AIP Chaos
An Interdisciplinary Journal of Nonlinear Science

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Citation: Chaos 23, 013118 (2013); doi: 10.1063/1.4791589
View online: http://dx.doi.org/10.1063/1.4791589
View Table of Contents: http://chaos.aip.org/resource/1/CHAOEH/v23/i1
Published by the American Institute of Physics.

Generalized variable projective synchronization of time delayed systems

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(Received 13 October 2012; accepted 28 January 2013; published online 14 February 2013)

We study generalized variable projective synchronization between two unified time delayed systems with constant and modulated time delays. A novel Krasovskii-Lyapunov functional is constructed and a generalized sufficient condition for synchronization is derived analytically using the Lyapunov stability theory and adaptive techniques. The proposed scheme is valid for a system of n-numbers of first order delay differential equations. Finally, a new neural oscillator is considered as a numerical example to show the effectiveness of the proposed scheme. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4791589]

The phenomenon of chaos synchronization has been investigated in detail for the last two decades. Various synchronization techniques have been discovered for a set of coupled time delayed systems. Due to both theoretical interest and practical applications, several types of synchronization have been found, such as complete synchronization (CS),[1] phase synchronization (PS),[2] lag synchronization (LS),[3] multiplexing synchronization (MS),[4] and generalized synchronization (GS).[5] Among them, there are few articles on projective and functional projective synchronization on time delayed systems. Here we propose the most generalized case for variable projective synchronization $y=\alpha(t)x(t-\sigma(t))$ between two time delayed systems, where the factor $\alpha(t)$ is a time dependent function. The Krasovskii-Lyapunov functional has been constructed and sufficient condition for variable projective synchronization derived. The scheme is also valid for lag and anticipatory synchronization. To show the effectiveness and feasibility of our scheme, numerical simulations are presented with a new delayed neural oscillator.

I. INTRODUCTION

Chaos synchronization is an interesting phenomenon, which has wide applications in Science, Engineering, and Communications.[6–8] Till now, many kinds of synchronization have been found in interacting chaotic systems, among them projective synchronization[9,10] has received much attention due to its faster communication[11,12] and proportionality between the dynamical systems. In case of projective synchronization, the drive and the response system can be synchronized up to a scaling factor. The scaling factor is a constant transformation between the driving and the response variables. In application to secure communications, this proportional feature can be used to extend binary digital to M-nary digital communication for getting communication much faster. Recently, function $\alpha(t)$ projective synchronization has been investigated,[13] where the driving and the response systems could be synchronized up to a scaling function $\alpha(t)$ leading to the synchronization manifold[14] $y=\alpha(t)x$. In case of variable projective synchronization, due to the unpredictability of the scaling function, it can additionally enhance the level of security during communication. The projective synchronization phenomenon is also investigated in time delayed systems with constant and variable scaling factors.[15–17] In this article, we propose the most general variable projective synchronization criterion between two time delayed systems where the synchronization manifold is of the form $y=\alpha(t)x(t-\sigma(t))$.

The coupling function can be derived by the active nonlinear control technique for the response dynamics. It is important to mention here that no results exist in the literature mentioning the synchronization system of n delayed differential equations.

The usual notations are followed in this article. The superscript T denotes the transposition and the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that X–Y is positive semi-definite (respectively, positive definite). $\Re^n$ and $\Re^{n\times n}$ denote n-dimensional Euclidean space and the set of all $n \times n$ real matrices, respectively. I is the identity matrix. The notation $\star$ always denotes the symmetric block in one symmetric matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions. Let $\tau_M > 0$ and $\mathcal{C}([-\tau_M, 0]; \Re^n)$ denotes the family of continuous functions φ from $[-\tau_M, 0]$ to $\Re^n$ with norm $||\varphi|| = \sup_{-\tau_M \leq \theta \leq 0}||\varphi||$. $\mathfrak{L}_2[0, \infty)$ stands for the space of square integrable functions on $[0, \infty)$.

The article is organized as follows: In Sec. II, we construct the generalized variable projective synchronization phenomenon for a coupled time delayed systems. A sufficient condition for synchronization is derived analytically

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using a novel Krasovskii-Lyapunov (K-L) functional and Lyapunov stability theory in Sec. III. The nonlinear controller for synchronization is obtained by an adaptive control technique. Section IV is the numerical simulation with a new delayed oscillator for constant and modulated time delay. Conclusion and further investigations are mentioned in Sec. V.

II. PROBLEM FORMULATION

We consider the following coupled chaotic systems with time varying delay:

$$\dot{x}(t) = A_1 x + A_2 x_{\tau(t)} + f(x, x_{\tau(t)}), \tag{1}$$

$$\dot{y}(t) = A_1 y + A_2 y_{\tau(t)} + f(y, y_{\tau(t)}) + u(x, y), \tag{2}$$

where $x(t), y(t) \in \mathbb{R}^n, A_1 \in \mathbb{R}^{n\times n}, A_2 \in \mathbb{R}^{n\times n}, f$ is a nonlinear vector field, $u(x, y)$ is the active nonlinear control term defined appropriately, and $x_{\tau(t)} = x(t - \tau(t))$. Here $\tau(t)$ is a time varying delay such that $0 \le \tau(t) \le \tau_m < \infty$. The main objective of this study is to achieve generalized variable projective synchronization (GVPS) between Eqs. (1) and (2).

Let the error between two dynamics be $e(t) = y(t) -\alpha(t)x_{\sigma(t)}$, where $\sigma(t)$ is the time varying delay during the information lag or anticipation such that $0 \le \sigma(t) \le \sigma_m < \infty \, \dot{\sigma}(t) = \delta (< 1)$. If $e(t) \to 0$, that is $y \to \alpha(t)x_{\sigma(t)}$ as $t \to \infty$, GVPS between the coupled system is achieved with respect to the nonlinear control $u(x, y)$ to be designed. Thus, the synchronization manifold of the systems (1) and (2) become $y = \alpha(t)x_{\sigma(t)}$.

To achieve the main focus of this paper—GVPS, a new nonlinear active control is defined as

$$\begin{aligned} u(x,y) = {} & K(\alpha(t)x_{\sigma(t)} - y) + \delta A_1 y + \delta A_2 y_{\tau(t)} \\ & - f(y, y_{\tau(t)}) + (1-\delta)\alpha(t) f(x_{\sigma(t)}, x_{\sigma(t)+\tau(t)}) \\ & + \dot{\alpha}(t) x_{\sigma(t)}, \end{aligned} \tag{3}$$

where $K \in \mathbb{R}^{n\times n}$ is the coupling strength to be determined later. Thus, the error dynamic can be obtained as

$$\begin{aligned} \dot{e} & = \dot{y} - \dot{\alpha}(t)x_{\sigma(t)} - \alpha(t)(1-\delta)\dot{x}_{\sigma(t)}, \\ & = A_1 y + A_2 y_{\tau(t)} + f(y, y_{\tau(t)}) - (1-\delta)\alpha(t)A_1 x_{\sigma(t)} \\ & \quad - (1-\delta)\alpha(t)A_2 x_{\sigma(t)+\tau(t)} - (1-\delta)\alpha(t) \\ & \quad \times f(x_{\sigma(t)}, x_{\sigma(t)+\tau(t)}) - \dot{\alpha}(t)x_{\sigma(t)} + K(\alpha(t)x_{\sigma(t)} - y) \\ & \quad - \delta A_1 y + \dot{\alpha}(t)x_{\sigma(t)} - \delta A_2 y_{\tau(t)} - f(y, y_{\tau(t)}) \\ & \quad + (1-\delta)\alpha(t) f(x_{\sigma(t)}, x_{\sigma(t)+\tau(t)}) \\ & = (1-\delta)A_1 e + (1-\delta)A_2 e_{\tau(t)} - Ke. \end{aligned} \tag{4}$$

The initial condition corresponding to the error dynamical system (4) is given as $e(\theta) = \varphi(\theta), \theta \in [-\tau_M, 0]$, $\varphi \in \mathcal{C}([-\tau_M, 0], \mathbb{R}^n)$, where $\mathcal{C}([-\tau_M, 0], \mathbb{R}^n)$ denotes the Banach space of the continuous vector valued functions mapping the interval $[-\tau_M, 0]$ into $\mathbb{R}$. Here $\tau_M = \max\{\tau_m, \sigma_m\}$.

If the asymptotic stability of the error dynamics (4) is assured at equilibrium then the asymptotic synchronization exists between both Eqs. (1) and (2) under the proposed control. In the above, the GVPS problem has been formulated using a nonlinear control. The difference between existing generalized projective synchronization[15,16,18,19] and the above proposed problem is the use of nonlinear control and the variable projective scaling factor $\alpha(t)$. In addition, the problem stated here can be used for n-systems of first order delayed differential equations, which can contain non-differentiable time varying delays unlike in Refs. 15 and 16. We have used an active nonlinear control technique for the slave dynamics in order to achieve the GVPS, which is similar to the one proposed in Ref. 15. The existing conditions do not depend on the derivative of $\sigma(t)$ whereas the error dynamics (4) itself contains δ so that we can have a bound for the rate of lag or anticipation by solving the condition. It is important to mention here that if the condition does not have proper parameter δ, it means that for any $\sigma(t)$ synchronization is achieved, which is not possible in the physical sense, and no results exist in the literature, which are dependent on the derivative bound of $\sigma(t)$. We use the following lemma on deriving sufficient lower matrix inequality (LMI) conditions:

Lemma 1: (Ref. 20). For any constant matrix $X \in \mathbb{R}^{n\times n}, X = X^T > 0$, there exists a positive scalar τ_M such that $0 \le \tau(t) \le \tau_M$, and a vector-valued function $\dot{x} : [-\tau_M, 0] \to \mathbb{R}^n$, such that the integral $-\tau_M \int_{t-\tau_M}^{t} \dot{x}^T (s)X\dot{x}(s)ds$ is well defined, and fulfills the following inequality:

$$\begin{aligned} & -\tau_M \int_{t-\tau_M}^{t} \dot{x}^T(s)X\dot{x}(s)ds \\ & \le \begin{bmatrix} x(t) \\ x(t-\tau_M) \end{bmatrix}^T \begin{bmatrix} -X & X \\ \star & -X \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau_M) \end{bmatrix}. \end{aligned} \tag{5}$$

III. ANALYTICAL CONDITIONS

In this section, we present various conditions for projective synchronization based on the stability theory of time varying delayed systems. For this objective, we consider the following Lyapunov functional:

$$\begin{aligned} V(t, e_t, \dot{e}_t) = {} & e^T(t)Pe(t) + \int_{-\tau_M}^{0} e^T(t+\xi) S_1 e(t+\xi) d\xi \\ & + \tau_M \int_{-\tau_M}^{0} \int_{t+\theta}^{t} \dot{e}^T(\xi) S_2 \dot{e}(\xi) d\xi d\theta \\ & + \int_{t-\sigma(t)}^{t} e^T(\xi) S_3 e(\xi) d\xi, \end{aligned} \tag{6}$$

where $e_t = e(t+s), \forall s \in [-\tau_M, 0]$, and P, S_1, S_2, S_3 are positive definite matrices of appropriate dimensions. It is clear that $V(t, e_t, \dot{e}_t)$ is a quadratic functional depending on derivatives. The asymptotic stability of the closed-loop error system described by Eq. (4) is assured if there exist $\varepsilon_i > 0 \, (i = 1, 2, 3)$ such that

$$\varepsilon_1||e(t)||^2 \le V(t, e_t, \dot{e}_t) \le \varepsilon_2||e_t||^2_{\mathbb{M}},$$
$$\dot{V}(t, e_t, \dot{e}_t) \le -\varepsilon_3||e(t)||^2.$$

By using the Lyapunov functional given in Eq. (6), conditions for GVPS are obtained and stated as the following preposition.

A. Generalized variable projective synchronization condition

Preposition 1: For the scalar $\tau_M > 0, \delta > 0$, the error dynamical system (4) is globally asymptotically stable with the prescribed active nonlinear control (3), if there exist symmetric positive definite matrices P, S_1, S_2, S_3, and any matrix M with compatible dimensions such that the following inequality holds:

$$\Xi = \begin{bmatrix} \varpi_{11} & \varpi_{12} & 0 & 0 & \varpi_{15} \\ \star & -2S_2 & S_2 & 0 & \varpi_{25} \\ \star & \star & -S_1 - S_2 & 0 & 0 \\ \star & \star & \star & \varpi_{44} & 0 \\ \star & \star & \star & \star & \varpi_{55} \end{bmatrix} < 0, \quad (7)$$

where

$$\varpi_{11} = S_1 - S_2 + S_3 + 2(1-\delta)MA_1 - 2L$$
$$\varpi_{12} = S_2 + (1-\delta)MA_2;$$
$$\varpi_{15} = -M^T + (1-\delta)(MA_1)^T - L^T + P;$$
$$\varpi_{25} = (1-\delta)(MA_2)^T;$$
$$\varpi_{44} = -(1-\delta)S_3;$$
$$\varpi_{55} = \tau_M^2 S_2 - 2M;$$

and the control gain is given by $K = M^{-1}L$.

Proof: see Appendix. Preposition 1 presents a condition for asymptotic stability of the closed-loop error dynamics (4), which eventually lead to GVPS of Eqs. (1) and (2). It must be noted that the condition is dependent on the derivative of the time varying delay $\sigma(t)$ in the synchronization manifold but does not depend on the derivative of the time varying delay $\tau(t)$. Thus, the condition provides an upper bound of $\tau(t)$, τ_M, and the time varying delay can be any bounded continuous function not necessary to have a derivative. Thus, modulated delays and non-differentiable delays can be considered in the synchronization problem and still the generalized condition is feasible to achieve synchronization. Now, we assume that the derivative of the time varying delay exists, that is, $\dot{\tau}(t) = \varrho$. Then, the following less conservative condition is proposed.

B. GVPS condition for differentiable time varying delays

Preposition 2: For the scalar $\tau_M > 0, \delta > 0$, and $\varrho > 0$, the system described by Eq. (4) is globally asymptotically stable with the prescribed active nonlinear control (3), if there exist symmetric positive definite matrices P, S_1, S_2, S_3, S_4, and any matrix M with compatible dimensions such that the following inequality holds:

$$\Xi = \begin{bmatrix} \varpi_{11} & \varpi_{12} & 0 & 0 & \varpi_{15} \\ \star & \varpi_{22} & S_2 & 0 & \varpi_{25} \\ \star & \star & -S_1 - S_2 & 0 & 0 \\ \star & \star & \star & \varpi_{24} & 0 \\ \star & \star & \star & \star & \varpi_{55} \end{bmatrix} < 0, \quad (8)$$

where

$$\varpi_{11} = S_1 - S_2 + S_3 + S_4 + 2(1-\delta)MA_1 - 2L$$
$$\varpi_{12} = S_2 + (1-\delta)MA_2;$$
$$\varpi_{15} = -M^T + (1-\delta)(MA_1)^T - L^T + P;$$
$$\varpi_{22} = -S_2 - (1-\varrho)S_4;$$
$$\varpi_{24} = -(1-\delta)S_3;$$
$$\varpi_{25} = (1-\delta)(MA_2)^T;$$
$$\varpi_{55} = \tau_M^2 S_2 - 2M;$$

and the control gain is given by $K = G^{-1}L$.

Proof: see Appendix. In the above two prepositions, we have considered variable projective synchronization, i.e., projective scaling factor as $\alpha(t)$ along with the non-differentiable time varying delay and rate of lag/anticipation variations. The above conditions generalize the projective synchronization proposed in Refs. 15, 18–21. In Refs. 15 and 16, authors derived a generalized projective synchronization criterion for a constant scaling factor α. In fact, the result obtained in Refs. 15 and 16 is not suitable for n—system of delay differential equations (DDEs), and there is the requirement to obtain a new condition. For this objective, we consider generalized projective synchronization (GPS) between Eqs. (1) and (2) without a variable scaling factor and the error between both dynamics is $e(t) = y(t) - \alpha x_{\sigma(t)}$, where α is constant. Thus, the synchronization manifold of systems (1) and (2) is $y = \alpha x_{\sigma(t)}$. The corresponding nonlinear active control can be defined as

$$u(x, y) = K(\alpha x_{\sigma(t)} - y) + \delta A_1 y + \delta A_2 y_{\tau(t)}$$
$$-f(y, y_{\tau(t)}) + \alpha(1-\delta) f(x_{\sigma(t)}, x_{\sigma(t)+\tau(t)}),$$

where $K \in \mathbb{R}^{n\times n}$ is the coupling strength. Thus, the error dynamics can further be reduced to Eq. (4), and the corresponding LMI conditions for GPS are given in Prepositions 1 and 2. It can be noted that the derived result is the most generalized one.[15,16] We now proceed to obtain anticipate-complete-lag synchronization with constant σ. For $\sigma = 0$, projective complete synchronization is achieved; for $\sigma > 0$, projective lag synchronization can be achieved and for $\sigma < 0$, anticipation projective synchronization is achieved. Thus the synchronization manifold is $y = \alpha x_\sigma$. We define a nonlinear control to obtain the error dynamical system (4) and the corresponding criteria are given below.

Preposition 3: For the scalar $\tau_M > 0$, the error dynamical system is globally asymptotically stable, if there exist symmetric positive definite matrices P, S_1, S_2, S_3, S_4, and any matrix M with compatible dimensions such that the following inequality holds:

$$\Xi = \begin{bmatrix} \varpi_{11} & \varpi_{12} & 0 & 0 & \varpi_{15} \\ \star & -S_2 & S_2 & 0 & (MA_2)^T \\ \star & \star & -S_1 - S_2 & 0 & 0 \\ \star & \star & \star & -S_3 & 0 \\ \star & \star & \star & \star & \varpi_{55} \end{bmatrix} < 0, \quad (9)$$

where

$$\varpi_{11} = S_1 - S_2 + S_3 + 2MA_1 - 2L;$$
$$\varpi_{12} = S_2 + MA_2;$$
$$\varpi_{15} = -M + (MA_1)^T - L^T;$$

$\varpi_{55} = \tau_M^2 S_2 - 2M$; and the control gain is given by $K = G^{-1}L$.

Proof: see Appendix.

IV. NUMERICAL EXAMPLES

A. Delayed neural oscillator

In recent years, synchronization of neural oscillators, especially chaotic neural networks have discussed by many authors.[18,20,21] Potential implementation of neural network for field programmable gate array (FPGA) has been demonstrated in Ref. 22. Bursting synchronization in neuronal network has been studied in Ref. 23. For the current study, to test the GVPS conditions found, the following time varying delayed neural oscillator is introduced. The delayed neural network can be realized in FPGA by the method proposed in Ref. 22. The dynamics of neural oscillator is given as follows:

$$\dot{x}(t) = A_1 x(t) + A_2 x_{\tau(t)} + f(x, x_{\tau(t)}), \quad (10)$$

where $A_1 = diag\{-1, -1\}, A_2 = \begin{bmatrix} 0.1 & 0 \\ -0.2 & -1 \end{bmatrix}$,

$$f(x, x_{\tau(t)}) = \begin{bmatrix} 1.8\varphi(x^1) - 0.1\varphi(x^2) - 1.6\varphi(x^1_{\tau(t)}) - 0.6\varphi(x^2_{\tau(t)}) \\ -2.5\varphi(x^1) + 0.3\varphi((x^2) + 0.1\varphi(x^1_{\tau(t)}) - 2.4\varphi(x^2_{\tau(t)}) \end{bmatrix},$$

where x^i are the nodes of neural oscillator for $i = 1,2$ and $\varphi(\cdot) = \tanh(\cdot)$. The richness of its chaotic formulation has been shown as a gallery of chaotic attractors in Fig. 1 by considering various time varying delays for the neural oscillator (10). For GVPS, we consider the identical neural oscillator as a slave dynamics with the same parameter defined for (10)

$$\dot{y}(t) = A_1 y(t) + A_2 y_{\tau(t)} + f(y, y_{\tau(t)}) + u(x, y). \quad (11)$$

By making use of the control (3), the error dynamics (4) is formed. Now, we solve the matrix inequality (7) presented in the Preposition 1 using the MATLAB LMI toolbox to show the asymptotic stability of the error dynamical system. Taking $\delta = 0.5$, we get the following solutions: $P = 10^8 \begin{bmatrix} 3.1850 & -0.0000 \\ -0.0000 & 0.0000 \end{bmatrix}$, $M = 10^8 \begin{bmatrix} 4.7007 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}$, and the control gain $K = 10^4 \begin{bmatrix} 0.0001 & 0.0000 \\ -2.4187 & 0.0001 \end{bmatrix}$. The upper bound of the time varying delay $\tau(t)$ is $\tau_M = 2.7872$. Also we have presented the upper bounds τ_M corresponding to each δ, which is depicted in Fig. 2(a). It can be seen that the upper bound increases as the rate of $\sigma(t)$ increases to unity. Nevertheless, $\dot{\sigma}(t)$ cannot be unity and thus $\delta < 1$.

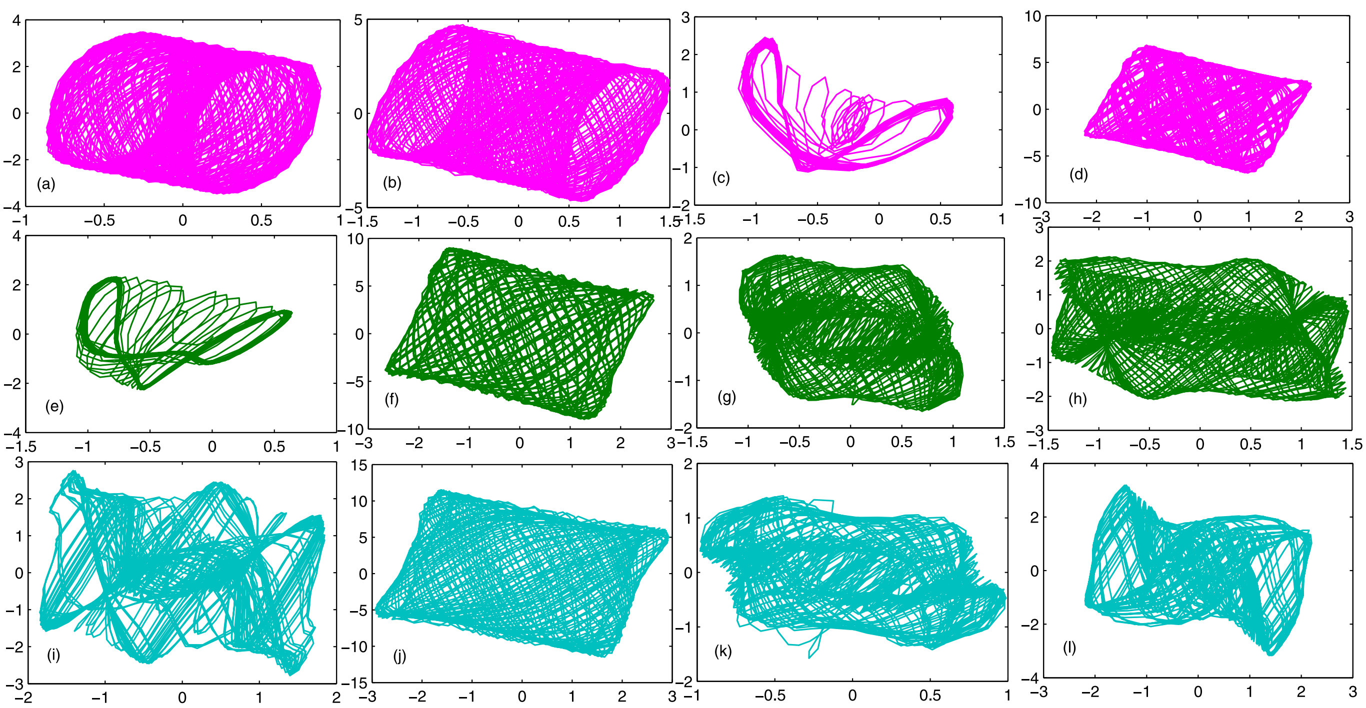

FIG. 1. Gallery of chaotic attractors formed by the neural oscillator (10) by the Phase-space portrait with $x^1(t)$ vs $x^2(t)$. The chaotic attractors are formed due to different types of delays especially time varying delays. (a) $\tau(t) = 1$; (b) $\tau(t) = 1.2$; (c) $\tau(t) = 1.5|\sin t|$; (d) $\tau(t) = 1.6$; (e) $\tau(t) = 1.9|\sin t|$; (f) $\tau(t) = 2$ (g); $\tau(t) = 2.1|\sin t|$; (h) $\tau(t) = 2.2|\sin t|$; (i) $\tau(t) = 2.3|\sin t|$; (j). $\tau(t) = 2.4|\sin t|$; (k) $\tau(t) = 2.05|\sin t|$; (l). $\tau(t) = 2.5|\sin t|$.

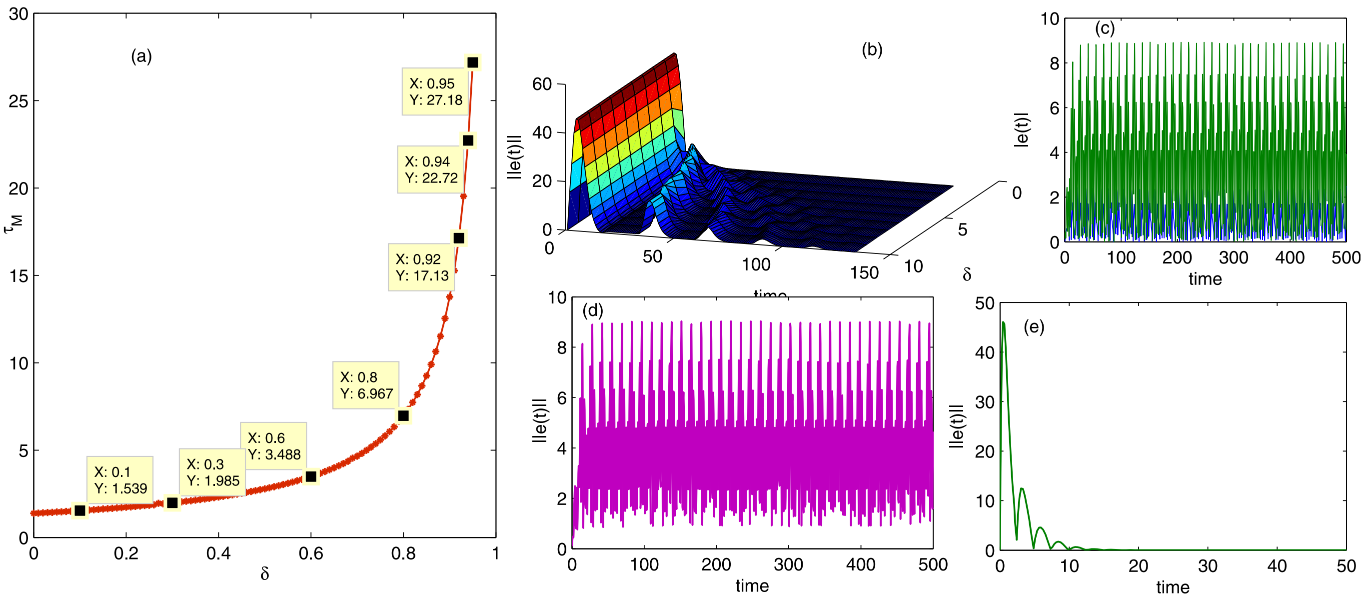

FIG. 2. Numerical simulations of GVPS for the neural oscillator (10). (a) represents the maximum upper bound in y scale for different values of δ in x scale. It is seen that an increase in the rate of $\sigma(t)$ increases the upper bound τ_M. (b) represents the norm of error trajectories taken as a function of δ after control applied to achieve the GVPS. (c) and (d) represent the absolute of the error states and norm of error states before applying the nonlinear control. The error trajectories approaches zero as the time progresses, which proves the existence of GVPS which can be seen from (e).

Fig. 2(b) shows the error trajectories of closed-loop error dynamics to prove the existence of GVPS between both neural oscillators and the error is given as a function of δ ranging from 0 to 10. Here the variable projective scaling factor is $\alpha = t^2$, and the time varying delay $\tau(t) = 1 + 0.1|\sin t|$ and $\sigma(t) = t/2$. It is observed that the synchronization manifold $y = t^2 x$ is achieved and can be seen from Fig. 2(e) in which the norm of error state trajectories is presented. We see that the obtained conditions effectively are applied to achieve GVPS between identical neural oscillators.

V. CONCLUSION

We have introduced a generalized projective synchronization criterion for a coupled time delayed systems of n first order equations, which can contain modulated non-differentiable time delays. A synchronization condition is derived by the Krasovskii-Lyapunov theory and the nonlinear controller is designed by an observer like control technique. The scheme is also valid for projective lag and projective anticipatory synchronization. We have proposed a new delayed neural oscillator with rich chaotic dynamics and the proposed scheme is verified numerically with the neural oscillator. The scheme of GVPS can be used in signal and digital communication with a more generalized synchronization manifold and which has faster communication than projective synchronization.

APPENDIX: PROOF OF PREPOSITION 1

Consider the Krasovskii-Lyapunov functional given in Eq. (6). Taking the time derivative of Eq. (6) along the trajectories of Eq. (4) yields

$$\begin{aligned}\dot V(t, e_t, \dot e_t) &= 2e^T(t)P\dot e(t) + e^T(t)S_1 e(t) + \tau_M^2 \dot e^T(t) S_2 \dot e(t)\\ &\quad - e^T(t-\tau_M)S_1 e(t-\tau_M) + e^T(t)S_3 e(t)\\ &\quad - \tau_M \int_{t-\tau_M}^{t} \dot e^T(\theta)\, S_2\, \dot e(\theta) d\theta\\ &\quad - (1-\delta)e^T(t-\sigma(t))S_3 e(t-\sigma(t)). \end{aligned} \tag{A1}$$

In view of Lemma 1, the integral term in Eq. (A1) can be written as

$$-\tau_M \int_{t-\tau_M}^{t} \dot e^T(\theta) S_2 \dot e(\theta) d\theta \le \tilde\zeta^T(t)\Omega\tilde\zeta(t), \tag{A2}$$

where $\tilde\zeta(t) = \begin{bmatrix} e \\ e_{\tau(t)} \\ e_{\tau_M} \end{bmatrix}$, and $\Omega = \begin{bmatrix} -S_2 & S_2 & 0 \\ \star & -2S_2 & S_2 \\ \star & \star & -S_2 \end{bmatrix}$. Consider the following equation for appropriately dimensioned matrix M:

$$\begin{aligned} 0 &= [2e^T M + 2\dot e^T M]\\ &\quad \times [-\dot e + (1-\delta)A_1 e + (1-\delta)A_2 e_{\tau(t)} - Ke]. \end{aligned} \tag{A3}$$

Thus, the Eq. (A1) can be written as

$$\begin{aligned}\dot V(e_t) &\le 2e^T(t)P\dot e(t) + e^T(t)S_1 e(t) - e^T(t-\tau_M)S_1 e(t-\tau_M)\\ &\quad + \tau_M^2 \dot e^T(t)S_2 \dot e(t) + \tilde\zeta^T(t)\Omega\tilde\zeta(t) + e^T(t)S_3 e(t)\\ &\quad - (1-\delta)e^T(t-\sigma(t))S_3 e(t-\sigma(t)) - 2e^T M\dot e\\ &\quad + 2e^T M(1-\delta)A_1 e + 2(1-\delta)e^T M A_2 e_{\tau(t)}\\ &\quad - 2e^T MKe - 2\dot e^T M\dot e + 2(1-\delta)\dot e^T M A_1 e\\ &\quad + 2(1-\delta)\dot e^T M A_2 e_{\tau(t)} - 2\dot e^T MKe. \end{aligned}$$

Letting $L = MK$ for $j = 1,2,3$, and $\zeta(t) = [\tilde\zeta(t) \quad e_{\sigma(t)} \quad \dot e(t)]$, we have

$$\dot{V}(t,\, e_t,\, \dot{e}_t) \le \zeta^T(t)\,\Xi\,\zeta(t) < 0, \tag{A4}$$

for all $\zeta(t) \neq 0$ with $\Xi < 0$. Thus, there exists a positive constant $\varepsilon > 0$ such that $\dot{V}(t, e_t, \dot{e}_t) \le -\varepsilon||e(t)||^2$. This completes the proof.

Proof of Preposition 2: Consider the Lyapunov functional

$$\begin{aligned} V(t, e_t, \dot{e}_t) = e^T(t)Pe(t) + \int_{-\tau_M}^{0} e^T(t+\xi)\,S_1\,e(t+\xi)d\xi \\ + \tau_M \int_{-\tau_M}^{0}\int_{t+\theta}^{t} \dot{e}^T(\xi)\,S_2\,\dot{e}(\xi)d\xi d\theta \\ + \int_{t-\sigma(t)}^{t} e^T(\xi)\,S_3\,e(\xi)d\xi \end{aligned} \tag{A5}$$

Following the similar procedure presented in the proof of Preposition 1, the LMI condition (8) can be obtained.

Proof of Preposition 3: Consider the Lyapunov functional

$$\begin{aligned} V(t, e_t, \dot{e}_t) = e^T(t)Pe(t) + \int_{-\tau_M}^{0} e^T(t+\xi)\,S_1\,e(t+\xi)d\xi \\ + \tau_M \int_{-\tau_M}^{0}\int_{t+\theta}^{t} \dot{e}^T(\xi)\,S_2\,\dot{e}(\xi)d\xi d\theta, \end{aligned} \tag{A6}$$

$$+\int_{t-\sigma}^{t} e^T(\xi)\,S_3\,e(\xi)d\xi. \tag{A7}$$

Following the similar procedure presented in the proof of Preposition 1, the LMI condition (8) can be obtained.

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