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Exponentially asymptotic synchronization of uncertain complex time-delay dynamical networks

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Abstract. This paper investigates the problem of exponentially asymptotic synchronization of complex time-delay dynamical networks with multi-links and structure uncertainty. The structure uncertainty belongs to the uncertain coupling strength and unknown topologies structures, which appear typically in networks environment. In order to synchronize complex networks with structure uncertainty, the adaptive controller is designed, and some general synchronization criteria of the controllers are proposed and proved based on the Lyapunov stability theory and the Lipschitz hypothesis. Finally, numerical simulations of dynamical networks with different topological structures are presented to demonstrate the feasibility and the effectiveness of the results.

1 Introduction

Generally speaking, a complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. With the small world (SW) property and the scale-free (SF) property being found, recently, the study of various complex networks has attracted increasing attention from researchers in various fields such as physics, mathematics, engineering, biology and sociology [1–5]. Synchronization in networks is a very common phenomenon in real systems. Investigation of synchronization in networks has drawn great attention recently [6–15]. Synchronization in specific kinds of complex networks such as small-world networks, random networks and scale-free networks has been achieved [16–25].

Most previous studies assume that interaction strengths or link weights can be exactly measured. Consequently, the synchronization problem is significantly simplified, naturally translating into a spectral graph theoretic problem, e.g. the spectrum of the Laplacian matrix of the network [16,20]. However, the assumption that the link weights can be exactly measured is not realistic in many real-world networks, because the measurement error and the uncertainties cannot be avoided in real systems. The study of the synchronization problem with structure uncertainty becomes an interesting and challenging topic.

Multi-links means that there are more than one connection between two nodes and each of them has its own property. For instance, there are relationship networks, transportation networks, world wide web, etc. [1–4]. Figure 1 shows a transportation network as an example of a network with multi-links, which is made up by combining the corresponding airline network, railway network and highway network [26]. When dealing with these complex networks with multi-links, we can split them into sub-networks. There are many different ways to split a complex network with multi-links. The most common principle of splitting them is based on the property of the connections. For most of the networks, the phenomenon that there are different transmission speeds between connections is widespread. For a transportation network, the transmission speed is different among airline network, railway network and highway network. In most situations, time-delay is an important aspect since time-delays of different connections are often not all identical [27–29]. Therefore, time-delay is a proper parameter that could be used to split the networks.

In our previous work [26], time-delay was introduced to split complex dynamical networks into sub-networks, upon which a model of complex dynamical networks with multi-links has been constructed. Asymptotic synchronization results of complex networks were also given in reference [26]. The speed of convergence towards synchrony provides a fundamental collective time scale for synchronizing networks [30]. For many real network systems, it

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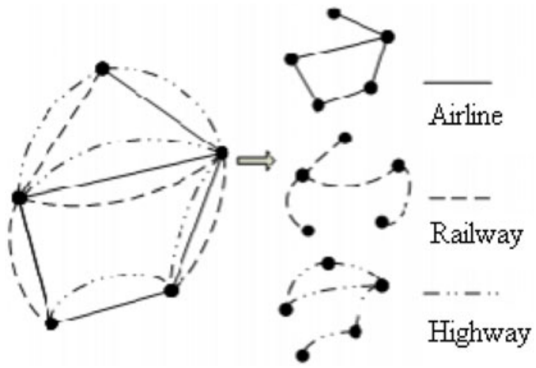

Fig. 1. Transportation network and its division. According to different transmission speeds, the transportation network can be split into airline network, railway network and highway network.

equally matters how fast the units synchronize or whether the network interactions fail to coordinate the unit's dynamics on time scales [30]. However, the important issue of synchronization speed for complex dynamical networks with multi-links has so far received little attention. Moreover, we often know very little information on the network structure, which makes network synchronization design very difficult. To overcome these difficulties, this paper will further investigate the adaptive feedback synchronization of complex dynamical networks with multi-links. In particular, we obtain several novel criteria for globally exponentially asymptotic synchronization of uncertain complex time-delay dynamical networks with multi-links. The results are proved to be able to guarantee different exponential convergence rates for the controlled states as well as the uncontrolled states of the error systems. The proposed method is efficient to control the convergence rates of synchronization and easy to be verified in engineering applications.

2 Model of complex time-delay dynamical network with multi-links

We consider a complex dynamical network model consisting of N identical nodes with linear couplings [31,32], which is characterized by

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} a_{ij} H x_j, \quad i = 1, 2, \ldots, N, \qquad (1)$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T$ is the state vector of the ith node, f is a smooth nonlinear vector field, H is the inner-coupling matrix and $A = (a_{ij})_{N\times N}$ represents the coupling strength and the topological structure of the network. If there is a connection from node i to node j $(j \neq i)$, then the coupling weight $a_{ij} \neq 0$; otherwise, $a_{ij} = 0$.

Note that the inner coupling matrix $H \in R^{n\times n}$ is a constant matrix linking the coupled variables. In general, H is selected as an identity matrix, and therefore it is omitted in our following models.

Networks with multi-links are very common in the real world. For a transportation network, the transmission speed is different among airline network, railway network and highway network which could be considered as that the connections of transportation network may have different time-delays. Recently in reference [26], time-delays were introduced to split the complex dynamical network into sub-networks in order to describe the time-delay property of the networks with multi-links, upon which a model of complex time-delay dynamical networks with multi-links has been generalized from the network model (1), which is described as follows:

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} a_{ij}^0 x_j(t) + \sum_{j=1}^{N} a_{ij}^1 x_j(t-\tau_1) + \ldots + \sum_{j=1}^{N} a_{ij}^{m-1} x_j(t-\tau_{m-1}), \quad i = 1, 2, \ldots N, \qquad (2)$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in R^n$ is the state vector of the ith node, $f : R^n \to R^n$ is a smooth nonlinear function, $A_l = (a_{ij}^l)_{N\times N} \in R^{N\times N}(l = 0, 1, \ldots, m-1)$ is the coupling strength and the topological structure of the lth sub-network, $\tau_l(l = 0, 1, \ldots, m-1)$ is time-delay of the lth sub-network compared to the zero sub-network $(\tau_0 = 0)$ which is without time-delay. The definition of a_{ij}^l is that in the lth sub-network, if there is a connection between node i and node j $(j \neq i)$, then $a_{ij}^l \neq 0$; otherwise, $a_{ij}^l = 0$, and we define $a_{ii}^l = -\sum_{j=1, j\neq i}^{N} a_{ij}^l$.

In dynamical network (2), some sub-networks may have small world characteristics, others random characteristics, etc. Taking into account different characteristics of each sub-network, the complex time-delay dynamical networks with multi-links may present some interesting dynamical phenomena.

3 Synchronization of uncertain time-delay dynamical networks with multi-links

In this paper, we consider the synchronization problem of network (2) with structure uncertainty. The structure uncertainty means that $A_l(l = 0, 1, \ldots, m-1)$ are unknown or uncertain coupling configuration matrix. Note that, for most random networks, the degree distribution of the networks can be measured according to some statistics, although exactly known each link is improbable. So the coupling configuration matrices $A_l(l = 0, 1, \ldots, m-1)$ can be random matrices with certain distribution.

Assumption 1: there always exist positive constants $k_i > 0$ satisfying the following Lipschitz condition:

$$||f(y) - f(x)|| \leq k_i ||y - x||, \qquad (3)$$

where x and y are the time-varying vectors.

Many dynamical systems meet the Lipschitz condition of assumption 1, especially chaotic systems, such as Chen's system, the Lorenz system, Chua's circuit and the Rössler system (cf. [32,33]).

We take the dynamical network given by equation (2) as the driving network, and the response network with a control scheme which is given by:

$$\dot{y}_i = f(y_i) + \sum_{j=1}^{N} \tilde{a}_{ij}^0 y_j(t) + \sum_{j=1}^{N} \tilde{a}_{ij}^1 y_j(t-\tau_1) + \dots + \sum_{j=1}^{N} \tilde{a}_{ij}^{m-1} y_j(t-\tau_{m-1}) + u_i, \quad (4)$$

where $i, j = 1, 2, \dots, N$, $y_i = (y_{i1}, y_{i2}, \dots, y_{in})^T \in R^n$ is the state vector of the ith node, $\tilde{a}_{ij}^l \in R^n$ is the estimation of a_{ij}^l, $l = 0, 1, 2, \dots, m-1$, f and $\tau_l (l = 1, \dots, m-1)$ have the same meaning as those in equation (2), and u_i is the controller for node i to be designed.

Definition. Networks (2) and (4) are globally exponentially asymptotically synchronous if there exist constants $M_i > 0$ and $\alpha > 0$, such that for any initial condition, we have:

$$||e_i(t)|| \le M_i \exp(-\alpha t), \quad (5)$$

where $i = 1, 2, \dots, N$ and $e_i(t) = y_i - x_i$ is the synchronization error.

The method of adaptive control based on Lyapunov stability theory has been considered widely and proved to be effective to solve problems about synchronization [34–37]. Now, we discuss the synchronization of dynamical networks (2) and (4) based on Lyapunov stability theory. The aim of this paper is to design a proper adaptive controller u_i and the principle of parameter estimation so that $||x_i - y_i|| \to 0$ as $t \to \infty$.

Theorem 1: let the controller $u = \varepsilon(y(t) - x(t))$, and the feedback strength $\varepsilon = -L$. The adaptive laws of $\tilde{a}_{ij}^l (l = 0, 1, \dots, m-1)$ are chosen as follows:

$$\begin{cases} \dot{\tilde{a}}_{ij}^0 = -\alpha_{ij}^0 e_i^T(t) y_j(t) \exp(\mu t), \\ \dot{\tilde{a}}_{ij}^1 = -\alpha_{ij}^1 e_i^T(t) y_j(t-\tau_1) \exp(\mu t), \\ \quad \vdots \\ \dot{\tilde{a}}_{ij}^{m-1} = -\alpha_{ij}^{m-1} e_i^T(t) y_j(t-\tau_{m-1}) \exp(\mu t), \end{cases} \quad (6)$$

where $i, j = 1, 2, \dots, N$, $\mu \ge 0$ is a sufficiently small positive constant and α_{ij}^l is the arbitrary constant for $l = 0, 1, \dots, m-1$. If assumption 1 holds, and if there exists a positive number L such that

$$p_i \Bigg\{ (k_i + \frac{1}{2}\mu - L + (a_{ii}^0)^+) + \sum_{j=1, i\ne j}^{n} |a_{ij}^0| + \frac{1}{2}(\exp(\mu\tau_1) + 1) \sum_{j=1}^{n} |a_{ij}^1| + \dots + \frac{1}{2}(\exp(\mu\tau_{m-1}) + 1) \sum_{j=1}^{n} |a_{ij}^{m-1}| \Bigg\} < 0, \quad (7)$$

where $(a_{ii}^0)^+ = \max\{a_{ii}^0, 0\}$, $p_i (i = 1, 2, \dots, N)$ is the positive constant and $|\dots|$ means the sign of the absolute value. Then the exponential synchronization is achieved between the driving network (2) and the response network (4) with

$$||e_i(t)|| \le M_i \exp(-0.5\mu t), \quad (8)$$

where $M_i > 0$. For the proof of theorem 1, please see Appendix for details.

Furthermore, if the state vector $x_i \in R$ and the smooth nonlinear vector $f : R \to R$, then we have the following results.

Theorem 2: let the controller $u = \varepsilon(y(t) - x(t))$ and the feedback strength $\varepsilon = -L$. The adaptive laws of $\tilde{a}_{ij}^l (l = 0, 1, \dots, m-1)$ are chosen as follows:

$$\begin{cases} \dot{\tilde{a}}_{ij}^0 = -\alpha_{ij}^0 \mathrm{Sgn}(e_i(t)) y_j(t) \exp(\mu t), \\ \dot{\tilde{a}}_{ij}^1 = -\alpha_{ij}^1 \mathrm{Sgn}(e_i(t)) y_j(t-\tau_1) \exp(\mu t), \\ \quad \vdots \\ \dot{\tilde{a}}_{ij}^{m-1} = -\alpha_{ij}^{m-1} \mathrm{Sgn}(e_i(t)) y_j(t-\tau_{m-1}) \exp(\mu t), \end{cases} \quad (9)$$

where $i, j = 1, 2, \dots, N$, and $\mu \ge 0$ is a sufficiently small positive constant, and α_{ij}^l is the arbitrary constant for $l = 0, 1, \dots, m-1$, $\mathrm{Sgn}(x)$ is the sign function which is defined as follows:

$$\mathrm{Sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases} \quad (10)$$

If assumption 1 holds, and if there exists a positive number L such that

$$p_i \Bigg\{ [k_i + \mu - L + (a_{ii}^0)^+] + \sum_{i=1, j\ne i}^{N} |a_{ij}^0| + \dots + (\exp(\mu\tau_1) + 1) \sum_{j=1}^{N} |a_{ij}^1| + \dots + (\exp(\mu\tau_{m-1}) + 1) \sum_{j=1}^{N} |a_{ij}^{m-1}| \Bigg\} < 0, \quad (11)$$

where $(a_{ii}^0)^+ = \max\{a_{ii}^0, 0\}$ and $p_i (i = 1, 2, \dots, N)$ is the positive constant. Then the synchronization is achieved between the driving network (2) and the response network (4). Moreover,

$$|e_i(t)| \le M_i \exp(-\mu t), \quad (12)$$

where $M_i > 0$. For the proof of theorem 2, please see Appendix for details.

Through theorems 1 and 2, we succeed to achieve globally exponentially asymptotically synchronization of uncertain complex time-delay dynamical networks with multi-links.

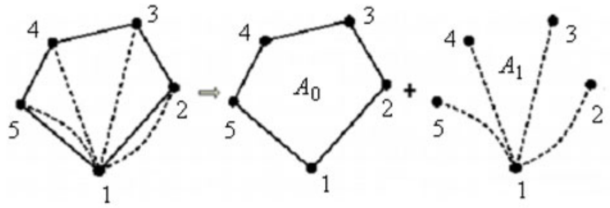

Fig. 2. Topological structure of the network for example 1.

4 Numerical simulations

In this section, we present several numerical simulation examples to illustrate the effectiveness of the proposed methods.

Example 1: our first example is to consider a five-node network which is shown in Figure 2. The network is composed of two different sub-networks. Sub-network A_0 is the one without time-delay, while the other sub-network A_1 has the time-delay $\tau_1 = 0.05$. So we get the following weight configuration matrices:

$$A_0 = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Generally, chaotic systems are more difficult to synchronize than non-chaotic ones. Many different chaotic systems have been investigated in the past decades [38–40]. Here, the famous Lorenz system is taken at the nodes of the networks, and node i is described by:

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} + \begin{bmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{bmatrix}. \tag{13}$$

According to equation (2), we have the following driving network

$$\dot{x_i} = f(x_i) + \sum_{j=1}^{5} a_{ij}^0 x_j(t) + \sum_{j=1}^{5} a_{ij}^1 x_j(t-\tau_1), \tag{14}$$

where $x_i = (x_{i1}, x_{i2}, x_{i3})^T$, $i = 1, 2, \ldots, 5$.

And the response network is:

$$\dot{y_i} = f(x_i) + \sum_{j=1}^{5} \tilde{a}_{ij}^0 y_j(t) + \sum_{j=1}^{5} \tilde{a}_{ij}^1 y_j(t-\tau_1), \tag{15}$$

where $y_i(t) = (y_{i1}(t), y_{i2}(t), y_{i3}(t))^T$, $i = 1, 2, \ldots, 5$.

According to theorem 1, we can construct the following feedback parameter adaptive laws:

$$\begin{cases} -\varepsilon = L = 2501, \\ \dot{\tilde{a}}_{ij}^0 = -\alpha_{ij}^0 e_i(t)^T y_j(t) \exp(0.0035t), \\ \dot{\tilde{a}}_{ij}^1 = -\alpha_{ij}^1 e_i(t)^T y_j(t-\tau_1) \exp(0.0035t), \end{cases} \tag{16}$$

where $i, j = 1, 2, \ldots, 5$.

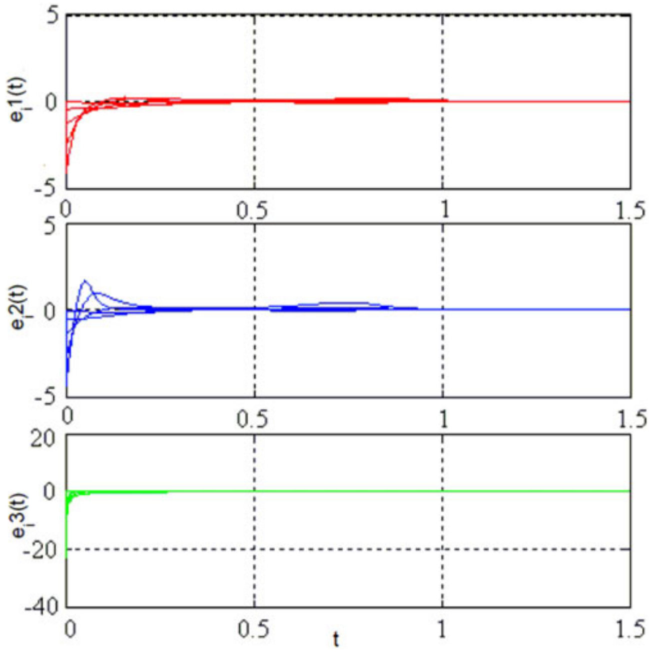

Fig. 3. (Color online) Synchronization errors between the driving and the response networks for example 1, where the time-delay $\tau_1 = 0.05$. The first panel is error $e_i 1(t)$, the second panel is error $e_i 2(t)$, and the last panel is $e_i 3(t)$.

It is clear that equation (7) is also satisfied with $p_i = 1, k_i = 71\ (i = 1, 2 \ldots, 5)$ and $\mu = \frac{1}{2}$ when $L > 2500$. From theorem 1, we can conclude that the response network is globally synchronous with the driving network. The synchronization errors from the simulations are shown in Figure 3.

Example 2: the second example is to consider a network with six different sub-networks which have different time-delays. The topological structure of the network is shown in Figure 4. Assuming the network is composed of four nodes, we can easily get the weight configuration matrixes A_0, A_1, A_2, A_3, A_4, A_5, in which matrix A_0 has no time-delay, and matrix A_5 has the maximal time-delay. The weighted configuration matrices are described by:

$$A_0 = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We suppose that the dynamics at the node of the network is a linear system whose states are expressed as $\dot{x}_i = -2x_i$.

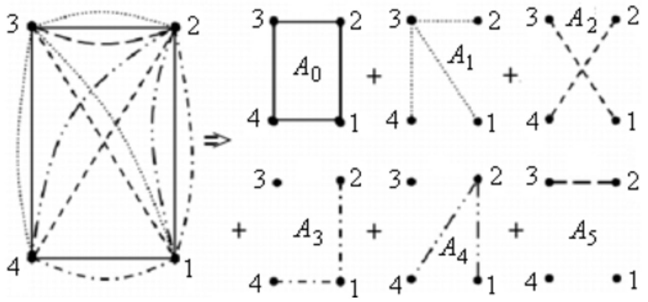

Fig. 4. Topological structure of the six-feature-edge network. According to different time-delays, the original network can be split into six sub-networks.

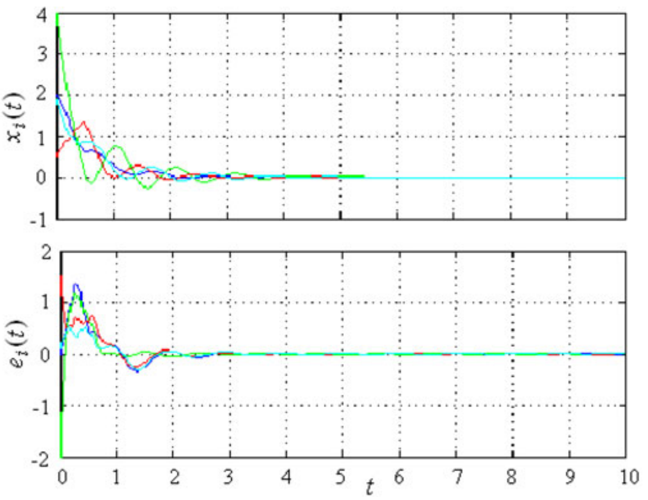

Fig. 5. (Color online) States of driving network and the synchronization errors between the driving and the response networks for example 2, where $\tau_1 = 0.05$, $\tau_2 = 0.1$, $\tau_3 = 0.15$, $\tau_4 = 0.2$ and $\tau_5 = 0.25$, respectively.

According to equation (2), we have the following driving network:

$$\dot{x}_i = -2x_i + \sum_{j=1}^{4} a_{ij}^0 x_j(t) + \sum_{j=1}^{4} a_{ij}^1 x_j(t-\tau_1) + \dots$$
$$+ \sum_{j=1}^{4} a_{ij}^5 x_j(t-\tau_5), \quad i = 1,\, 2,\, 3,\, 4; \tag{17}$$

where $\tau_1 = 0.05, \tau_2 = 0.1, \tau_3 = 0.15, \tau_4 = 0.2$ and $\tau_5 = 0.25$, respectively.

Similarly as in example 1, we construct the corresponding response network and the feedback parameters adaptive control laws. Equation (7) is also satisfied with $p_i = 1$, $k_i = 6\,(i = 1, 2, \dots, 4)$ and $\mu = \frac{1}{2}$ when $L > 12$. From theorem 1, we can conclude that the response network is globally synchronous with the driving network. The states of the driving network $x_i(t)$ and the synchronization errors $e_i(t)$ between the driving network and the response network are shown in Figure 5.

Example 3: our third example is to consider the network consisting of 50 nodes and describe the network using Lü systems. The network is composed of two different

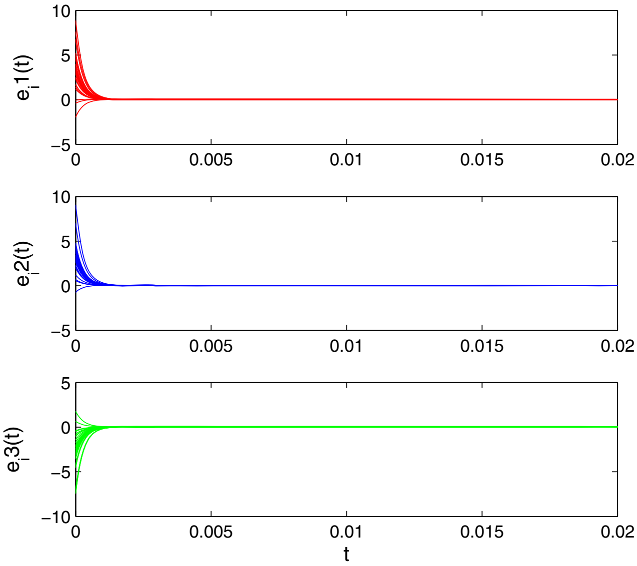

Fig. 6. (Color online) Separate synchronous error variables $e_i1(t)$, $e_i2(t)$, $e_i3(t)$ $(1 \le i \le 50)$ of network with links owning 2 properties. Both A_0 and A_1 are famous E-R random network model, the connection probability among nodes is 0.3, and $\tau = 0.01$.

sub-networks which can be described as:

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{50} a_{ij}^0 x_j(t) + \sum_{j=1}^{50} a_{ij}^1 x_j(t-\tau_1), \tag{18}$$

where $\tau_1 = 0.05$, and the node dynamical systems is: $\dot{x}_i = (-36x_{i1} + 36x_{i2};\ 20x_{i2} - x_{i1}x_{i3};\ -3x_{i3} + x_{i1}x_{i2})$.

Assume that the structures of these two sub-networks of (25) obey the random network model, i.e., the weight configuration matrices A_0 and A_1 are random matrixes. Similarly as in example 1 and example 2, we construct the corresponding response network and the feedback parameters adaptive control laws, and select the parameters as $p_i = 100$, $k_i = 65\,(i = 1, 2, \dots, 50)$, $\mu = 2$. When $L > 2400$, equation (7) is satisfied. From theorem 1, we can conclude that the response network is globally synchronous with the driving network. Figure 6 plots synchronous errors with connection probability 0.5, and where the famous Erdös-Rényi random network model is considered. Figure 7 plots synchronous errors with different connection probability 0.1, 0.4, 0.6 and 0.9, indicating that the proposed method is effective in suppressing the influence of the structure uncertainty. To be more persuadable, sub-networks with scale-free structure and small-world structure are also considered. Figure 8 plots the synchronous errors with different sub-network models, such as the E-R random network model, the B-A scale-free network model and the small-world network model. From Figure 6 to Figure 8, we attain that our theorem is feasible in different uncertain network structure models.

From Figures 3 and 5–8, we can see that parameters p_i and μ influence the synchronization speed, i.e., the larger these two parameters are, the faster the synchronization

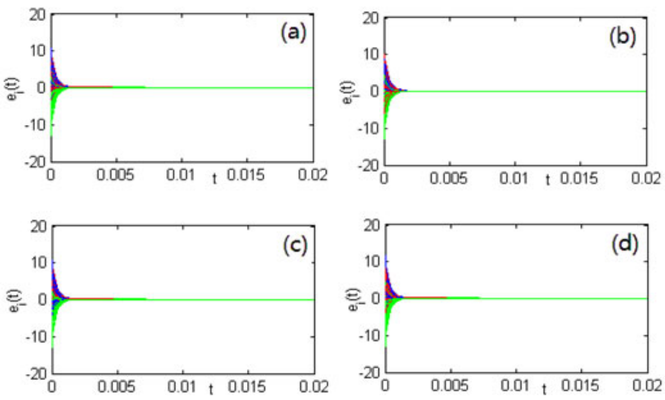

Fig. 7. (Color online) Synchronization errors $e_i(t) = (e_i1(t), e_i2(t), e_i3(t))$ with different connection probability q_i among nodes, where (a) $q_i = 0.1$, (b) $q_i = 0.4$, (c) $q_i = 0.6$, (d) $q_i = 1$. Both A_0 and A_1 are famous E-R random models, and these two models have the same connection probability.

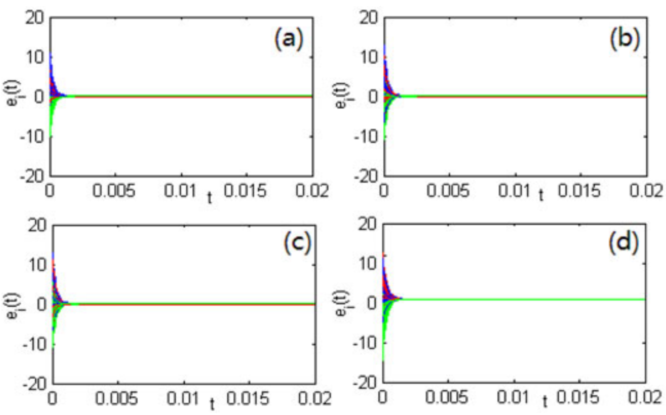

Fig. 8. (Color online) Synchronous errors $e_i1(t), e_i2(t), e_i3(t)$ of different network models where $\tau = 0.01$. (a) A_0: B-A scale-free model, A_1: B-A scale-free model, (b) A_0: B-A scale-free model, A_1: random network model and where $l = 5$, (c) A_0: random network model, A_1: small-world model, (d) A_0: small-world model, A_1: B-A scale-free model. For random networks, the connection probability among nodes is 0.3. For scale-free network, the initial graph is complete with three nodes, and two edges are added in the network when a new node is introduced. For small-world networks, each node in initial regular ring lattice has $K = 8$ neighbors, and the rewiring probability of each edge is 0.2.

speed is. Furthermore, we can see that whether A is certain or random, since we use the adaptive control, the synchronization can be obtained well. In addition, from Figures 7 and 8, we can see that different random subnetwork structures have little influence on the synchronization performance, i.e., the network structures with different probability distributions have very small influences on the synchronization performance which shows the advantage of the proposed parameter adaptive law.

5 Conclusion

In this paper, exponentially asymptotic synchronization between two complex time-delay multi-links dynamical networks with uncertain coupling configuration has been studied both theoretically and numerically. In order to synchronize complex networks with unknown topological structures, the adaptive controller is given based on the Lyapunov stability theory. Furthermore, sufficient conditions for synchronization between the driving and the response networks are obtained. Finally, several numerical simulations demonstrate the effectiveness of the proposed results. Moreover, there are some further significant directions to be investigated such as the pinning control of uncertain complex dynamical network with multi-links and the topological structures identification of complex networks with multi-links based on the adaptive method.

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Appendix

Proof of theorem 1: we can get the following error dynamical system:

$$\begin{aligned}\dot{e}_i(t) &= f(y_i) - f(x_i) + \sum_{j=1}^{N} a_{ij}^{0} e_j(t) \\ &\quad + \sum_{j=1}^{N} a_{ij}^{1} e_j(t-\tau_1) + \ldots \\ &\quad + \sum_{j=1}^{N} a_{ij}^{m-1} e_j(t-\tau_{m-1}) + \sum_{j=1}^{N} (\tilde{a}_{ij}^{0} - a_{ij}^{0}) y_j(t) \\ &\quad + \sum_{j=1}^{N} (\tilde{a}_{ij}^{1} - a_{ij}^{1}) y_j(t-\tau_1) + \ldots \\ &\quad + \sum_{j=1}^{N} (\tilde{a}_{ij}^{m-1} - a_{ij}^{m-1}) y_j(t-\tau_{m-1}) - Le_i(t),\end{aligned} \quad \text{(A.1)}$$

where $e_i(t) = (e_{i1}(t), \ldots, e_{in}(t))^T$ and we define $g_i(e_i(t)) = f(y_i) - f(x_i)$ $(i = 1, 2, \ldots, N)$. According to equation (3), we obtain

$$||g_i(e_i(t))|| \leq k_i ||e_i(t)||, \quad i = 1, 2, \ldots, N. \quad \text{(A.2)}$$

Now, we construct the Lyapunov candidate as:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} p_i \Bigg\{ e_i^T(t)e_i(t)\exp(\mu t) + \sum_{j=1}^{N} \frac{1}{\alpha_{ij}^0}\left(\tilde{a}_{ij}^0 - a_{ij}^0\right)^2 + \sum_{j=1}^{N} \frac{1}{\alpha_{ij}^1}\left(\tilde{a}_{ij}^1 - a_{ij}^1\right)^2 + \ldots + \sum_{j=1}^{N} \frac{1}{\alpha_{ij}^{m-1}}\left(\tilde{a}_{ij}^{m-1} - a_{ij}^{m-1}\right)^2 + \sum_{j=1}^{N} |a_{ij}^1| \int_{t-\tau_1}^{t} e_j^T(s)e_j(s)\exp[\mu(s+\tau_1)]ds + \ldots + \sum_{j=1}^{N} |a_{ij}^{m-1}| \int_{t-\tau_{m-1}}^{t} e_j^T(s)e_j(s) \times \exp\left[\mu(s+\tau_{m-1})\right] ds \Bigg\}, \tag{A.3}$$

where $p_i (i = 1, 2, \ldots, N)$ is the positive constant. Then we have:

$$\dot{V}(t) = \sum_{i=1}^{N} p_i \Bigg\{ e_i^T(t)\exp(\mu t)\Bigg[g_i(e_i(t)) + \sum_{j=1}^{N} a_{ij}^0 e_j(t) + \sum_{j=1}^{N} a_{ij}^1 e_j(t-\tau_1) + \ldots + \sum_{j=1}^{N} a_{ij}^{m-1} e_j(t-\tau_{m-1}) + \sum_{j=1}^{N} \left(\tilde{a}_{ij}^0 - a_{ij}^0\right) y_j(t) + \sum_{j=1}^{N} (\tilde{a}_{ij}^1 - a_{ij}^1) y_j(t-\tau_1) + \ldots + \sum_{j=1}^{N} \left(\tilde{a}_{ij}^{m-1} - a_{ij}^{m-1}\right) y_j(t-\tau_{m-1}) - Le_i(t)\Bigg] + \frac{\mu}{2} e_i^T(t)e_i(t)\exp(\mu t) + \sum_{j=1}^{N} \frac{1}{a_{ij}^0}\left(\tilde{a}_{ij}^0 - a_{ij}^0\right)\dot{\tilde{a}}_{ij}^0 + \sum_{j=1}^{N} \frac{1}{a_{ij}^1}\left(\tilde{a}_{ij}^1 - a_{ij}^1\right)\dot{\tilde{a}}_{ij}^1 + \ldots + \sum_{j=1}^{N} \frac{1}{a_{ij}^{m-1}}\left(\tilde{a}_{ij}^{m-1} - a_{ij}^{m-1}\right)\dot{\tilde{a}}_{ij}^{m-1} + \frac{1}{2}\sum_{j=1}^{N} |a_{ij}^1| (e_j^T(t)e_j(t)\exp(\mu\tau_1) - e_j^T(t-\tau_1)e_j(t-\tau_1))\exp(\mu t) + \ldots + \frac{1}{2}\sum_{j=1}^{N} |a_{ij}^{m-1}| (e_j^T(t)e_j(t)\exp(\mu\tau_{m-1}) - e_j^T(t-\tau_{m-1})e_j(t-\tau_{m-1}))\exp(\mu t) \Bigg\}. \tag{A.4}$$

According to equation (22), we get

$$\dot{V}(t) \le \sum_{i=1}^{N} p_i \exp(\mu t) \Bigg\{ \left(k_i + \frac{\mu}{2} - L\right) e_i^T(t)e_i(t) + \sum_{j=1}^{N} a_{ij}^0 e_i^T(t)e_j(t) + \sum_{j=1}^{N} a_{ij}^1 e_i^T(t)e_j(t-\tau_1) + \ldots + \sum_{j=1}^{N} a_{ij}^{m-1} e_i^T(t)e_j(t-\tau_{m-1}) + \frac{1}{2}\sum_{j=1}^{N} |a_{ij}^1| (e_j^T(t)e_j(t)\exp(\mu\tau_1) - e_j^T(t-\tau_1)e_j(t-\tau_1)) + \ldots + \frac{1}{2}\sum_{j=1}^{N} |a_{ij}^{m-1}| \left(e_j^T(t)e_j(t)\exp(\mu\tau_{m-1}) - e_j^T(t-\tau_{m-1})e_j(t-\tau_{m-1})\right) \Bigg\}. \tag{A.5}$$

Let $z^T = e_i^T(t)$ and $w = e_j(t - \tau_l)$ $(l = 1, 2, \ldots, m-1)$, and according to $z^T w \le \frac{1}{2}(z^T z + w^T w)$, we attain

$$\dot{V}(t) \le \sum_{i=1}^{N} p_i \exp(\mu t) \Bigg\{ \left(k_i + \frac{\mu}{2} - L\right) e_i^T(t)e_i(t) + a_{ii}^0 e_i^T(t)e_i(t) + \sum_{j=1, i\neq j}^{N} |a_{ij}^0| e_i^T(t)e_j(t) + \frac{1}{2}\sum_{j=1}^{N} |a_{ij}^1| \left(e_j^T(t-\tau_1)e_j(t-\tau_1) + e_i^T(t)e_i(t)\right) + \ldots + \frac{1}{2}\sum_{j=1}^{N} |a_{ij}^{m-1}| \left(e_j^T(t-\tau_{m-1})e_j(t-\tau_{m-1}) + e_i^T(t)e_i(t)\right) + \frac{1}{2}\sum_{j=1}^{N} |a_{ij}^1| (e_j^T(t)e_j(t)\exp(\mu\tau_1) - e_j^T(t-\tau_1)e_j(t-\tau_1)) + \ldots + \frac{1}{2}\sum_{j=1}^{N} |a_{ij}^{m-1}| \left(e_j^T(t)e_j(t)\exp(\mu\tau_{m-1}) - e_j^T(t-\tau_{m-1})e_j(t-\tau_{m-1})\right) \Bigg\}$$
$$\le \sum_{i=1}^{N} p_i \exp(\mu t) e_i^T(t)e_i(t) \Bigg\{ \left(k_i + \frac{\mu}{2} - L + (a_{ii}^0)^+\right) + \sum_{j=1}^{N} |a_{ij}^0| + \frac{1}{2}(\exp(\mu\tau_1) + 1)\sum_{j=1}^{N} |a_{ij}^1| + \ldots + \frac{1}{2}(\exp(\mu\tau_{m-1}) + 1)\sum_{j=1}^{N} |a_{ij}^{m-1}| \Bigg\}. \tag{A.6}$$

If there exists a large number L such that

$$p_i\Bigg\{\left(k_i+\frac{1}{2}\mu-L+\left(a_{ii}^0\right)^+\right)+\sum_{j=1,i\neq j}^{N}\left|a_{ij}^0\right| +\frac{1}{2}(\exp(\mu\tau_1)+1)\sum_{j=1}^{N}\left|a_{ij}^1\right|+\ldots +\frac{1}{2}(\exp(\mu\tau_{m-1})+1)\sum_{j=1}^{N}\left|a_{ij}^{m-1}\right|\Bigg\}<0, \quad \text{(A.7)}$$

then we get $\dot{V}(t)\leq 0$. It follows that $V(t)\leq V(0)$ for any $t\geq 0$.

Using the Lyapunov function (A.3), we have

$$1/2p_i||e_i||^2\exp(\mu t)=1/2p_ie_i^Te_i\exp(\mu t)\leq V(t)\leq V(0).$$

Therefore, we obtain $||e_i(t)||\leq M_i\exp(-0.5\mu t)$ with $M_i=\sqrt{2V(0)/p_i}\geq 0$. Thus, according to the Lyapunov theorem [26,31,34–37,41], we yield $||e_i(t)||\to 0$ as $t\to\infty$. That is, networks (2) and (4) are globally exponentially asymptotically synchronous.

Proof of theorem 2: just as the proof of theorem 1, construct a Lyapunov candidate as follows:

$$V(t)=\sum_{i=1}^{N}p_i\Bigg\{|e_i(t)|\exp(\mu t)+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\alpha_{ij}^0}\left(\tilde{a}_{ij}^0-a_{ij}^0\right)^2 +\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\alpha_{ij}^1}(\tilde{a}_{ij}^1-a_{ij}^1)^2+\ldots +\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\alpha_{ij}^{m-1}}\left(\tilde{a}_{ij}^{m-1}-a_{ij}^{m-1}\right)^2 +\sum_{j=1}^{N}\left|a_{ij}^1\right|\int_{t-\tau_1}^{t}|e_j(s)|\exp[\mu(s+\tau_1)]ds+\ldots +\sum_{j=1}^{N}\left|a_{ij}^{m-1}\right|\int_{t-\tau_{m-1}}^{t}|e_j(s)|\exp[\mu(s+\tau_{m-1})]ds\Bigg\},$$

where $p_i(i=1,2,\ldots,N)$ is the positive constant.

Then we have:

$$\dot{V}(t)=\sum_{i=1}^{N}p_i\Bigg\{\mathrm{Sgn}(e_i(t))\dot{e}_i(t)\exp(\mu t)+\mu|e_i(t)|\exp(\mu t) +\sum_{j=1}^{N}\frac{1}{\alpha_{ij}^0}\left(\tilde{a}_{ij}^0-a_{ij}^0\right)\dot{\tilde{a}}_{ij}^0+\ldots +\sum_{j=1}^{N}\frac{1}{\alpha_{ij}^{m-1}}(\tilde{a}_{ij}^{m-1}-a_{ij}^{m-1})\dot{\tilde{a}}_{ij}^{m-1}+\exp(\mu t) \times\Bigg[\sum_{j=1}^{N}|a_{ij}^1|(|e_j(t)\exp(\mu\tau_1)|-|e_j(t-\tau_1)|)+\ldots +\sum_{j=1}^{N}|a_{ij}^{m-1}|(|e_j(t)\exp(\mu\tau_{m-1})|-|e_j(t-\tau_{m-1})|)\Bigg]\Bigg\}.$$

According to $\mathrm{Sgn}(e_i(t))e_i(t)=|e_i(t)|$, $|g_i(e_i(t))|\leq k_i|e_i(t)|$, $\mathrm{Sgn}(g_i(e_i(t)))g_i(e_i(t))=|g_i(e_i(t))|$, we obtain

$$\dot{V}(t)\leq\sum_{i=1}^{N}p_i\exp(\mu t)\Bigg\{\mathrm{Sgn}(e_i(t))\left[k_i|e_i(t)|-L|e_i(t)|+\mu|e_i(t)|\right]+\sum_{i=1}^{N}\left|a_{ij}^0\right||e_i(t)|+\mathrm{Sgn}(e_i(t)) \times\Bigg[\sum_{j=1}^{N}\left|a_{ij}^1\right|(e_i(t)+e_j(t-\tau_1))+\ldots +\sum_{j=1}^{N}\left|a_{ij}^{m-1}\right|(e_i(t)+e_j(t-\tau_{m-1}))\Bigg] +\Bigg[\sum_{j=1}^{N}\left|a_{ij}^1\right|(|e_j(t)\exp(\mu\tau_1)|-|e_j(t-\tau_1)|)+\ldots +\sum_{j=1}^{N}\left|a_{ij}^{m-1}\right|(|e_j(t)\exp(\mu\tau_{m-1})| -|e_j(t-\tau_{m-1})|)\Bigg]\Bigg\}$$

$$\leq\sum_{i=1}^{N}p_i\exp(\mu t)|e_i(t)|\Bigg\{\left[k_i+\mu-L+\left(a_{ii}^0\right)^+\right] +\sum_{i=1,j\neq i}^{N}\left|a_{ij}^0\right|+(\exp(\mu\tau_1)+1)\sum_{j=1}^{N}|a_{ij}^1|+\ldots +(\exp(\mu\tau_{m-1})+1)\sum_{j=1}^{N}\left|a_{ij}^{m-1}\right|\Bigg\}.$$

If there exists a large number L such that

$$p_i\Bigg\{\left[k_i+\mu-L+\left(a_{ii}^0\right)^+\right]+\sum_{i=1,j\neq i}^{N}\left|a_{ij}^0\right|+\ldots +(\exp(\mu\tau_1)+1)\sum_{j=1}^{N}\left|a_{ij}^1\right|+\ldots +(\exp(\mu\tau_{m-1})+1)\sum_{j=1}^{N}\left|a_{ij}^{m-1}\right|\Bigg\}<0,$$

then we achieve that $\dot{V}(t)\leq 0$. Similar to the proof of theorem 1, we have $|e_i(t)|\leq M_i\exp(-\mu t)$ with $M_i=\sqrt{V(0)/p_i}\geq 0$. Thus, we have $||e_i(t)||\to 0$ as $t\to\infty$. That is, networks (2) and (4) are globally exponentially asymptotically synchronous. The proof is thus completed.

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