n>\alpha_{n-1}>\cdots>\alpha_1$. In the same way, it is supposed that the initial-value problem (3.7)–(3.8) has a solution $x(t)\in C[0,b]$ for some $b>0$.

Similarly, there exist $p_i, q_i\in\mathbb{Z}_+$ such that $\alpha_i=p_i/q_i$, where $(p_i,q_i)=1$. Let M be the lower common multiple of the denominators $q_i, i=1,2,\ldots,n$, and take $\gamma=1/M, N=\alpha_n M$. Then, the equivalent system of (3.7)–(3.8) is given in the following corollary, which is somewhat different from that discussed in [15]. Since we show interest in stability analysis, we prefer to study an MRO system like (3.7)–(3.8).

Corollary 3.2. *Equation* (3.7) *with the initial-value conditions* (3.8) *is equivalent to the N-dimensional system of fractional differential equations*

$$ \left.\begin{aligned} &{}_{\mathrm{C}}D_{0,t}^{\gamma}x(t) = x_1(t),\\ &{}_{\mathrm{C}}D_{0,t}^{\gamma}x_1(t) = x_2(t),\\ &\vdots\\ &{}_{\mathrm{C}}D_{0,t}^{\gamma}x_{\alpha_1 M-1}(t) = x_{\alpha_1 M}(t),\\ &{}_{\mathrm{C}}D_{0,t}^{\gamma}x_{\alpha_1 M}(t) = x_{\alpha_1 M+1}(t),\\ &\vdots\\ &{}_{\mathrm{C}}D_{0,t}^{\gamma}x_{\alpha_2 M}(t) = x_{\alpha_2 M+1}(t),\\ &\vdots\\ \text{and}\quad &{}_{\mathrm{C}}D_{0,t}^{\gamma}x_{\alpha_n M-1}(t) = f(x,t) - b_1 x_{\alpha_{n-1}M}(t) - \cdots - b_{n-1}x_{\alpha_1 M}(t) - b_n, \end{aligned}\right\} \tag{3.9} $$

with the initial-value conditions

$$ x_i(0) = \begin{cases} x_0^{(k)}, & i=kM; k=0,1,\ldots,m_n-1,\\ 0, & \text{otherwise}, \end{cases} \tag{3.10} $$

where $x_0(t)=x(t)$, that is,

- *whenever $[x(t), x_1(t),\ldots,x_M(t),\ldots,x_{2M}(t),\ldots,x_{\alpha_n M-1}(t)]^{\mathrm{T}}, x\in C^{m_n}[0,b]$ is a solution to system* (3.9), *equipped with the initial-value conditions* (3.10), *then $x(t)$ solves equation* (3.7) *and satisfies its corresponding initial-value conditions* (3.8);
- *whenever $x(t), x\in C^{m_n}[0,b]$ is a solution to equation* (3.7) *with the initial-value conditions* (3.8), *then $[x(t),x_1(t),\ldots,x_M(t),\ldots,x_{2M}(t),\ldots,x_{\alpha_n M-1}(t)]^{\mathrm{T}} = [x(t), {}_{\mathrm{C}}D_{0,t}^{\gamma}x(t),\ldots, {}_{\mathrm{C}}D_{0,t}^{M\gamma}x(t),\ldots, {}_{\mathrm{C}}D_{0,t}^{2M\gamma}x(t),\ldots, {}_{\mathrm{C}}D_{0,t}^{(\alpha_n M-1)\gamma}x(t)]^{\mathrm{T}}$ satisfies system* (3.9) *and its initial-value conditions* (3.10).

Corollary 3.2 still holds for $\alpha_i=m_i\in\mathbb{Z}_+$. From the above corollary, there is a strong connection between the ordinary differential equation (ODE) and the fractional ODE. For example,

$$ \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t), \quad t>0, $$

and

$$x(0) = x_0$$

is equivalent to

$$
\begin{aligned}
{}_{C}D_{0,t}^{1/n} x(t) &= x_1(t), \\
{}_{C}D_{0,t}^{1/n} x_1(t) &= x_2(t), \\
&\vdots \\
{}_{C}D_{0,t}^{1/n} x_{n-2}(t) &= x_{n-1}(t) \\
\text{and} \quad {}_{C}D_{0,t}^{1/n} x_{n-1}(t) &= f(x,t),
\end{aligned}
$$

with the initial-value conditions

$$
\begin{aligned}
x(0) &= x_0, \\
x_1(0) &= 0, \\
&\vdots \\
\text{and} \quad x_{n-1}(0) &= 0,
\end{aligned}
$$

for $\forall n \in \mathbb{Z}_+$.

From the above example, for a given function $x(t)$ whose first-order derivative exists, we can find another way to numerically compute its arbitrary order $\alpha = m/n \in (0,1)$ by constructing an equation and its equivalent system. For any irrational number $\beta \in (0,1)$, the function ${}_{C}D_{0,t}^{\beta} x(t)$ can be numerically approximated according to the fact that an arbitrary irrational number can be approached by a rational number series to arbitrary accuracy.

(b) Stability analysis

In this subsection, we always presume that the solution to a given system can be extended to $+\infty$. In the following, we study the stability of the zero solution to the autonomous system:

$$
\left.
\begin{aligned}
{}_{C}D_{0,t}^{\alpha_1} x_{1_1}(t) &= g_1(\bar{x}(t)), \\
{}_{C}D_{0,t}^{\alpha_2} x_{2_1}(t) &= g_2(\bar{x}(t)), \\
&\vdots \\
\text{and} \quad {}_{C}D_{0,t}^{\alpha_n} x_{n_1}(t) &= g_n(\bar{x}(t)),
\end{aligned}
\right\} \tag{3.11}
$$

with the initial-value conditions

$$x_{i_1}^{(k)}(0) = x_{i0}^{(k)} \quad (i = 1, 2, \ldots, n;\ k = 0, 1, \ldots, m_i - 1), \tag{3.12}$$

where $\bar{x}$, t and α_i $(i = 1, 2, \ldots, n)$ are the same as those in theorem 3.1, $g_i : \mathbb{D} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are continuous, $\mathbb{D} \subseteq \mathbb{R}^n$ is a domain that contains the origin $\bar{x} = 0$.

Next, we give the definition of the stability of the Caputo-type differential equation as (3.1) [1,2,9].

Definition 3.3. The autonomous system (3.11) is said to be

— stable if and only if $\forall \bar{x}_0, \exists \epsilon > 0, \forall t \geq 0, \|\bar{x}(t)\| \leq \epsilon$;
— asymptotically stable if and only if $\lim_{t \to +\infty} \|\bar{x}(t)\| = 0$,

where $\bar{x}_0 = [x_{10}^{(0)}, x_{10}^{(1)}, \ldots, x_{10}^{(m_1-1)}, x_{20}^{(0)}, x_{20}^{(1)}, \ldots, x_{20}^{(m_2-1)}, \ldots, x_{n0}^{(0)}, x_{n0}^{(1)}, \ldots, x_{n0}^{(m_n-1)}]^{\mathrm{T}}$.

By using theorem 3.1, one obtains the following stability result, which can be regarded as a direct application of theorem 3.1.

Theorem 3.4. *Assume that g_i satisfy $g_i(0) = 0, i = 1, 2, \ldots, n$, and the initial-value problem (3.11)–(3.12) has a unique solution $\bar{x} \in C([0, +\infty)^n)$. Then, the zero solution to system (3.11) is asymptotically stable if $|\arg(\lambda)| > \gamma\pi/2$, where λ is the solution to the characteristic equation*

$$\det(\lambda E - A) = 0, \tag{3.13}$$

$\gamma = 1/M$ is the same as that of theorem 3.1, E is the identity matrix with order $N = \sum_{i=1}^{n} \alpha_i M$, and A is the Jacobian matrix at the zero point of the equivalent system of (3.11)

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix},$$

$$A_{ii} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ \left.\dfrac{\partial g_i}{\partial x_{i_1}(t)}\right|_{\bar{x}=0} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{\alpha_i M \times \alpha_i M} \quad (i = 1, 2, \ldots, n)$$

and

$$A_{ij} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \left.\dfrac{\partial g_i}{\partial x_{j_1}(t)}\right|_{\bar{x}=0} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{\alpha_i M \times \alpha_j M} \quad (i \neq j, i, j = 1, 2, \ldots, n).$$

In particular, if system (3.11) is a linear system, i.e. $g = [g_1, g_2, \ldots, g_n]^{\mathrm{T}} = B\bar{x}$, where the $n \times n$ matrix $B = (b_{ij})$, then

— *the zero solution to system (3.11) is asymptotically stable if and only if any solution to equation (3.13) satisfies $|\arg(\lambda)| > \gamma\pi 2$;*
— *the zero solution to system (3.11) is stable if and only if either it is asymptotically stable (i.e. $|\arg(\lambda)| > \gamma\pi/2$), or $|\arg(\lambda)| \geq \gamma\pi/2$ and those critical solutions to equation (3.13) that satisfy $|\arg(\lambda)| = \gamma\pi/2$ have the same algebraic and geometric multiplicities, and the zero solution to equation (3.13) has the same algebraic and geometric multiplicities if there exists the zero solution. Here,*

$$A_{ii} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ b_{ii} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{\alpha_i M \times \alpha_i M} \quad (i = 1, 2, \ldots, n)$$

and

$$A_{ij} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ b_{ij} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{\alpha_i M \times \alpha_j M} \quad (i \neq j, i, j = 1, 2, \ldots, n).$$

Proof. Based on theorem 3.1, the MRO fractional differential system (3.11) and (3.12) can be changed into a higher-dimensional fractional differential system with the same order lying in (0, 1). Then, combining with [1, theorem 2], [2, remark 3.4.(b)] and the linearization method of stability analysis for fractional differential equations [26–33], one can obtain the conclusions. ■

Remark 3.5. By applying the properties of the determinant, equation (3.13) is equivalent to the following equation:

$$\det(\mathrm{diag}(\lambda^{\alpha_1 M}, \lambda^{\alpha_2 M}, \ldots, \lambda^{\alpha_n M}) - G) = 0, \tag{3.14}$$

where

$$G = \begin{pmatrix} \dfrac{\partial g_1}{\partial x_{1_1}} & \dfrac{\partial g_1}{\partial x_{2_1}} & \cdots & \dfrac{\partial g_1}{\partial x_{n_1}} \\ \dfrac{\partial g_2}{\partial x_{1_1}} & \dfrac{\partial g_2}{\partial x_{2_1}} & \cdots & \dfrac{\partial g_2}{\partial x_{n_1}} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial g_n}{\partial x_{1_1}} & \dfrac{\partial g_n}{\partial x_{2_1}} & \cdots & \dfrac{\partial g_n}{\partial x_{n_1}} \end{pmatrix}_{\bar{x}=0}$$

is an $n \times n$ matrix.

Likewise, we consider the stability of the following autonomous MRO fractional differential equation:

$$_{\mathrm{C}}D_{0,t}^{\alpha_n} x(t) + b_1 {}_{\mathrm{C}}D_{0,t}^{\alpha_{n-1}} x(t) + \cdots + b_{n-1} {}_{\mathrm{C}}D_{0,t}^{\alpha_1} x(t) + b_n = f(x(t)), \quad t > 0, \tag{3.15}$$

with the initial-value conditions

$$x^{(k)}(0) = x_0^{(k)} \quad (k = 0, 1, \ldots, m_n - 1), \tag{3.16}$$

where x, m_n, b_i, α_i, $(i = 1, 2, \ldots, n)$ are the same as those in corollary 3.2, $f : \mathbb{D} \to \mathbb{R}$ is continuous, $\mathbb{D} \subseteq \mathbb{R}$ is a domain that contains the origin $x(t) = 0$. One can then derive the following result.

Corollary 3.6. *Suppose that $f(y(t))$ is a real-valued continuous function such that $f(0) = b_n$, and equation* (3.15) *with the initial-value conditions* (3.16) *has a unique solution $x(t) \in C[0, +\infty)$. Then, the zero solution to equation* (3.15) *is asymptotically stable if $|\arg(\lambda)| > \gamma\pi/2$, where λ is the solution to the characteristic equation*

$$\det(\lambda E - A) = 0, \tag{3.17}$$

$\gamma = 1/M$ is the same as that of theorem 3.1, E is the identity matrix with order $N = \alpha_n M$, and A is the Jacobian at the zero point of the equivalent system with the same rational order of (3.15) *as follows*:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_{N1} & a_{N2} & a_{N3} & a_{N4} & \cdots & a_{NN} \end{pmatrix}$$

and

$$a_{Nj} = \begin{cases} \left.\dfrac{\partial f}{\partial x}\right|_{x=0}, & j = 1, \\ -b_{n-i}, & j = \alpha_i M + 1, i = 1, 2, \ldots, n-1, \\ 0, & \text{otherwise}. \end{cases} \tag{3.18}$$

In particular, if equation (3.15) *is a linear equation and $b_n = 0$, i.e. $f(x(t)) = b_0 x(t)$, where the constant number $b_0 \in \mathbb{R}$, then*

— *the zero solution to equation* (3.15) *is asymptotically stable if and only if any solution λ to equation* (3.17) *satisfies $|\arg(\lambda)| > \gamma\pi/2$;*

— *the zero solution to equation (3.15) is stable if and only if either it is asymptotically stable (i.e.* $|\arg(\lambda)| > \gamma\pi/2$) *or* $|\arg(\lambda)| \geq \gamma\pi/2$ *and those critical solutions to equation (3.17) that satisfy* $|\arg(\lambda)| = \gamma\pi/2$ *have the same algebraic and geometric multiplicities, and the zero solution to equation (3.17) has the same algebraic and geometric multiplicities if there exists a zero solution. Here,*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_{N1} & a_{N2} & a_{N3} & a_{N4} & \cdots & a_{NN} \end{pmatrix}$$

and

$$a_{Nj} = \begin{cases} b_0, & j = 1, \\ -b_{n-i}, & j = \alpha_i M + 1, i = 1, 2, \ldots, n-1, \\ 0, & \text{otherwise}. \end{cases} \tag{3.19}$$

Remark 3.7. By applying the properties of the determinant, equation (3.17) is equivalent to the following equation:

$$\lambda^N - a_{NN}\lambda^{N-1} - a_{N,N-1}\lambda^{N-2} - \ldots - a_{N2}\lambda - a_{N1} = 0. \tag{3.20}$$

(c) Several examples

In the sequel, we will give several concrete examples to illustrate theorem 3.1 and corollary 3.2.

Example 3.8. We consider the following MRO system of fractional differential equations:

$$\left.\begin{aligned} {}_{\mathrm{C}}D_{0,t}^{3/2} x_{1_1}(t) &= f_1(x_{1_1}, x_{2_1}, t) \\ \text{and} \quad {}_{\mathrm{C}}D_{0,t}^{4/3} x_{2_1}(t) &= f_2(x_{1_1}, x_{2_1}, t), \end{aligned}\right\} \tag{3.21}$$

with the initial-value conditions

$$\left.\begin{aligned} x_{1_1}(0) = x_{10}^{(0)}, \quad & x'_{1_1}(0) = x_{10}^{(1)}, \\ x_{2_1}(0) = x_{20}^{(0)} \quad \text{and} \quad & x'_{2_1}(0) = x_{20}^{(1)}. \end{aligned}\right\} \tag{3.22}$$

By using theorem 3.1, system (3.21), together with (3.22), is equivalent to a 17-dimensional system that reads

$$\left.\begin{aligned} {}_{\mathrm{C}}D_{0,t}^{1/6} x_{1_1}(t) &= x_{1_2}(t), \\ {}_{\mathrm{C}}D_{0,t}^{1/6} x_{1_2}(t) &= x_{1_3}(t), \\ &\vdots \\ {}_{\mathrm{C}}D_{0,t}^{1/6} x_{1_9}(t) &= f_1(x_{1_1}, x_{2_1}, t), \\ {}_{\mathrm{C}}D_{0,t}^{1/6} x_{2_1}(t) &= x_{2_2}(t), \\ {}_{\mathrm{C}}D_{0,t}^{1/6} x_{2_2}(t) &= x_{2_3}(t), \\ &\vdots \\ \text{and} \quad {}_{\mathrm{C}}D_{0,t}^{1/6} x_{2_8}(t) &= f_2(x_{1_1}, x_{2_1}, t), \end{aligned}\right\} \tag{3.23}$$

with the initial-value conditions

$$\left.\begin{aligned}
&x_{1_1}(0)=x_{10}^{(0)},\\
&x_{1_2}(0)=x_{1_3}(0)=\ldots=x_{1_6}(0)=0,\\
&x_{1_7}(0)=x_{10}^{(1)},\\
&x_{1_8}(0)=x_{1_9}(0)=0,\\
&x_{2_1}(0)=x_{20}^{(0)},\\
&x_{2_2}(0)=x_{2_3}(0)=\ldots=x_{2_6}(0)=0,\\
&x_{2_7}(0)=x_{20}^{(1)}\\
\text{and}\quad &x_{2_8}(0)=0.
\end{aligned}\right\}\tag{3.24}$$

Next, we will give another example to illustrate how to obtain the equivalent system with the same order of an MRO fractional differential equation.

Example 3.9. We study the MRO fractional differential equation

$$_{\mathrm{C}}D_{0,t}^{5/2}x(t)+b_1{}_{\mathrm{C}}D_{0,t}^{4/3}x(t)+b_2{}_{\mathrm{C}}D_{0,t}^{1/6}x(t)+b_3=f(x,t),\quad t>0,\tag{3.25}$$

with the initial-value conditions

$$x(0)=x_0^{(0)},\quad x^{(1)}(0)=x_0^{(1)}\quad\text{and}\quad x^{(2)}(0)=x_0^{(2)}.\tag{3.26}$$

Applying corollary 3.2, one has that system (3.25) with the corresponding initial-value conditions (3.26) is equivalent to the following system in $\mathbb{R}^{15}$:

$$\left.\begin{aligned}
&{}_{\mathrm{C}}D_{0,t}^{1/6}x(t)=x_1(t),\\
&{}_{\mathrm{C}}D_{0,t}^{1/6}x_1(t)=x_2(t),\\
&\vdots\\
&{}_{\mathrm{C}}D_{0,t}^{1/6}x_{13}(t)=x_{14}(t)\\
\text{and}\quad &{}_{\mathrm{C}}D_{0,t}^{1/6}x_{14}(t)=f(x,t)-b_1x_8(t)-b_2x_1(t)-b_3,
\end{aligned}\right\}\tag{3.27}$$

with the initial-value conditions

$$\left.\begin{aligned}
&x(0)=x_0^{(0)},\\
&x_1(0)=x_2(0)=\ldots=x_5(0)=0,\\
&x_6(0)=x_0^{(1)},\\
&x_7(0)=x_8(0)=\ldots=x_{11}(0)=0,\\
&x_{12}(0)=x_0^{(2)}\\
\text{and}\quad &x_{13}(0)=x_{14}(0)=0.
\end{aligned}\right\}\tag{3.28}$$

At last, we consider an interesting model in vibration mechanics.

Example 3.10. Consider the famous Bagley–Torvik equation [34]

$$x''(t)+a{}_{\mathrm{C}}D_{0,t}^{3/2}x(t)+bx(t)=f(t),\quad t>0,\tag{3.29}$$

with the initial-value conditions

$$x(0)=x_0^{(0)}\quad\text{and}\quad x'(0)=x_0^{(1)}.\tag{3.30}$$

This model was originally established by Bagley and Torvik. They considered the motion of a half-space Newtonian viscous fluid induced by a prescribed transverse motion of a rigid plate on the surface. Their aim was to demonstrate that the resulting shear stress at any point in the fluid can be characterized directly in terms of a fractional derivative of the fluid velocity profile. In the above model, we assume that the mass of the plate, which is immersed in the Newtonian fluid with density ρ and viscosity μ, is a unit. This thin rigid plate is connected by a massless spring of stiffness K to a fixed point outside the fluid. $f(t)$ relates to the force, the constant coefficient a depends upon the area of the plate, the fluid density ρ, and viscosity μ, and b relies on the stiffness K of the spring outside the Newtonian fluid.

It is easy to know that equation (3.29) with initial conditions (3.30) is equivalent to

$$\left.\begin{aligned} {}_{C}D_{0,t}^{1/2}x(t) &= x_1(t),\\ {}_{C}D_{0,t}^{1/2}x_1(t) &= x_2(t),\\ {}_{C}D_{0,t}^{1/2}x_2(t) &= x_3(t)\\ \text{and}\quad {}_{C}D_{0,t}^{1/2}x_3(t) &= f(t) - ax_3(t) - bx(t), \end{aligned}\right\} \tag{3.31}$$

with the initial-value conditions

$$\left.\begin{aligned} x(0) &= x_0^{(0)},\\ x_1(0) &= 0,\\ x_2(0) &= x_0^{(1)}\\ \text{and}\quad x_3(0) &= 0, \end{aligned}\right\} \tag{3.32}$$

by utilizing corollary 3.2.

Next, we consider the stability of the Bagley–Torvik equation without the external forcing term, i.e. $f(t) = 0$. That is to say, we consider the stability of the following system with the same order:

$$\left.\begin{aligned} {}_{C}D_{0,t}^{1/2}x(t) &= x_1(t),\\ {}_{C}D_{0,t}^{1/2}x_1(t) &= x_2(t),\\ {}_{C}D_{0,t}^{1/2}x_2(t) &= x_3(t)\\ \text{and}\quad {}_{C}D_{0,t}^{1/2}x_3(t) &= -ax_3(t) - bx(t). \end{aligned}\right\} \tag{3.33}$$

Here, $M = 2, \gamma = \frac{1}{2}, \alpha_1 = 1, \alpha_2 = \frac{3}{2}, \alpha_3 = 2$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -b & 0 & 0 & -a \end{pmatrix}.$$

If the zero solution to the characteristic equation of (3.33) satisfies the condition of corollary 3.6, then the stability problem will be settled. The characteristic equation can be written as

$$\begin{vmatrix} \lambda & -1 & 0 & 0\\ 0 & \lambda & -1 & 0\\ 0 & 0 & \lambda & -1\\ b & 0 & 0 & \lambda + a \end{vmatrix} = 0.$$

We obtain the solutions to the characteristic equation satisfying $\lambda^3(\lambda + a) + b = 0$.

If $b = 0$ but $a \neq 0$ (i.e. the thin rigid plate is immersed in the fluid, but is free from the spring), we can see that the characteristic equation has a zero solution, whose algebraic multiplicity is not equal to the geometric multiplicity. So, in this situation, the zero solution of equation (3.29) with the force $f(t) = 0$, $b = 0$, $a \neq 0$ is unstable. Such a theoretical result fits well the real situation.

4. Analysis of a multiple-rational-order fractional differential system with a Riemann–Liouville derivative

For simplicity, we first study the MRO with fractional order lying in (0, 1) as

$$\left.\begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{\alpha_1}x_{1_1}(t) &= f_1(\bar{x},t), \\ {}_{\mathrm{RL}}D_{0,t}^{\alpha_2}x_{2_1}(t) &= f_2(\bar{x},t), \\ &\vdots \\ \text{and}\quad {}_{\mathrm{RL}}D_{0,t}^{\alpha_n}x_{n_1}(t) &= f_n(\bar{x},t), \end{aligned}\right\} \tag{4.1}$$

with the initial-value conditions

$$[{}_{\mathrm{RL}}D_{0,t}^{\alpha_i-1}x_{i_1}(t)]_{t=0} = x_{i0} \quad (i=1,2,\ldots,n), \tag{4.2}$$

where the time variable $t>0$, $\bar{x}(t)=[x_{1_1}(t),\ldots,x_{n_1}(t)]^{\mathrm{T}}$, $[f_1,f_2,\ldots,f_n]^{\mathrm{T}} \in \mathbb{R}^n$. All α_i, $i=1,2,\ldots,n$, are rational numbers satisfying $0<\alpha_i<1$. Also, we assume that the initial-value problem (4.1)–(4.2) has a unique solution $\bar{x}(t) \in (C^1[0,T])^n$.

(a) Equivalent system

For the rational number $\alpha_i \in (0,1)$, $i=1,2,\ldots,n$, we note that there exist $p_i, q_i \in \mathbb{Z}_+$ such that $\alpha_i = p_i/q_i$, where p_i and q_i are two co-prime numbers, $i=1,2,\ldots,n$. Let M be the lower common multiple of the denominators q_i, $i=1,2,\ldots,n$. Let us take $\gamma = 1/M$ and $N = M(\alpha_1+\alpha_2+\cdots+\alpha_n)$, then one can obtain the following equivalence result.

Theorem 4.1. *System* (4.1) *with the initial-value conditions* (4.2) *is equivalent to the N-dimensional system* (4.3) *of fractional differential equations with order* γ,

$$\left.\begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{1_1}(t) &= x_{1_2}(t), \\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{1_2}(t) &= x_{1_3}(t), \\ &\vdots \\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{1_{\alpha_1 M}}(t) &= f_1(\bar{x},t), \\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{2_1}(t) &= x_{2_2}(t), \\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{2_2}(t) &= x_{2_3}(t), \\ &\vdots \\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{2_{\alpha_2 M}}(t) &= f_2(\bar{x},t), \\ &\vdots \\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{n_1}(t) &= x_{n_2}(t), \\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{n_2}(t) &= x_{n_3}(t), \\ &\vdots \\ \textit{and}\quad {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{n_{\alpha_n M}}(t) &= f_n(\bar{x},t), \end{aligned}\right\} \tag{4.3}$$

subject to the initial-value conditions

$$[{}_{\mathrm{RL}}D_{0,t}^{\gamma-1}x_{i_j}(t)]_{t=0} = \begin{cases} x_{i0}, & j=\alpha_i M, \\ 0, & \textit{otherwise}, \end{cases} \tag{4.4}$$

where $i=1,2,\ldots,n$, that is,

— *whenever* $[x_{1_1}(t), x_{1_2}(t), \ldots, x_{1_{\alpha_1 M}}(t), x_{2_1}(t), x_{2_2}(t), \ldots, x_{2_{\alpha_2 M}}(t), \ldots, x_{n_1}(t), x_{n_2}(t), \ldots, x_{n_{\alpha_n M}}(t)]^{\mathrm{T}}$ *with* $[x_{1_1}(t), x_{2_1}(t), \ldots, x_{n_1}(t)]^{\mathrm{T}} \in (C^1[0,b])^n$ *is a solution to system (4.3), equipped with the initial-value conditions (4.4), then* $[x_{1_1}(t), x_{2_1}(t), \ldots, x_{n_1}(t)]^{\mathrm{T}}$ *solves system (4.1) and satisfies its corresponding initial-value conditions (4.2);*

— *whenever* $[x_{1_1}(t), x_{2_1}(t), \ldots, x_{n_1}(t)]^{\mathrm{T}} \in (C^1[0,b])^n$ $i=1,2,\ldots,n$, *is a solution to system (4.1) with the initial-value conditions (4.2), then the vector* $[x_{1_1}(t), x_{1_2}(t), \ldots, x_{1_{\alpha_1 M}}(t), x_{2_1}(t), x_{2_2}(t), \ldots, x_{2_{\alpha_2 M}}(t), \ldots, x_{n_1}(t), x_{n_2}(t), \ldots, x_{n_{\alpha_n M}}(t)]^{\mathrm{T}} = [x_{1_1}(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma} x_{1_1}(t), \ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_1 M-1)\gamma} x_{1_1}(t), x_{2_1}(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma} x_{2_1}(t), \ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_2 M-1)\gamma} x_{2_1}(t), \ldots, x_{n_1}(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma} x_{n_1}(t), \ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_n M-1)\gamma} x_{n_1}(t)]^{\mathrm{T}}$ *satisfies system (4.3) and its initial-value conditions (4.4).*

Proof. (1) Suppose that $[x_{1_1}(t), x_{1_2}(t), \ldots, x_{1_{\alpha_1 M}}(t), x_{2_1}(t), x_{2_2}(t), \ldots, x_{2_{\alpha_2 M}}(t), \ldots, x_{n_1}(t), x_{n_2}(t), \ldots, x_{n_{\alpha_n M}}(t)]^{\mathrm{T}}$ is a solution to system (4.3) with the initial-value conditions (4.4), then the following relations hold:

$$\left.\begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{\gamma} x_{1_{\alpha_1 M}}(t) &= f_1(\bar{x}, t),\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma} x_{2_{\alpha_2 M}}(t) &= f_2(\bar{x}, t),\\ &\vdots\\ \text{and}\qquad {}_{\mathrm{RL}}D_{0,t}^{\gamma} x_{n_{\alpha_n M}}(t) &= f_n(\bar{x}, t). \end{aligned}\right\} \tag{4.5}$$

First, using repeatedly the composition formula of the fractional integral operator and the Riemann–Liouville derivative operator (2.8) and initial-value conditions (4.4), we have

$$\begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{\gamma-1} x_{1_{\alpha_1 M}}(t) &= {}_{\mathrm{RL}}D_{0,t}^{\gamma-1}({}_{\mathrm{RL}}D_{0,t}^{\gamma} x_{1_{\alpha_1 M-1}}(t))\\ &= {}_{\mathrm{RL}}D_{0,t}^{2\gamma-1} x_{1_{\alpha_1 M-1}}(t) - [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1} x_{1_{\alpha_1 M-1}}(t)]_{t=0} \frac{t^{-\gamma}}{\Gamma(1-\gamma)}\\ &= {}_{\mathrm{RL}}D_{0,t}^{2\gamma-1} x_{1_{\alpha_1 M-1}}(t)\\ &= {}_{\mathrm{RL}}D_{0,t}^{2\gamma-1}({}_{\mathrm{RL}}D_{0,t}^{\gamma} x_{1_{\alpha_1 M-2}}(t))\\ &= {}_{\mathrm{RL}}D_{0,t}^{3\gamma-1} x_{1_{\alpha_1 M-2}}(t) - [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1} x_{1_{\alpha_1 M-2}}(t)]_{t=0} \frac{t^{-2\gamma}}{\Gamma(1-2\gamma)}\\ &= {}_{\mathrm{RL}}D_{0,t}^{3\gamma-1} x_{1_{\alpha_1 M-2}}(t)\\ &\vdots\\ &= {}_{\mathrm{RL}}D_{0,t}^{\alpha_1 M\gamma-1} x_{1_1}(t)\\ &= {}_{\mathrm{RL}}D_{0,t}^{\alpha_1-1} x_{1_1}(t), \end{aligned}$$

i.e.

$$[{}_{\mathrm{RL}}D_{0,t}^{\alpha_1-1} x_{1_1}(t)]_{t=0} = [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1} x_{1_{\alpha_1 M}}(t)]_{t=0} = x_{10}. \tag{4.6}$$

In the same manner, we have

$$\begin{aligned} &[{}_{\mathrm{RL}}D_{0,t}^{\alpha_2-1} x_{2_1}(t)]_{t=0} = [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1} x_{2_{\alpha_2 M}}(t)]_{t=0} = x_{20},\\ &[{}_{\mathrm{RL}}D_{0,t}^{\alpha_3-1} x_{3_1}(t)]_{t=0} = [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1} x_{3_{\alpha_3 M}}(t)]_{t=0} = x_{30},\\ &\vdots\\ \text{and}\qquad &[{}_{\mathrm{RL}}D_{0,t}^{\alpha_n-1} x_{n_1}(t)]_{t=0} = [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1} x_{n_{\alpha_n M}}(t)]_{t=0} = x_{n0}. \end{aligned}$$

So, the initial-value conditions (4.2) are valid.

Second, using repeatedly the composition formula of fractional integral operator and the Riemann–Liouville derivative operator (2.5) and the initial-value conditions (4.4) yields

$$\begin{aligned}
{}_{\mathrm{RL}}D_{0,t}^{\gamma} x1_{\alpha_1 M}(t) &= {}_{\mathrm{RL}}D_{0,t}^{\gamma}({}_{\mathrm{RL}}D_{0,t}^{\gamma} x1_{\alpha_1 M-1}(t)) \\
&= {}_{\mathrm{RL}}D_{0,t}^{2\gamma} x1_{\alpha_1 M-1}(t) - [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1} x1_{\alpha_1 M-1}(t)]_{t=0} \frac{t^{-\gamma-1}}{\Gamma(-\gamma)} \\
&= {}_{\mathrm{RL}}D_{0,t}^{2\gamma} x1_{\alpha_1 M-1}(t) \\
&= {}_{\mathrm{RL}}D_{0,t}^{2\gamma}({}_{\mathrm{RL}}D_{0,t}^{\gamma} x1_{\alpha_1 M-2}(t)) \\
&= {}_{\mathrm{RL}}D_{0,t}^{3\gamma} x1_{\alpha_1 M-2}(t) - [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1} x1_{\alpha_1 M-2}(t)]_{t=0} \frac{t^{-2\gamma-1}}{\Gamma(-2\gamma)} \\
&= {}_{\mathrm{RL}}D_{0,t}^{3\gamma} x1_{\alpha_1 M-2}(t) \\
&\ \ \vdots \\
&= {}_{\mathrm{RL}}D_{0,t}^{\alpha_1 M\gamma} x1_1(t) \\
&= {}_{\mathrm{RL}}D_{0,t}^{\alpha_1} x1_1(t) \\
&= f_1(\bar{x}, t).
\end{aligned}$$

Similar to the above derivation, one can obtain

$$\begin{aligned}
&{}_{\mathrm{RL}}D_{0,t}^{\gamma} x2_{\alpha_2 M}(t) = {}_{\mathrm{RL}}D_{0,t}^{\alpha_2} x2_1(t) = f_2(\bar{x}, t), \\
&{}_{\mathrm{RL}}D_{0,t}^{\gamma} x3_{\alpha_3 M}(t) = {}_{\mathrm{RL}}D_{0,t}^{\alpha_3} x3_1(t) = f_3(\bar{x}, t), \\
&\ \ \vdots
\end{aligned}$$

and

$${}_{\mathrm{RL}}D_{0,t}^{\gamma} xn_{\alpha_n M}(t) = {}_{\mathrm{RL}}D_{0,t}^{\alpha_n} xn_1(t) = f_n(\bar{x}, t).$$

Therefore, the vector $[x1_1(t), x2_1(t), \ldots, xn_1(t)]^{\mathrm{T}}$ solves system (4.1) and satisfies its corresponding initial-value conditions (4.2), and the first part of this theorem is completed.

(2) Suppose that $[x1_1(t), x2_1(t), \ldots, xn_1(t)]^{\mathrm{T}}$, $xi_1(t) \in C_{1-\alpha_i}[0, b]$, $i = 1, 2, \ldots, n$, is a solution of system (4.1) with the initial-value conditions (4.2), then we have

$$\begin{aligned}
&{}_{\mathrm{RL}}D_{0,t}^{\alpha_1} x1_1(t) = f_1(\bar{x}, t), \\
&{}_{\mathrm{RL}}D_{0,t}^{\alpha_2} x2_1(t) = f_2(\bar{x}, t), \\
&\ \ \vdots
\end{aligned}$$

and

$${}_{\mathrm{RL}}D_{0,t}^{\alpha_n} xn_1(t) = f_n(\bar{x}, t),$$

i.e.

$$\left.\begin{aligned}
&{}_{\mathrm{RL}}D_{0,t}^{\alpha_1 M\gamma} x1_1(t) = f_1(\bar{x}, t), \\
&{}_{\mathrm{RL}}D_{0,t}^{\alpha_2 M\gamma} x2_1(t) = f_2(\bar{x}, t), \\
&\ \ \vdots \\
\text{and}\quad &{}_{\mathrm{RL}}D_{0,t}^{\alpha_n M\gamma} xn_1(t) = f_n(\bar{x}, t).
\end{aligned}\right\} \tag{4.7}$$

For $i = 1, 2, \ldots, n$, it follows from the initial-value conditions (4.2) that

$$[{}_{\mathrm{RL}}D_{0,t}^{k\gamma-1} xi_1(t)]_{t=0} = 0 \quad (k = 1, 2, \ldots, \alpha_i M - 1). \tag{4.8}$$

In fact, from (2.4), we have

$$ {}_{\mathrm{RL}}D_{0,t}^{k\gamma-1}x_{i_1}(t) = {}_{\mathrm{RL}}D_{0,t}^{-(\alpha_i-k\gamma)}({}_{\mathrm{RL}}D_{0,t}^{-(1-\alpha_i)}x_{i_1}(t)). $$

Taking into account $[{}_{\mathrm{RL}}D_{0,t}^{-(1-\alpha_i)}x_{i_1}(t)]_{t=0} = x_{i0}$, in other words, there exists $\delta > 0$ such that the function ${}_{\mathrm{RL}}D_{0,t}^{-(1-\alpha_i)}x_{i_1}(t)$ is bounded on the interval $[0, \delta]$. Then, we arrive at

$$ \begin{aligned} [{}_{\mathrm{RL}}D_{0,t}^{k\gamma-1}x_{i_1}(t)]_{t=0} &= [{}_{\mathrm{RL}}D_{0,t}^{-(\alpha_i-k\gamma)}({}_{\mathrm{RL}}D_{0,t}^{-(1-\alpha_i)}x_{i_1}(t))]_{t=0} \\ &= 0, \end{aligned} $$

with the conclusion of remark 2.7.

Next, using repeatedly (2.5) and (4.8), we obtain

$$ \begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{\gamma}({}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{i_1}(t)) &= {}_{\mathrm{RL}}D_{0,t}^{2\gamma}x_{i_1}(t) - [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1}x_{i_1}(t)]_{t=0}\frac{t^{-\gamma-1}}{\Gamma(-\gamma)} \\ &= {}_{\mathrm{RL}}D_{0,t}^{2\gamma}x_{i_1}(t), \\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}({}_{\mathrm{RL}}D_{0,t}^{\gamma}({}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{i_1}(t))) &= {}_{\mathrm{RL}}D_{0,t}^{3\gamma}x_{i_1}(t) - [{}_{\mathrm{RL}}D_{0,t}^{2\gamma-1}x_{i_1}(t)]_{t=0}\frac{t^{-\gamma-1}}{\Gamma(-\gamma)} \\ &= {}_{\mathrm{RL}}D_{0,t}^{3\gamma}x_{i_1}(t), \\ &\ \ \vdots \end{aligned} $$

and

$$ \begin{aligned} \underbrace{{}_{\mathrm{RL}}D_{0,t}^{\gamma}({}_{\mathrm{RL}}D_{0,t}^{\gamma}\cdots{}_{\mathrm{RL}}D_{0,t}^{\gamma}}_{\alpha_i M}x_{i_1}(t)) &= {}_{\mathrm{RL}}D_{0,t}^{\alpha_i M\gamma}x_{i_1}(t) - [{}_{\mathrm{RL}}D_{0,t}^{(\alpha_i M-1)\gamma-1}x_{i_1}(t)]_{t=0}\frac{t^{-\gamma-1}}{\Gamma(-\gamma)} \\ &= {}_{\mathrm{RL}}D_{0,t}^{\alpha_i M\gamma}x_{i_1}(t) \\ &= f_i(\bar{x}, t), \end{aligned} $$

where $i = 1, 2, \ldots, n$. In addition, for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, \alpha_i M - 2$,

$$ \begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{\gamma-1}({}_{\mathrm{RL}}D_{0,t}^{k\gamma}x_{i_1}(t)) &= {}_{\mathrm{RL}}D_{0,t}^{(k+1)\gamma-1}x_{i_1}(t) - [{}_{\mathrm{RL}}D_{0,t}^{k\gamma-1}x_{i_1}(t)]_{t=0}\frac{t^{-\gamma}}{\Gamma(1-\gamma)} \\ &= {}_{\mathrm{RL}}D_{0,t}^{(k+1)\gamma-1}x_{i_1}(t), \end{aligned} $$

that is to say,

$$ \begin{aligned} [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1}({}_{\mathrm{RL}}D_{0,t}^{k\gamma}x_{i_1}(t))]_{t=0} &= \left[{}_{\mathrm{RL}}D_{0,t}^{(k+1)\gamma-1}x_{i_1}(t) - [{}_{\mathrm{RL}}D_{0,t}^{k\gamma-1}x_{i_1}(t)]_{t=0}\frac{t^{-\gamma}}{\Gamma(1-\gamma)}\right]_{t=0} \\ &= [{}_{\mathrm{RL}}D_{0,t}^{(k+1)\gamma-1}x_{i_1}(t)]_{t=0} \\ &= 0 \end{aligned} $$

and

$$ \begin{aligned} [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1}({}_{\mathrm{RL}}D_{0,t}^{(\alpha_i M-1)\gamma}x_{i_1}(t))]_{t=0} &= \left[{}_{\mathrm{RL}}D_{0,t}^{\alpha_i-1}x_{i_1}(t) - [{}_{\mathrm{RL}}D_{0,t}^{(\alpha_i M-1)\gamma-1}x_{i_1}(t)]_{t=0}\frac{t^{-\gamma}}{\Gamma(1-\gamma)}\right]_{t=0} \\ &= [{}_{\mathrm{RL}}D_{0,t}^{\alpha_i-1}x_{i_1}(t)]_{t=0} \\ &= x_{i0}. \end{aligned} $$

So, the vector $[x_{1_1}(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{1_1}(t), \ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_1 M-1)\gamma}x_{1_1}(t), x_{2_1}(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{2_1}(t), \ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_2 M-1)\gamma}x_{2_1}(t), \ldots, x_{n_1}(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{n_1}(t), \ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_n M-1)\gamma}x_{n_1}(t)]^{\mathrm{T}}$ satisfies system (4.3) and its initial-value conditions (4.4). The proof is completed. ■

Next, we extend theorem 4.1 to the more general MRO system of the following form:

$$\left.\begin{aligned}{}_{\mathrm{RL}}D_{0,t}^{\alpha_1}x_{1_1}(t) &= f_1(\bar{x},t),\\ {}_{\mathrm{RL}}D_{0,t}^{\alpha_2}x_{2_1}(t) &= f_2(\bar{x},t),\\ &\vdots\\ {}_{\mathrm{RL}}D_{0,t}^{\alpha_n}x_{n_1}(t) &= f_n(\bar{x},t),\end{aligned}\right\} \tag{4.9}$$

and

with the initial-value conditions

$$[{}_{\mathrm{RL}}D_{0,t}^{\alpha_i-k_i}x_{i_1}(t)]_{t=0} = \begin{cases} x_{i0}, & k_i = 1,\\ 0, & k_i = 2,3,\ldots,m_i,\end{cases} \tag{4.10}$$

where $i=1,2,\ldots,n$, the time variable $t>0$, $\bar{x}(t)=[x_{1_1}(t),x_{2_1}(t),\ldots,x_{n_1}(t)]^{\mathrm{T}}$, $[f_1,f_2,\ldots,f_n]^{\mathrm{T}}\in\mathbb{R}^n$. All α_i, $i=1,2,\ldots,n$, are rational numbers satisfying $m_i-1<\alpha_i<m_i\in\mathbb{Z}_+$.

With almost the similar reasoning as theorem 4.1, we obtain the following theorem.

Theorem 4.2. *System* (4.9) *with the initial-value condition* (4.10) *is equivalent to the N-dimensional system of equations with derivative order* γ

$$\left.\begin{aligned}{}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{1_1}(t) &= x_{1_2}(t),\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{1_2}(t) &= x_{1_3}(t),\\ &\vdots\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{1_{(\alpha_1-m_1+1)M}}(t) &= x_{1_{(\alpha_1-m_1+1)M+1}}(t),\\ &\vdots\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{1_{\alpha_1 M}}(t) &= f_1(\bar{x},t),\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{2_1}(t) &= x_{2_2}(t),\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{2_2}(t) &= x_{2_3}(t),\\ &\vdots\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{2_{(\alpha_2-m_2+1)M}}(t) &= x_{2_{(\alpha_2-m_2+1)M+1}}(t),\\ &\vdots\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{2_{\alpha_2 M}}(t) &= f_2(\bar{x},t),\\ &\vdots\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{n_1}(t) &= x_{n_2}(t),\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{n_2}(t) &= x_{n_3}(t),\\ &\vdots\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{n_{(\alpha_n-m_n+1)M}}(t) &= x_{n_{(\alpha_n-m_n+1)M+1}}(t),\\ &\vdots\\ {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{n_{\alpha_n M}}(t) &= f_n(\bar{x},t),\end{aligned}\right\} \tag{4.11}$$

and

subject to the initial-value conditions

$$[{}_{\mathrm{RL}}D_{0,t}^{\gamma-1}x_{i_j}(t)]_{t=0}=\begin{cases}x_{i0}, & j=\alpha_i M,\\ 0, & \text{otherwise},\end{cases} \tag{4.12}$$

where $\alpha_i = p_i/q_i$, p_i *and* q_i *are two co-prime numbers,* $i=1,2,\ldots,n$. M *is the lower common multiple of the denominators* q_i, $i=1,2,\ldots,n$, *and* $\gamma=1/M$, $N=M(\alpha_1+\alpha_2+\ldots+\alpha_n)$. *That is*

- *whenever* $[x_{1_1}(t), x_{1_2}(t),\ldots,x_{1_{\alpha_1 M}}(t), x_{2_1}(t), x_{2_2}(t),\ldots,x_{2_{\alpha_2 M}}(t),\ldots,x_{n_1}(t),x_{n_2}(t),\ldots,x_{n_{\alpha_n M}}(t)]^{\mathrm{T}}$ *with* $[x_{1_1}(t),x_{2_1}(t),\ldots,x_{n_1}(t)]^{\mathrm{T}}\in C^{m_1}[0,b]\times C^{m_2}[0,b]\times\cdots\times C^{m_n}[0,b]$ *is a solution to system (4.11), equipped with the initial-value conditions (4.12), then* $[x_{1_1}(t),x_{2_1}(t),\ldots,x_{n_1}(t)]^{\mathrm{T}}$ *solves system (4.9) and satisfies its corresponding initial-value conditions (4.10);*
- *whenever* $[x_{1_1}(t),x_{2_1}(t),\ldots,x_{n_1}(t)]^{\mathrm{T}}\in C^{m_1}[0,b]\times C^{m_2}[0,b]\times\cdots\times C^{m_n}[0,b]$ *is a solution to system (4.9) with the initial-value conditions (4.10), then the vector* $[x_{1_1}(t),x_{1_2}(t),\ldots,x_{1_{\alpha_1 M}}(t), x_{2_1}(t), x_{2_2}(t),\ldots,x_{2_{\alpha_2 M}}(t),\ldots,x_{n_1}(t),x_{n_2}(t),\ldots,x_{n_{\alpha_n M}}(t)]^{\mathrm{T}}=[x_{1_1}(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{1_1}(t),\ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_1 M-1)\gamma}x_{1_1}(t), x_{2_1}(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{2_1}(t),\ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_2 M-1)\gamma}x_{2_1}(t),\ldots,x_{n_1}(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{n_1}(t),\ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_n M-1)\gamma}x_{n_1}(t)]^{\mathrm{T}}$ *satisfies system (4.11) and its initial-value conditions (4.12).*

Now, we study the equivalent system with the same order of the following MRO fractional differential equation:

$${}_{\mathrm{RL}}D_{0,t}^{\alpha_n}x(t)+a_1\,{}_{\mathrm{RL}}D_{0,t}^{\alpha_{n-1}}x(t)+\cdots+a_{n-1}\,{}_{\mathrm{RL}}D_{0,t}^{\alpha_1}x(t)+a_n=f(x,t),\quad t>0, \tag{4.13}$$

with the initial-value condition

$$[{}_{\mathrm{RL}}D_{0,t}^{\alpha_n-1}x(t)]_{t=0}=x^0, \tag{4.14}$$

where $x\in\mathbb{R}$, function $f\in\mathbb{R}$, and a_i, $i=1,2,\ldots,n$, are constant numbers. The orders α_i, $i=1,2,\ldots,n$, are rational numbers such that $0<\alpha_i<1$ and $\alpha_n>\alpha_{n-1}>\cdots>\alpha_1$. Here, we assume that the initial-value problem (4.13)–(4.14) has a solution $x(t)\in L^{\alpha_n}(0,b)$ for some $b>0$.

Similarly, there exist $p_i,q_i\in\mathbb{Z}_+$ such that $\alpha_i=p_i/q_i$, where $(p_i,q_i)=1$. Let M be the lower common multiple of the denominators q_i, $i=1,2,\ldots,n$, and take $\gamma=1/M$, $N=\alpha_n M$.

Corollary 4.3. *Equation (4.13) with the initial-value conditions (4.14) is equivalent to the N-dimensional system of fractional differential equations,*

$$\left.\begin{aligned}
&{}_{\mathrm{RL}}D_{0,t}^{\gamma}x(t)=x_1(t),\\
&{}_{\mathrm{RL}}D_{0,t}^{\gamma}x_1(t)=x_2(t),\\
&\vdots\\
&{}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{\alpha_1 M-1}(t)=x_{\alpha_1 M}(t),\\
&{}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{\alpha_1 M}(t)=x_{\alpha_1 M+1}(t),\\
&\vdots\\
&{}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{\alpha_2 M}(t)=x_{\alpha_2 M+1}(t),\\
&\vdots\\
\text{and}\quad &{}_{\mathrm{RL}}D_{0,t}^{\gamma}x_{\alpha_n M-1}(t)=f(x,t)-a_1x_{\alpha_{n-1}M}(t)-\cdots-a_{n-1}x_{\alpha_1 M}(t)-a_n,
\end{aligned}\right\} \tag{4.15}$$

with the initial-value conditions

$$[{}_{\mathrm{RL}}D_{0,t}^{\gamma-1}x_i(t)]_{t=0}=\begin{cases}x^0, & i=\alpha_n M-1,\\ 0, & \textit{otherwise},\end{cases} \tag{4.16}$$

where $x_0(t) = x(t)$, *that is,*

— *whenever* $[x(t), x_1(t), \ldots, x_{\alpha_n M-1}(t)]^{\mathrm{T}}, x \in C^1[0,b]$ $x(t) \in C_{1-\alpha_n}[0,b]$, *for some* $b > 0$, *is a solution to system* (4.15), *equipped with the initial-value conditions* (4.16), *then* $x(t)$ *solves equation* (4.13) *and satisfies its corresponding initial-value conditions* (4.14);

— *whenever* $x(t) \in C^1[0,b]$ *is a solution to equation* (4.13) *with the initial-value conditions* (4.14), *then* $[x(t), x_1(t), \ldots, x_{\alpha_n M-1}(t)]^{\mathrm{T}} = [x(t), {}_{\mathrm{RL}}D_{0,t}^{\gamma}x(t), {}_{\mathrm{RL}}D_{0,t}^{2\gamma}x(t), \ldots, {}_{\mathrm{RL}}D_{0,t}^{(\alpha_n M-1)\gamma}x(t)]^{\mathrm{T}}$ *satisfies system* (4.15) *and its initial-value conditions* (4.16).

Remark 4.4. In suitable conditions, the solutions to Riemann–Liouville-type fractional differential equations can be extended to $+\infty$ [35].

(b) Stability analysis

In the following, we study the stability of the zero solution of the linear MRO fractional differential system that is widely used in control processing:

$$\left.\begin{aligned}
{}_{\mathrm{RL}}D_{0,t}^{\alpha_1}x_{1_1}(t) &= \sum_{i=1}^{n} a_{1i}x_{i_1}(t),\\
{}_{\mathrm{RL}}D_{0,t}^{\alpha_2}x_{2_1}(t) &= \sum_{i=1}^{n} a_{2i}x_{i_1}(t),\\
&\vdots\\
\text{and}\quad {}_{\mathrm{RL}}D_{0,t}^{\alpha_n}x_{n_1}(t) &= \sum_{i=1}^{n} a_{ni}x_{i_1}(t),
\end{aligned}\right\} \tag{4.17}$$

with the initial-value conditions

$$[{}_{\mathrm{RL}}D_{0,t}^{\alpha_i-1}x_{i_1}(t)]_{t=0} = x_{i0} \quad (i = 1, 2, \ldots, n), \tag{4.18}$$

where $\bar{x}$, t and α_i $(i = 1, 2, \ldots, n)$ are the same as those in theorem 4.1, and $A = (a_{ij}) \in \mathbb{R}^{n\times n}$.

In the following, we introduce the stability definition of system (4.17) [2,9].

Definition 4.5. The linear fractional differential system (4.17) is said to be

— stable if and only if $\forall \bar{x}_0 = [x_{10}, x_{20}, \ldots, x_{n0}]^{\mathrm{T}}$, there exist $\varepsilon > 0$ and $\delta > 0$ such that $\|\bar{x}(t)\| \le \varepsilon$ for $t \ge \delta$;

— asymptotically stable if and only if system (4.17) is stable and $\lim_{t\to+\infty} \|\bar{x}\| = 0$.

When $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, the stability of system (4.17) has been studied in [2], the corresponding conclusion is as follows. Since system (4.17) is a linear one with a constant coefficient matrix, we can obtain the necessary and sufficient condition of the stability of the solution to this system.

Lemma 4.6. *The linear fractional differential system* (4.17) *equipped with the initial-value conditions* (4.18), *where* $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$, $0 < \alpha < 1$, *is*

— *asymptotically stable if and only if all the non-zero eigenvalues of A satisfy* $|\arg(\mathrm{spec}(A))| > \alpha\pi/2$, *or A has k-multiple zero eigenvalues corresponding to a Jordan block* $\mathrm{diag}(J_1, J_2, \ldots, J_i)$, *where* J_l *is a Jordan canonical form with order* n_l, $\sum_{l=1}^{i} n_l = k$, *and* $n_l\alpha < 1, 1 \le l \le i$.

— *stable if and only if either it is asymptotically stable, or those critical eigenvalues that satisfy* $|\arg(\mathrm{spec}(A))| = \alpha\pi/2$ *have the same algebraic and geometric multiplicities, or A has k-multiple zero eigenvalues corresponding to a Jordan block matrix* $\mathrm{diag}(J_1, J_2, \ldots, J_i)$, *where* J_l *is a Jordan canonical form with order* n_l, $\sum_{l=1}^{i} n_l = k$, *and* $n_l\alpha \le 1, 1 \le l \le i$.

By using theorem 4.1 and lemma 4.6, one has the following stability result.

Theorem 4.7. *If the solution of system (4.17) with the initial-value conditions (4.18) satisfies theorem 3.1, then system (4.17) is*

— *asymptotically stable if and only if all the non-zero eigenvalues of* $\bar{A}$ *satisfy* $|\arg(\mathrm{spec}(\bar{A}))| > \gamma\pi/2$, *or* $\bar{A}$ *has k-multiple zero eigenvalues corresponding to a Jordan block* $\mathrm{diag}(J_1, J_2, \ldots, J_i)$, *where* J_l *is a Jordan canonical form with order* n_l, $\sum_{l=1}^{i} n_l = k$, *and* $n_l\gamma < 1, 1 \le l \le i$;
— *stable if and only if either it is asymptotically stable, or those critical eigenvalues that satisfy* $|\arg(\mathrm{spec}(\bar{A}))| = \gamma\pi/2$ *have the same algebraic and geometric multiplicities, or* $\bar{A}$ *has k-multiple zero eigenvalues corresponding to a Jordan block matrix* $\mathrm{diag}(J_1, J_2, \ldots, J_i)$, *where* J_l *is a Jordan canonical form with order* n_l, $\sum_{l=1}^{i} n_l = k$, *and* $n_l\gamma \le 1, 1 \le l \le i$,

where $\gamma = 1/M$ *is the same as that of theorem 4.1,* $\mathrm{spec}(\bar{A})$ *denotes the eigenvalues of matrix* $\bar{A}$. $\bar{A} = (A_{ij})$,

$$A_{ii} = \begin{pmatrix} \mathbf{0} & E_{ii} \\ a_{ii} & \mathbf{0} \end{pmatrix}_{\alpha_i M \times \alpha_i M}$$

and

$$A_{ij} = \begin{pmatrix} \mathbf{0} & O_{ij} \\ a_{ij} & \mathbf{0} \end{pmatrix}_{\alpha_i M \times \alpha_j M} \qquad (i \ne j;\ i, j = 1, 2, \ldots, n).$$

where E_{ii} *are the identity matrices with orders* $\alpha_i M - 1$ *and* O_{ij} *are* $(\alpha_i M - 1) \times (\alpha_j M - 1)$ *zero matrices,* $i, j = 1, 2, \ldots, n$.

Proof. Based on theorem 4.1, the MRO fractional differential system (4.17) and (4.18) can be changed into a higher-dimensional fractional differential system with the same order γ lying in (0,1),

$$_{\mathrm{RL}}D_{0,t}^{\gamma} X(t) = \bar{A}X(t), \quad t > 0,$$

with the initial-value condition

$$[_{\mathrm{RL}}D_{0,t}^{\gamma-1} X(t)]_{t=0} = X_0,$$

where the vector $X(t) = [x_{1_1}(t), x_{1_2}(t), \ldots, x_{1_{\alpha_1 M}}(t), x_{2_1}(t), x_{2_2}(t), \ldots, x_{2_{\alpha_2 M}}(t), \ldots, x_{n_1}(t), x_{n_2}(t), \ldots, x_{n_{\alpha_n M}}(t)]^{\mathrm{T}}$ and $X_0 = [0, 0, \ldots, x_{10}, 0, 0, \ldots, x_{20}, \ldots, 0, 0, \ldots, x_{n0}]^{\mathrm{T}}$.

Then, according to lemma 4.6, one can obtain conclusions. ∎

Remark 4.8. By applying the properties of the determinant, the eigenvalues of matrix $\bar{A}$ in theorem 4.7, i.e. the zero solutions λ of

$$\det(\lambda E - \bar{A}) = 0,$$

satisfy the following equation:

$$\det(\mathrm{diag}(\lambda^{\alpha_1 M}, \lambda^{\alpha_2 M}, \ldots, \lambda^{\alpha_n M}) - A) = 0,$$

where E is the identity matrix with order $N = \sum_{k=1}^{n} \alpha_i M$.

In the following, we study the stability of the linear nonautonomous differential system associated with system (4.9)

$$\left.\begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{\alpha_1} x_{1_1}(t) &= \sum_{i=1}^{n} (a_{1i} + b_{1i}(t)) x_{i_1}(t), \\ {}_{\mathrm{RL}}D_{0,t}^{\alpha_2} x_{2_1}(t) &= \sum_{i=1}^{n} (a_{2i} + b_{2i}(t)) x_{i_1}(t), \\ &\vdots \\ {}_{\mathrm{RL}}D_{0,t}^{\alpha_n} x_{n_1}(t) &= \sum_{i=1}^{n} (a_{ni} + b_{ni}(t)) x_{i_1}(t), \end{aligned}\right\} \tag{4.19}$$

and

subject to the initial-value conditions (4.10), where $i=1,2,\ldots,n$, the time variable $t>0$, $\bar{x}(t)=[x_{1_1}(t),x_{2_1}(t),\ldots,x_{n_1}(t)]^{\mathrm{T}}$, $a_{ji}\in\mathbb{R}$, $b_{ji}(t):[0,+\infty)\to\mathbb{R}$ are continuous functions, $j=1,2,\ldots,n$. All α_i, $i=1,2,\ldots,n$, are rational numbers satisfying $m_i-1<\alpha_i<m_i\in\mathbb{Z}_+$.

From theorem 4.1, we know that system (4.19) equipped with the initial-value conditions (4.10) is equivalent to the N-dimensional differential system with the same order γ,

$$_{\mathrm{RL}}D^{\gamma}_{0,t}X(t)=AX(t)+B(t)X(t),\quad t>0, \tag{4.20}$$

with the initial-value condition

$$[_{\mathrm{RL}}D^{\gamma-1}_{0,t}X(t)]_{t=0}=X_0, \tag{4.21}$$

where γ and N are the same as those of theorem 4.2, the vector $X(t)=[x_{1_1}(t),x_{1_2}(t),\ldots,x_{1_{\alpha_1 M}}(t), x_{2_1}(t),x_{2_2}(t),\ldots,x_{2_{\alpha_2 M}}(t),\ldots,x_{n_1}(t),x_{n_2}(t),\ldots,x_{n_{\alpha_n M}}(t)]^{\mathrm{T}}$, $X_0=[0,0,\ldots,x_{10},0,0,\ldots,x_{20},\ldots,0,0,\ldots,x_{n0}]^{\mathrm{T}}$. $A=(A_{ij})$, $B(t)=(B_{ij}(t))$,

$$A_{ii}=\begin{pmatrix}\mathbf{0} & E_{ii}\\ a_{ii} & \mathbf{0}\end{pmatrix}_{\alpha_i M\times\alpha_i M}\qquad (i=1,2,\ldots,n),$$

$$A_{ij}=\begin{pmatrix}\mathbf{0} & O_{ij}\\ a_{ij} & \mathbf{0}\end{pmatrix}_{\alpha_i M\times\alpha_j M}\qquad (i\neq j;\ i,j=1,2,\ldots,n)$$

and

$$B_{ij}(t)=\begin{pmatrix}\mathbf{0} & O_{ij}\\ b_{ij}(t) & \mathbf{0}\end{pmatrix}_{\alpha_i M\times\alpha_j M}\qquad (i,j=1,2,\ldots,n),$$

where E_{ii} are the identity matrices with orders $\alpha_i M-1$ and O_{ij} are $(\alpha_i M-1)\times(\alpha_j M-1)$ zero matrices.

Equation (4.20) is a linear system but with a variable coefficient matrix, so we only obtain the sufficient condition of the stability of its solution.

Theorem 4.9. *Suppose that the matrix A satisfies $|\mathrm{spec}(A)|\neq 0$, $|\arg(\mathrm{spec}(A))|\geq\gamma\pi/2$, the critical eigenvalues that satisfy $|\arg(\mathrm{spec}(A))|=\gamma\pi/2$ have the same algebraic and geometric multiplicities, and $\int_0^\infty\|B(t)\|\,\mathrm{d}t$ is bounded. Then, the zero solution of* (4.19) *is stable, where* spec(A) *denotes the eigenvalues of matrix A.*

Likewise, we consider the stability of the following autonomous MRO fractional differential equation:

$$_{\mathrm{RL}}D^{\alpha_n}_{0,t}x(t)+a_1\,{}_{\mathrm{RL}}D^{\alpha_{n-1}}_{0,t}x(t)+\cdots+a_{n-1}\,{}_{\mathrm{RL}}D^{\alpha_1}_{0,t}x(t)+a_nx(t)=0,\quad t>0, \tag{4.22}$$

with the initial-value conditions

$$[_{\mathrm{RL}}D^{\alpha_n-1}_{0,t}x(t)]_{t=0}=x^0, \tag{4.23}$$

where x,a_i,α_i, $i=1,2,\ldots,n$, are the same as those in corollary 4.3.

It follows from corollary 4.3 that equation (4.22) with the initial-value conditions (4.23) is equivalent to the following system:

$$\begin{aligned}
&_{\mathrm{RL}}D^{\gamma}_{0,t}x(t)=x_1(t),\\
&_{\mathrm{RL}}D^{\gamma}_{0,t}x_1(t)=x_2(t),\\
&\vdots\\
&_{\mathrm{RL}}D^{\gamma}_{0,t}x_{\alpha_1 M}(t)=x_{\alpha_1 M+1}(t),\\
&\vdots
\end{aligned}$$

and

$$_{\mathrm{RL}}D^{\gamma}_{0,t}x_{\alpha_n M-1}(t)=-a_1x_{\alpha_{n-1}M}(t)-\cdots-a_{n-1}x_{\alpha_1 M}(t)-a_nx(t),$$

with the initial-value conditions

$$[_{\mathrm{RL}}D^{\gamma-1}_{0,t}x_i(t)]_{t=0}=\begin{cases}x^0, & i=\alpha_n M-1,\\ 0, & \text{otherwise},\end{cases}$$

i.e.

$$ {}_{\mathrm{RL}}D_{0,t}^{\gamma}X(t) = BX(t), \quad t > 0, $$

with the initial-value condition

$$ [{}_{\mathrm{RL}}D_{0,t}^{\gamma-1}X(t)]_{t=0} = [0, 0, \ldots, 0, x^0]^{\mathrm{T}}, $$

where

$$ B = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ b_{N1} & b_{N2} & b_{N3} & b_{N4} & \cdots & b_{NN} \end{pmatrix}, $$

$$ b_{Nj} = \begin{cases} -a_n, & j = 1, \\ -a_{n-i}, & j = \alpha_i M, \ i = 1, 2, \ldots, n-1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.24} $$

and $N = \alpha_n M$.

One can derive the following result.

Corollary 4.10. *The zero solution to equation* (4.22) *is*

— *asymptotically stable if and only if* $|\arg(\lambda)| > \gamma\pi/2$, *where* λ *is the solution of the characteristic equation*

$$ \det(\lambda E - B) = 0, \tag{4.25} $$

$\gamma = 1/M$ *is the same as that of corollary 4.3,* E *is an identity matrix with order* $N = \alpha_n M$; *or equation* (4.25) *has k-multiple zero eigenvalues corresponding to a Jordan block* $\mathrm{diag}(J_1, J_2, \ldots, J_i)$, *where* J_l *is a Jordan canonical form with order* n_l, $\sum_{l=1}^{i} n_l = k$, *and* $n_l\gamma < 1$, $1 \le l \le i$;

— *stable if and only if either* $|\arg(\lambda)| > \gamma\pi/2$ *or* $|\arg(\lambda)| \ge \gamma\pi/2$ *and those critical solutions of equation* (4.25) *that satisfy* $|\arg(\lambda)| = \gamma\pi/2$ *have the same algebraic and geometric multiplicities, or equation* (4.25) *has k-multiple zeros corresponding to a Jordan block matrix* $\mathrm{diag}(J_1, J_2, \ldots, J_i)$, *where* J_l *is a Jordan canonical form with order* n_l, $\sum_{l=1}^{i} n_l = k$, *and* $n_l\gamma \le 1, 1 \le l \le i$.

Proof. This corollary can be proved in the same manner as that in the proof of theorem 4.7, so is omitted here. ■

Remark 4.11. By applying the properties of the determinant, equation (4.25) is equivalent to the following equation:

$$ \lambda^N - b_{NN}\lambda^{N-1} - b_{N,N-1}\lambda^{N-2} - \cdots - b_{N2}\lambda - b_{N1} = 0. \tag{4.26} $$

(c) Several examples

In this subsection, we will give several numerical simulations to illustrate the main results derived in this section.

Example 4.12. Consider the following fractional differential system:

$$\left.\begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{1/2}x_{1_1}(t) &= a_{11}x_{1_1}(t) + a_{12}x_{2_1}(t) \\ \text{and} \quad {}_{\mathrm{RL}}D_{0,t}^{1/3}x_{2_1}(t) &= a_{21}x_{1_1}(t) + a_{22}x_{2_1}(t), \end{aligned}\right\} \tag{4.27}$$

with the initial-value conditions

$$\left.\begin{aligned} [{}_{\mathrm{RL}}D_{0,t}^{-1/2}x_{1_1}(t)]_{t=0} &= x_{10} \\ \text{and} \quad [{}_{\mathrm{RL}}D_{0,t}^{-2/3}x_{2_1}(t)]_{t=0} &= x_{20}. \end{aligned}\right\} \tag{4.28}$$

It is obvious that $\gamma = \frac{1}{6}$, $N=5$, and it follows from theorem 4.1 that system (4.27) equipped with the initial-value conditions (4.28) is equivalent to a five-dimensional system that reads

$$\left.\begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{1/6}x_{1_1}(t) &= x_{1_2}(t), \\ {}_{\mathrm{RL}}D_{0,t}^{1/6}x_{1_2}(t) &= x_{1_3}(t), \\ {}_{\mathrm{RL}}D_{0,t}^{1/6}x_{1_3}(t) &= a_{11}x_{1_1}(t) + a_{12}x_{2_1}(t), \\ {}_{\mathrm{RL}}D_{0,t}^{1/6}x_{2_1}(t) &= x_{2_2}(t) \\ \text{and} \quad {}_{\mathrm{RL}}D_{0,t}^{1/6}x_{2_2}(t) &= a_{21}x_{1_1}(t) + a_{22}x_{2_1}(t), \end{aligned}\right\} \tag{4.29}$$

with the initial-value conditions

$$\left.\begin{aligned} [{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{1_1}(t)]_{t=0} &= 0, \\ [{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{1_2}(t)]_{t=0} &= 0, \\ [{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{1_3}(t)]_{t=0} &= x_{10}, \\ [{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{2_1}(t)]_{t=0} &= 0 \\ \text{and} \quad [{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{2_2}(t)]_{t=0} &= x_{20}. \end{aligned}\right\} \tag{4.30}$$

Now, taking $a_{11} = -2$, $a_{12} = 0.2$, $a_{21} = 0$ and $a_{22} = -1.3$, we obtain

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1.3 & 0 \end{pmatrix}.$$

Using a simple calculation yields the eigenvalues λ_k $(k=1,2,3,4,5)$ of $\bar{A}$,

$$\begin{aligned} \lambda_1 &= \sqrt[3]{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right), \\ \lambda_2 &= \sqrt[3]{2}(\cos\pi + \mathrm{i}\sin\pi), \\ \lambda_3 &= \sqrt[3]{2}\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right), \\ \lambda_4 &= \sqrt[2]{1.3}\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) \\ \text{and} \quad \lambda_5 &= \sqrt[2]{1.3}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right), \end{aligned}$$

which satisfy $|\arg(\lambda_k)| > \gamma\pi/2 = \pi/12$, so system (4.27) with the initial-value condition (4.28) is asymptotically stable from theorem 4.7. At the same time, we give a figure to demonstrate this, see figure 1.

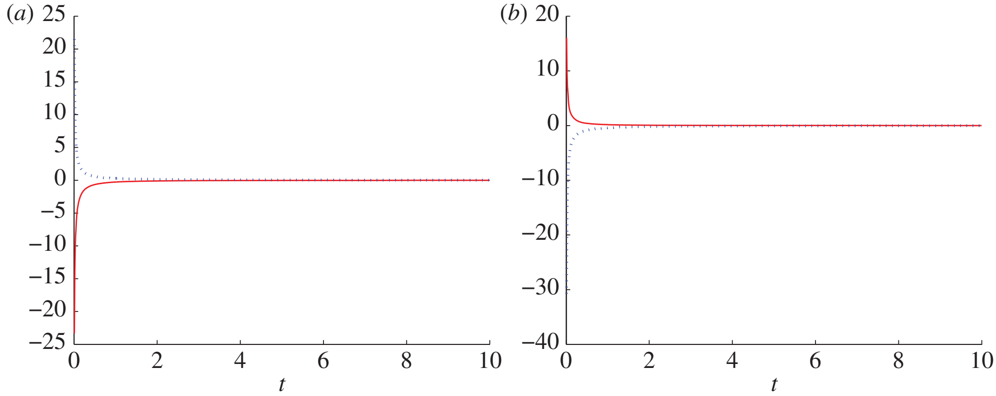

Figure 1. (*a*) Numerical solution of system (4.27) with the initial values $x_{10} = -5$ and $x_{20} = 3.5$ and (*b*) numerical solution of system (4.27) with the initial values $x_{10} = 3.5$ and $x_{20} = -5$. (*a*,*b*) Solid line, $x_{1_1}(t)$; dotted line, $x_{2_1}(t)$. (Online version in colour.)

Example 4.13. Consider the following MRO fractional differential equation:

$$_{\mathrm{RL}}D_{0,t}^{1/3}x(t) + a_1\,{}_{\mathrm{RL}}D_{0,t}^{1/6}x(t) + a_2x(t) = 0, \quad t > 0, \tag{4.31}$$

with the initial-value condition

$$[_{\mathrm{RL}}D_{0,t}^{-2/3}x(t)]_{t=0} = x_0. \tag{4.32}$$

In the same way, based on corollary 4.3, we see that $\gamma = \frac{1}{6}$ and $N = 2$ from equation (4.31). Furthermore, equation (4.31) with the initial-value condition (4.32) is equivalent to the following system in $\mathbb{R}^2$:

$$\left.\begin{aligned} &{}_{\mathrm{RL}}D_{0,t}^{1/6}x(t) = x_1(t) \\ \text{and} \quad &{}_{\mathrm{RL}}D_{0,t}^{16}x_1(t) = -a_1x_1(t) - a_2x(t), \end{aligned}\right\} \tag{4.33}$$

with the initial-value conditions

$$\left.\begin{aligned} &[_{\mathrm{RL}}D_{0,t}^{\gamma-1}x(t)]_{t=0} = 0 \\ \text{and} \quad &[_{\mathrm{RL}}D_{0,t}^{\gamma-1}x_1(t)]_{t=0} = x^0. \end{aligned}\right\} \tag{4.34}$$

Next, we take $a_1 = 0.002$, $a_2 = 0.05$ and investigate the stability of equation (4.31) with the initial-value condition (4.32). According to corollary 4.10, it is needed to compute the eigenvalues of the coefficient matrix of system (4.33). The coefficient matrix of system (4.33) can be written as

$$B = \begin{pmatrix} 0 & 1 \\ -0.05 & -0.002 \end{pmatrix},$$

and it is easy to obtain the eigenvalues of B,

$$\lambda_1 = \frac{-0.002 + \sqrt{0.199996}\mathrm{i}}{2} \quad \text{and} \quad \lambda_2 = \frac{-0.002 - \sqrt{0.199996}\mathrm{i}}{2}.$$

We can see that λ_1 and λ_2 satisfy $|\arg(\lambda_k)| > \gamma\pi/2 = \pi/12$, $k = 1, 2$. Therefore, the zero solution to equation (4.31) is asymptotically stable. In figure 2, we numerically simulate the above stability result in the light of different initial values.

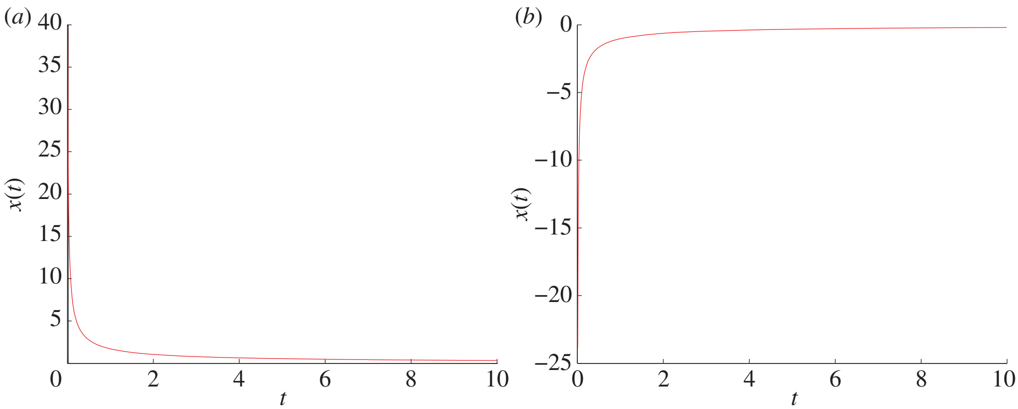

Figure 2. (*a*) Numerical solution of equation (4.31) with the initial values $x^0 = 5$ and (*b*) numerical solution of equation (4.31) with the initial values $x^0 = -3$. (Online version in colour.)

Example 4.14. For simplicity, we consider the following linear nonautonomous fractional differential system:

$$\left.\begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{3/2}x_{1_1}(t) &= -x_{1_1}(t) + \frac{1}{(t+1)^2}x_{2_1}(t) \\ \text{and}\quad {}_{\mathrm{RL}}D_{0,t}^{4/3}x_{2_1}(t) &= 2x_{1_1}(t) - 3x_{2_1}(t), \end{aligned}\right\} \tag{4.35}$$

with the initial-value conditions

$$\left.\begin{aligned} &[{}_{\mathrm{RL}}D_{0,t}^{1/2}x_{1_1}(t)]_{t=0} = x_{10}, \quad [{}_{\mathrm{RL}}D_{0,t}^{-1/2}x_{1_1}(t)]_{t=0} = 0, \\ &[{}_{\mathrm{RL}}D_{0,t}^{1/3}x_{2_1}(t)]_{t=0} = x_{20} \quad \text{and} \quad [{}_{\mathrm{RL}}D_{0,t}^{-2/3}x_{2_1}(t)]_{t=0} = 0. \end{aligned}\right\} \tag{4.36}$$

Next, we determine the stability of the zero solution to system (4.35). According to theorem 4.2, we can change it into the following equivalent system in R^{17}:

$$\left.\begin{aligned} &{}_{\mathrm{RL}}D_{0,t}^{1/6}x_{1_1}(t) = x_{1_2}(t), \\ &{}_{\mathrm{RL}}D_{0,t}^{1/6}x_{1_2}(t) = x_{1_3}(t), \\ &\vdots \\ &{}_{\mathrm{RL}}D_{0,t}^{1/6}x_{1_9}(t) = -x_{1_1}(t) + \frac{1}{(t+1)^2}x_{2_1}(t), \\ &{}_{\mathrm{RL}}D_{0,t}^{1/6}x_{2_1}(t) = x_{2_2}(t), \\ &{}_{\mathrm{RL}}D_{0,t}^{1/6}x_{2_2}(t) = x_{2_3}(t), \\ &\vdots \\ \text{and}\quad &{}_{\mathrm{RL}}D_{0,t}^{1/6}x_{2_8}(t) = 2x_{1_1}(t) - 3x_{2_1}(t), \end{aligned}\right\} \tag{4.37}$$

with the initial-value conditions

$$\left.\begin{aligned} &[{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{1_1}(t)]_{t=0} = \cdots = [{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{1_8}(t)]_{t=0} = 0, \\ &[{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{1_9}(t)]_{t=0} = x_{10}, \\ &[{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{2_1}(t)]_{t=0} = \cdots = [{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{2_7}(t)]_{t=0} = 0 \\ \text{and}\quad &[{}_{\mathrm{RL}}D_{0,t}^{-5/6}x_{2_8}(t)]_{t=0} = x_{20}. \end{aligned}\right\} \tag{4.38}$$

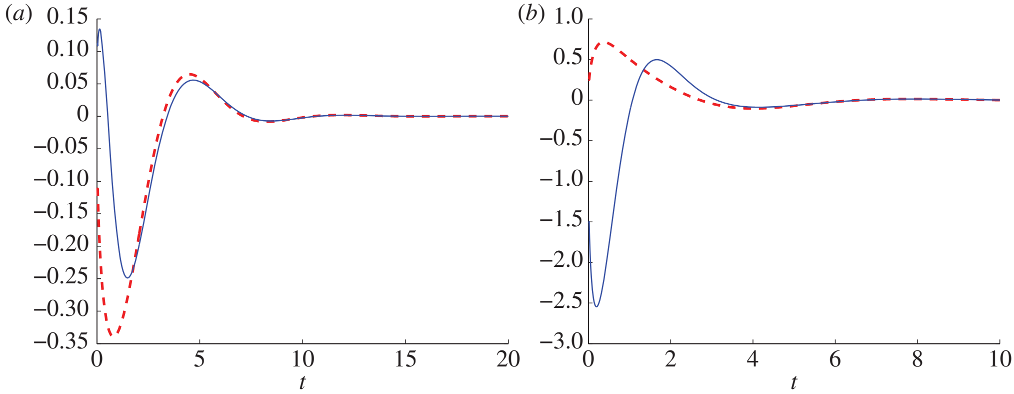

Figure 3. (*a*) Numerical solutions of system (4.35) with the initial values $x_{10} = -0.5$ and $x_{20} = 0.3$ and (*b*) numerical solutions of system (4.35) with the initial values $x_{10} = 1.6$ and $x_{20} = -5$. (Online version in colour.)

By tedious calculation, the matrix $B(t)$ and the eigenvalues λ_k $(k = 1, 2, \ldots, 17)$ of system matrix A in system (4.35) satisfy the conditions of theorem 4.9, so the zero solution of system (4.35) is stable. After numerical simulations, we also find that its zero solution is stable, see figure 3, which coincides with the theoretical analysis.

5. Conclusion

In this paper, we study Caputo-type and Riemann–Liouville-type MRO fractional differential systems. By using the properties of the fractional calculus, we can change the original systems in Caputo and Riemann–Liouville senses into their respective equivalent ones. Through these systems, we can conveniently study the stability of the equilibria to the original systems. Various examples are also displayed, which support the theoretical results.

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