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Delay coupling enhances synchronization in complex networks

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Abstract – The phenomenon of enhancement of synchronization due to time delay is investigated in an arbitrary delay coupled network with chaotic units. Using the master stability formalism for a delay coupled network, we elaborate that there always exists an extended regime of stable synchronous solutions of the network for appropriate coupling delays. Further, the stable synchronous state is achieved even at smaller values of coupling strength with delay, which can be only attained at much larger coupling strength without delay. This also facilitates the increase in the number of synchronized nodes in the delay coupled network beyond size instability of the same network without delay. Further, the largest transverse Lyapunov exponents in the master stability surface of the network clearly demarcates the stable synchronous solutions from the unstable ones. The generic nature of our results is also corroborated using three paradigmatic models, namely, Rössler and Lorenz systems as well as Hindmarsh-Rose neurons as nodes in the delay coupled network.

Introduction. – Understanding the dynamics of collective behaviors of networks or ensembles of oscillators continues to be an area of active research in view of their potential applications in various fields of science and technology. Examples include self-organization of flock of birds, animal gaits, coherent neural oscillations, laser arrays, etc. [1–15]. In addition to facilitating analytical/ semi-analytical treatment, a network of oscillators can also exhibit a plethora of rich dynamical behaviors, such as multistable states [9,10], amplitude death [8], chimera states [11,12], phase-flip [13] and Neimark-Sacker [14] type bifurcations, which the individual units are incapable of producing in isolation. In particular, synchronization of a network of dynamical systems is a fundamental nonlinear phenomenon observed in diverse natural systems [3–7]. Recent investigations on synchronization are focused on nontrivial effects of delay coupling due to a finite propagation time of information transfer in communication channels [14–27]. Delay coupling enhances the synchronizability of networks and interestingly leads to the emergence of a wide range of new collective behavior [14,15]. Further, it has also been shown that connection delays can actually be conductive to synchronization, so that it is possible for the delayed system to synchronize where the undelayed system does not [14].

Pecora and Carroll introduced an elegant approach, the master stability function (MSF), to investigate the stability properties of a network of oscillators, which allows one to separate the local dynamics of the individual oscillators from the coupling matrix characterizing the topology of the underlying network [28]. Despite of the flurry of independent research activities on both synchronization of delay coupled networks [14–22] and in analysing the stability of synchronization of a network of oscillators by exploiting the MSF [28–33], only a very few recent studies have been dealt with both MSF and delay coupled networks in probing the stability properties of the latter [23–27]. However, these reports limited their analysis either to phase oscillators or to maps except in ref. [23], which reported the enhancement of neural synchrony by delay in two diffusively coupled Hindmarsh-Rose (HR) neurons and in ref. [26], which analysed optically coupled semiconductor lasers modeled by Lang-Kobayshi equations using MSF. Nevertheless, the authors of ref. [23] admitted that their analysis could not establish the stable synchronized states in the master stability surface of the

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network of delay coupled HR neurons. Hence, the master stability formalism has not yet been successfully exploited in understanding the stability properties of synchronization transitions in a delay coupled network synthesized from even simple low-dimensional dynamical systems. Further, the effect of delay on the synchronizability of such a network has not yet been investigated using the MSF.

In this letter, we aim at filling this problem by analysing an arbitrary delay coupled network using the master stability formalism. In particular, we find that the maximal transverse Lyapunov exponent estimated from the MSF displays an extended regime of a stable synchronized state as a function of the coupling strength for appropriate values of the delay coupling. Further, the stable synchronous state is achieved even at smaller values of the coupling strength with delay, which is actually attained at larger coupling strength without delay. The extended regime of the stable synchronized state by delay facilitates an increase in the number of synchronized nodes in the delay coupled network, which can also be confirmed from the scaling relation of the MSF. We have also established the stable synchronized state in the master stability surface from the values of the maximal transverse Lyapunov exponent of the MSF of the delay coupled network. We demonstrate our results using several paradigmatic models.

Master stability formalism for a delay coupled network. – We consider an arbitrary delay coupled network with N identical chaotic units in accordance with the MSF [28] as follows:

$$\dot{x_i}(t) = f\left[x_i(t)\right] - \sigma \sum_{j=1}^{N} g_{ij} h\left[x_j(t-\tau)\right], \qquad (1)$$

where $x_i \in \mathbb{R}^m, i = 1, \ldots, N$, f is a nonlinear function describing the dynamics of the individual units, σ is the overall coupling strength, $G = g_{ij}$ is a $N \times N$ matrix, which determines the topology of an arbitrary network, h is a coupling function and τ is the coupling delay. The invariant synchronization manifold is assured by the zero row sum, that is $\sum_{j=1}^{N} g_{ij} = 0 \;\; \forall \;\; i$, of each row of the connectivity matrix G. At first, we consider the arbitrary network as undirected such that G is symmetric with real eigenvalues and then we extend our analysis to directed networks, in which case G is asymmetric leading to complex eigenvalues. It has also to be noted that our results remain unaffected in either cases, as the imaginary part of the complex eigenvalues has only a rotational effect [34], as we will discuss below in detail. If G is diagonalizable, then the stability of the synchronization manifold is governed by the eigenvalues of G and the following block diagonalized variational equation for the transverse modes,

$$\dot{\xi}(t) = Df\left[x(t)\right]\xi(t) - \varepsilon Dh\left[x(t-\tau)\right]\xi(t-\tau). \qquad (2)$$

Here, $x(t)$ is the dynamics within the synchronization manifold, $\xi(t)$ is the perturbation transverse to the synchronization manifold, which could be complex, $Df[x(t)]$ is the Jacobian evaluated on the synchronization manifold and $Dh[x(t-\tau)]$ is an $m \times m$ matrix that determines which components of the oscillator are coupled. The normalized coupling parameter $\varepsilon = \sigma\gamma_k$, where γ_k is an eigenvalue of the coupling matrix G, $k = 0, 1, 2, \ldots, N-1$. It is to be noted that the eigenvalue γ_k is more generic because of the arbitrary G and it could be the eigenvalue corresponding to any specific topology determined by G. Here, $k = 0$ corresponds to the synchronization manifold with $\gamma_0 = 0$, while all the other eigenvalues for $k > 0$ correspond to the transverse modes determining the stability of the synchronization manifold $x(t)$. Thus, the synchronized state is stable for a given network topology, characterized by G, when all the eigenvalues of G for $k > 0$ are negative (or when all the transverse Lyapunov exponents of (2) are negative). Considering the fact that G could have complex eigenvalues arising from asymmetric couplings [34], the generic variational equation, that is the MSF, can be written as

$$\dot{\xi}(t) = Df\left[x(t)\right]\xi(t) + (\alpha + i\beta)Dh\left[x(t-\tau)\right]\xi(t-\tau), \quad (3)$$

where, α and β are the real and imaginary parts of the eigenvalue. Separating $\xi(t)$ into the real part $\xi_r(t)$ and the imaginary part $\xi_i(t)$, we get

$$\begin{aligned}\dot{\xi_r}(t) &= Df\left[x(t)\right]\xi_r(t) + \alpha Dh\left[x(t-\tau)\right]\xi_r(t-\tau) \\ &\quad -\beta Dh\left[x(t-\tau)\right]\xi_i(t-\tau), \end{aligned} \qquad (4a)$$

$$\begin{aligned}\dot{\xi_i}(t) &= Df\left[x(t)\right]\xi_i(t) + \alpha Dh\left[x(t-\tau)\right]\xi_i(t-\tau) \\ &\quad +\beta Dh\left[x(t-\tau)\right]\xi_r(t-\tau). \end{aligned} \qquad (4b)$$

The maximal transverse Lyapunov exponent (λ_{max}) estimated from eqs. (4) yields a complex surface in the (α, β)-plane, namely the master stability surface of the considered network of oscillators. For a given value of the coupling strength σ and the delay time τ, one can locate a point in the complex surface and the sign of λ_{max} at that point determines the stability of the synchronization manifold.

Delay enhanced stable synchronization. – Recently, Huang *et al.* [29] classified the typical behaviors of MSFs into four categories by examining all possible one-component linear coupling configurations in several prototype three-dimensional systems. It is to be noted that the x-component coupling in most of the dynamical systems tends the network to remain synchronized with a single crossing of the maximal transverse Lyapunov exponent with the zero axis from positive. According to their classification, the λ_{max} of class Γ_2 possesses two successive crossings with the zero axis in a finite interval of the normalized coupling strength $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ indicating a finite range of the stable synchronized state. Dynamical systems with different component couplings fall under different classes [29].

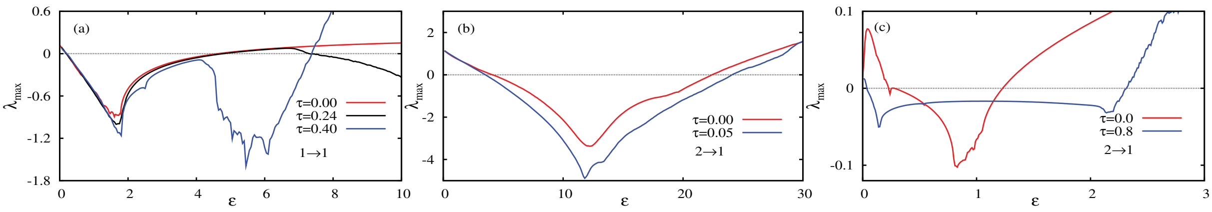

Fig. 1: (Colour on-line) The largest transverse Lyapunov exponents λ_{max} of the master stability function (2) of a delay coupled network as a function of the normalized coupling parameter ε for different values of the coupling delay τ. The notation $i \to j$ indicates the coupling as being from i-th component of one oscillator to the j-th component of another one. (a) Rössler system as nodes with $\tau = 0.0, 0.24$ and 0.4, (b) Lorenz system as nodes with $\tau = 0.0$ and 0.05 and (c) HR neurons as nodes with $\tau = 0.0$ and 0.8.

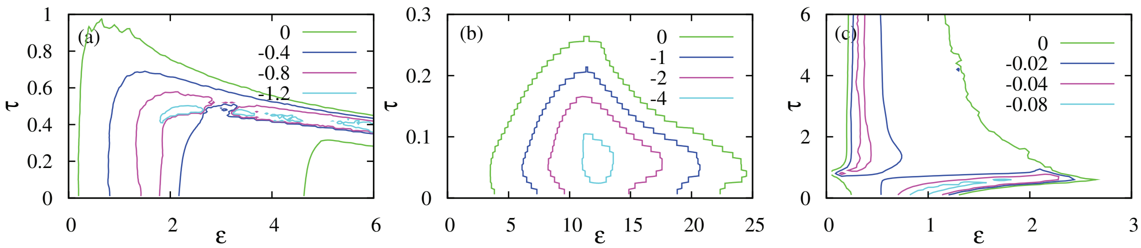

Fig. 2: (Colour on-line) Contours of the largest transverse Lyapunov exponent λ_{max} in the real plane of the master stability function (2) as a function of ε and the coupling delay τ. (a) $x \to x$ coupled Rössler systems; (b) $y \to x$ coupled Lorenz systems; and (c) $y \to x$ coupled Hindmarsh-Rose neurons. Contours correspond to different values of λ_{max} encompassing the stable synchronous states.

To demonstrate the existence of the extended regime of the stable synchronous state of a delay coupled network, it would be convenient if the original stable synchronized regime of the network without delay is limited within a finite range of the normalized coupling parameter ε. Therefore, we stick to the appropriate component couplings that come under the class Γ_2 for the following paradigmatic models that have been widely exploited in synchronization studies: $x \to x$ coupled Rössler systems governed by the equation of motions [28,29,35],

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x + 0.2y,\\ \dot{z} &= 0.2 + z(x - 9.0);\end{aligned}$$

$y \to x$ coupled Lorenz systems [29,36] represented as

$$\begin{aligned}\dot{x} &= 10.0(x - y),\\ \dot{y} &= x(28.0 - z) - y,\\ \dot{z} &= xy - 2.0z;\end{aligned}$$

and $y \to x$ coupled HR neurons described by the equation of motions [23,37]

$$\begin{aligned}\dot{x} &= y - x^3 + 3.0x^2 - z + 3.2,\\ \dot{y} &= 1.0 - 5.0x^2 - y,\\ \dot{z} &= 0.06(4.0(x + 1.6) - z).\end{aligned}$$

Details on the physical meaning of the components of the above dynamical systems can be found in their corresponding references. Now, we demonstrate our results by estimating the largest transverse Lyapunov exponent of the MSF (2) and (4) in both real axis and complex (α, β) planes, respectively. The Lyapunov exponents can be estimated using any standard methods as a function of $\varepsilon, \tau, \alpha$ and β. In particular, we have used the procedure suggested by Farmer [38] for time-delay systems.

The largest transverse Lyapunov exponent λ_{max} of the MSF (2) of the delay coupled network with the Rössler system as nodes is depicted in fig. 1(a) for three different values of the coupling delay τ. In the absence of the coupling delay, $\tau = 0$, the stable synchronous state exists only in the range of $\varepsilon \in (0.18, 4.61)$ as indicated by the sign of the λ_{max} (see fig. 1(a)). As the coupling delay is increased, the stable synchronous regime is gradually extended in the ε parameter space up to a certain threshold. Further increase in it leads to decrease in the regime of stable synchronous oscillations of the delay coupled network (see figs. 2). In particular, for $\tau = 0.24$ in the coupling, λ_{max} has a second stable regime for $\varepsilon > 7.38$ and for $\tau = 0.4$ there exists only one single stable synchronous regime but extended in the range of $\varepsilon \in (0.18, 7.38)$, which is almost twice compared to that without delay.

In order to understand the effect of delay coupling in the network of Rössler systems more clearly, we have shown λ_{max} as a function of $\varepsilon \in (0, 6)$ and the coupling delay $\tau \in (0, 1)$ in fig. 2(a). Contours in this figure correspond to different values of λ_{max}. It is evident from this figure that

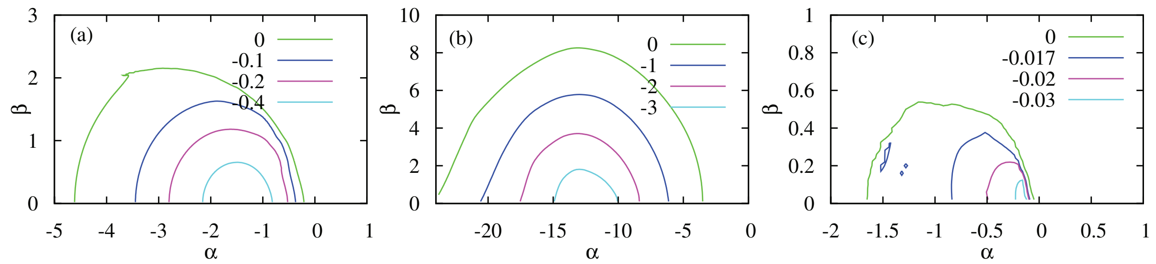

Fig. 3: (Colour on-line) Contours of the largest transverse Lyapunov exponent λ_{max} in the complex plane of the MSF (4). (a) $x \to x$ coupled Rössler systems with $\tau = 0.24$; (b) $y \to x$ coupled Lorenz systems with $\tau = 0.05$; and (c) $y \to x$ coupled Hindmarsh-Rose neurons systems with $\tau = 4.0$. Contours encompass the stable synchronous states.

as the coupling delay is increased from zero, the stable synchronous regime increases slowly towards the larger coupling strength up to a certain threshold as indicated by the contour corresponding to $\lambda_{max} = 0$. This confirms the extension of the interval of the stable synchronous state induced by the coupling delay. Further increase in the delay leads to decrease in the regime of stable synchronous state as indicated by the contour of $\lambda_{max} = 0$. Closer examination of the contour corresponding to $\lambda_{max} = 0$ in the range of coupling delay $\tau \in (0.1, 0.2)$ clearly indicates that the stable synchronous state occurs even for smaller values of ε in a narrow range in the presence of coupling delay. Nevertheless, the networks of Lorenz systems and HR neurons display more pronounced extended regime of stable synchronous state towards smaller coupling strengths (see figs. 1(b) and (c)), which is indeed attained for larger ε without delay in the network.

The stability issue of the synchronized state of the arbitrary network (1) is reduced to that of the block diagonalized variational equation, (2), for the transverse modes such that the sign of the largest Lyapunov exponent determines the stability. As the form of each block is the same with only the scalar multiplier $\varepsilon = \sigma\gamma_k$ differing for each [28] and hence it is straightforward to obtain a scaling relation between the eigenmodes of the connectivity matrix G. Further, one can estimate the maximal number of synchronized oscillators, N_{max}, of the network from the scaling relation in analogy with the one derived for a lattice [30]. However, it is to be noted that the scaling relation facilitates the estimation of N_{max} only from the explicit knowledge of the eigenvalues γ_k in terms of trigonometric functions and hence the estimation of N_{max} is restricted rather to regular networks [39], whereas for more complex networks it remains an open problem. As N_{max} in the regular networks depends solely on ε at the two successive zero crossings of λ_{max}, namely ε_1 and ε_2, the increase in the interval between ε_1 and ε_2 reflects in the increase in N_{max} [28,30–32,39]. As a consequence, the extended regime of stable synchronized state as a function of ε in the delay coupled network of Rössler systems facilitates an increase in the N_{max} when compared to the same network without delay.

The largest transverse Lyapunov exponent of the MSF (4) for $x \to x$ coupled Rössler systems is illustrated in the complex (α, β)-plane in fig. 3(a) for the value of the coupling delay $\tau = 0.24$. The contours encompasses the stable synchronous regimes corresponding to the negative values of λ_{max}. To the best of our knowledge, this is the first result showing stable synchronous states using the MSF of a delay coupled network of time continuous chaotic units. For $(\alpha, \beta) = (0, 0)$ in eq. (4), then $\lambda_{max} > 0$ as the nodes evolve independently. Equations (4a) and (4b) become exactly the same as eq. (2) for $\alpha < 0$ when $\beta = 0$ and the effect of coupling delay for this case is clearly illustrated in figs. 1 and 2, and explained in appropriate places. The network does not synchronize at all for $\alpha > 0$. Increasing β in eq. (4) for fixed α rotates the transverse modes as can be seen in figs. 3 and hence the complex surface defining stability is symmetric along the real axis. Further, λ_{max} becomes positive for larger (or smaller) values of β beyond a certain threshold even if $\alpha < 0$ as depicted in figs. 3 attributing to the fact that a large imaginary coupling can destabilize the system. In short, varying the real term in $\alpha + i\beta$ has the effect of damping the transverse perturbations to the synchronization manifold, whereas changes in the imaginary terms have a rotational effect between different transverse modes [34].

Figure 1(b) illustrates λ_{max} of the master stability function (2) of a delay coupled network with the Lorenz systems as its nodes. λ_{max} depicts a finite regime of stable synchronous chaos for the $y \to x$ coupled Lorenz systems in the range of $\varepsilon \in (4.17, 22.53)$ in the absence of delay coupling. On the other hand, the regime of stable synchronized state is extended in both directions of the coupling strength in the range of $\varepsilon \in (3.51, 24.14)$ for $\tau = 0.05$ corroborating the delay-induced extended regime of stable synchronous states. As a consequence, the number of synchronized oscillators in the delay coupled network is increased beyond that of the same network without delay. To elucidate this phenomenon of delay enhanced synchrony more clearly, λ_{max} is illustrated as contours in fig. 2(b) as a function of $\varepsilon \in (0, 25)$ and $\tau \in (0, 0.3)$. The contour for $\lambda_{max} = 0$ confirms the above phenomenon up to a certain threshold value of τ and then the stable regime decreases with further increase in τ. The master stability surface of $y \to x$ coupled Lorenz systems is depicted in fig. 3(b) for the coupling delay $\tau = 0.05$. Isocline lines of λ_{max} attributes to the stable synchronous state.

For the case of $y \to x$ coupled Hindmarsh-Rose neurons, λ_{max} is depicted in fig. 1(c). There exists a stable synchronized regime in the interval $\varepsilon \in (0.28, 1.23)$ for $\tau = 0.0$, whereas for $\tau = 0.8$ λ_{max} indeed shows a large regime of stable synchronized state in the range $\varepsilon \in (0.036, 2.31)$ extended for both smaller and larger values of ε. It is a strong evidence that stable synchronization can occur even for smaller values of the coupling strength in the presence of the coupling delay, which is obtained only for sufficiently larger values of ε without delay in the same network. N_{max} can also be increased proportionally as discussed above. Further, the isocline line corresponding to $\lambda_{max} = 0$ as a function of $\varepsilon \in (0, 3)$ and $\tau \in (0, 6)$ in fig. 2(c) also confirms the extended regime of stable synchronized state, compared to that for $\tau = 0$, almost in the entire range of the coupling delay we have considered. Contours of λ_{max} estimated from (4) for $y \to x$ coupled HR neurons depicted in fig. 3(c) clearly delineate the stable synchronous regimes in the complex plane of the MSF for the coupling delay $\tau = 4.0$.

Thus, we have demonstrated the existence of an extended regime of stable synchronized states at both smaller and larger values of the coupling strength in an arbitrary network induced by the coupling delay using the master stability formalism. The generic nature of our results is also exhibited using three paradigmatic models that have been widely exploited in the literature for synchronization studies.

Summary and conclusion. – We have investigated the phenomenon of enhancement of the stable synchronous state of an arbitrary delay coupled network using the MSF. We have demonstrated that there always exists an extended regime of stable synchronous state as a function of coupling strength for appropriate coupling delays, which cannot be observed without any delay in the coupling. Further, the stable synchronous state is achieved even at smaller values of the coupling strength with delay, which is actually attained at larger coupling strength without delay. The extended regime of synchronization also suggests that the number of synchronized nodes in the delay coupled networks can be increased proportionally beyond the size instability of the same networks without delay. Further, the contours of the largest transverse Lyapunov exponent in the complex plane corresponding to the MSF of the delay coupled network clearly demarcate the stable synchronous states from the unstable ones. The generic nature of our results is also corroborated using three paradigmatic models, namely, Rössler, Lorenz systems and Hindmarsh-Rose neurons as nodes of the network, that have been widely exploited in the literature for synchronization studies. The mechanism behind the phenomenon of delay enhanced stable synchronization remains unclear and is an open problem to be explored.

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