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Statistical characteristics of the Poincaré return times for a one-dimensional nonhyperbolic map[⋆]

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Abstract. Characteristics of the Poincaré return times are considered in a one-dimensional cubic map with a chaotic nonhyperbolic attractor. Two approaches, local one (Kac's theorem) and global one related with the AP-dimension estimation of return times, are used. The return times characteristics are studied in the presence of external noise. The characteristics of Poincaré recurrences are compared with the form of probability measure and the complete correspondence of the obtained results with the mathematical theory is shown. The influence of the attractor crisis on the return time characteristics is also analyzed. The obtained results have a methodical and educational significance and can be used for solving a number of applied tasks.

1 Introduction

Dynamical systems (DS) with a complicated character of their trajectories can be described from the geometrical viewpoint of limit sets in the phase space as well as by analysing the evlolution of phase trajectories in time. A so-called Poincaré recurrence is one of the fundamental peculiarities of the temporal dynamics of DS. The recurrence according to Poincaré means that any trajectory starting from a point $\boldsymbol{x}_0$ passes very closely to the initial point infinitely often in time. H. Poincaré named such motions as stable according to Poisson. The problem of Poincare recurrences for ergodic systems with a given probability measure has been completely resolved and described in the mathematical literature [1–4]. Kac's theorem [1] is a fundamental mathematical result in this research topic. It was proven that the mean time of the first Poincaré return to an ε-vicinity of a given point is inversely proportional to the probability

$$\langle \tau_r(\varepsilon) \rangle = \frac{1}{P(\varepsilon)}. \tag{1}$$

The angle brackets in (1) denote ensemble- or time-averaging for ergodic systems. In particular, for discrete-time systems a sequence of first return times will be $\tau_{rk} = n_{k+1} - n_k$ $(k = 1, 2, \ldots, k_{\max})$, and the discrete-time mean can be calculated as follows:

$$\langle \tau_r(\varepsilon) \rangle = \frac{1}{k_{\max}} \sum_{k=1}^{k_{\max}} (n_{k+1} - n_k), \quad k_{\max} \gg 1, \tag{2}$$

where index k denotes a discrete time moment of the trajectory falling in an ε-vicinity. Kac's theorem (1) was proven provided that $\varepsilon \to 0$, i.e. for the case of very large values of τ_r. It is known that in a general case [5,6]

$$P(\varepsilon) \simeq p(x)\varepsilon^{d_f}, \tag{3}$$

where $p(x)$ is the distribution density, d_f is the fractal dimension of the set. If the density $p(x)$ is a smooth function, then $d_f = N$ (the phase space dimension). For the one-dimensional case $N = 1$ we have $P(\varepsilon) = p(x)\varepsilon$.

The most important mathematical result consists in the proof that the distribution density of a random process of recurrences obeys the Poisson law [2]:

$$p(\tau_r) = \frac{1}{\langle \tau_r \rangle} \exp\left(-\frac{\tau_r}{\langle \tau_r \rangle} \right). \tag{4}$$

The law (4) describes the distribution of a random sequence of recurrences to an ε-vicinity of a point and is valid in the limit $\varepsilon \to 0$.

Expressions (1) and (4) concern the problem of recurrences in an open neighborhood ε of the studying set and in this sense characterize the local properties of Poincare recurrences.

[⋆] We are happy to devote this paper to the outstanding scientist, our teacher and colleague Prof. Dr. Werner Ebeling on the occasion of his 75th birthday.

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There is another approach to the problem of Poincaré recurrences when a set is divided into ε-elements and then the minimal returns are averaged over these partitions. Since the averaging is performed over all the set, such an approach can be called a global one. The mathematical theory of the global approach is presented in [3,4], where it was proven that for systems with zero topological entropy the asymptotical for $\varepsilon \to 0$ relation is valid

$$\langle \tau_{\text{rinf}}(\varepsilon)\rangle \sim \varepsilon^{-\frac{d}{\alpha_c}}. \tag{5}$$

Here $\langle \tau_{\text{rinf}}(\varepsilon)\rangle$ is the mean over the set of ε partition elements, ε is the size of a partition element, d is the phase space dimension of a system, α_c is the dimension of Poincaré recurrences, introduced by Afraimovich and Pesin (AP-dimension). The minimal return times are considered in order to avoid $\tau_r \to \infty$ in the limit $\varepsilon \to 0$ for that (5) was proven. If the topological entropy differs from zero, then a different relation is valid, that is typical for chaotic dissipative systems [3]:

$$\langle \tau_{\text{rinf}}(\varepsilon)\rangle \sim -\frac{d}{\alpha_c}\ln\varepsilon, \quad \varepsilon \ll 1. \tag{6}$$

For multifractal sets the relation (6) may have a more complicated form depending on ε.

The material presented above is a short description of the basic mathematical results on Poincaré recurrences. Besides the theoretical results, the literature on nonlinear dynamics covers numerous problems being important from the applied point of view. Among them one can mention the usage of Poincaré recurrences in controlling chaos (a so-called "targeting" task [7,8]), the application of the AP-dimension for diagnosing the synchronization of interacting chaotic oscillators [4,9], as well as for indicating the effect of stochastic resonance at noise-induced transitions in bistable systems [10]. Taking into account the fact that strict results were obtained for hyperbolic systems with a given probability measure, the analysis of Poincaré recurrences in nonhyperbolic systems becomes an important task. Note that Poincaré recurrences are numerically simulated by using algorithms and computer programs that do not require the knowledge of probability measure. In this connection the numerical results are not compared, as a rule, with the form and evolution of the probability measure when system parameters are varied and an external noise is added. This comparison is absolutely necessary because it is precisely the changes in the probability density $p(x)$ that define experimental results on the base of mathematical theory.

The purpose of this paper is to analyse in detail the problem of Poincaré recurrences for a one-dimensional nonhyperbolic map when system parameters and the external noise intensity are varied. We study a cubic map with a chaotic nonhyperbolic attractor, that is described as follows:

$$x_{n+1} = \left(\alpha x_n - x_n^3\right)\exp\left(-\frac{x_n^2}{B}\right) + \sqrt{2D}\xi(n), \tag{7}$$

where α is the control parameter, $B = 10$ is the coefficient in the exponential factor that bounds the rate of growth of the variable x_n, D is the noise intensity, $\xi(n)$ is a source of bounded white noise.

The map (7) is one of the basic models of a bistable system, that demonstrates the effect of stochastic resonance [11], enables to construct numerically the probability measure $p(x)$ and to compare in detail the characteristics of Poincaré recurrences with the theoretical data [1–3,12].

The bistability in system (7) gives the possibility of realizing a crisis of two symmetric attractors when the parameter α passes a certain critical value α^* [11]. The crisis can be induced by external noise with a certain intensity D for $\alpha < \alpha^*$. Since such an effect is quite important for studying the stochastic resonance, we pay special attention to the characteristics of Poincaré recurrences in the regime of noise-induced attractor crisis.

2 Local approach. Kac's theorem

The evolution of the probability density $p(x_n)$ in system (7) is illustrated in Figure 1 for $\alpha = 2.7$ and different values of the external noise intensity. As can be seen from Figure 1, for $D < 10^{-4}$ two symmetric ($x_n \to -x_n$) attractors are realized in system (7). When $D = 1.1 \times 10^{-4}$ these two attractors merge into a single one. As seen from the whole Figure 1, the form of function $p(x_n)$ is noticeably evolving as D increases. We will show that this fact is strongly responsible for the corresponding changes in the quantitative characteristics of return times. Note that in our numerical experiments the density $p(x_n)$ was approximately calculated as follows:

$$p(x_n) = \frac{\Delta P(x_n \pm \Delta x_n/2)}{\Delta x_n}, \tag{8}$$

where ΔP is the probability of the trajectory falling within the x_n range and $\Delta x_n \le 10^{-3}$ is the range length. The calculation results of $p(x_n)$ via (8) for all the x_n domain were normalized such that the sum of all the probabilities was equal to 1.

We choose five points on the one of the symmetric attractors of system (7): $x_0^1 = 0.37$, $x_0^2 = 0.65$, $x_0^3 = 0.95$, $x_0^4 = 1.25$ and $x_0^5 = 1.46$ and consider their vicinities of size $\varepsilon = 0.1$ (see Fig. 2). These regions are characterized by different distribution densities $p(x_0^i)$, $i = 1, 2, \ldots, 5$ that are also shown in Figure 2. We calculate the mean return times $\langle \tau_r(\varepsilon)\rangle$ in the ε-vicinity of the chosen points for $\varepsilon = 0.1$ as a function of the noise intensity D. The numerical results are presented in Figure 3. From this figure it follows that: (1) passing through the crisis point as $D > 10^{-4}$ increases results in a sharp growth of the mean return time for all the chosen points; (2) when the noise intensity grows, the mean return time behaves fundamentally in a nonlinear manner. One can observe both a decrease and an increase of the mean return time depending on the noise intensity. If to compare carefully Figures 3 and 1, then it is clearly seen that $\langle \tau_r(D)\rangle$ grows if $P(x_n)$ decreases, and the mean return time decreases when $P(x_n)$ increases. Such a behavior is fully described by expression (1).

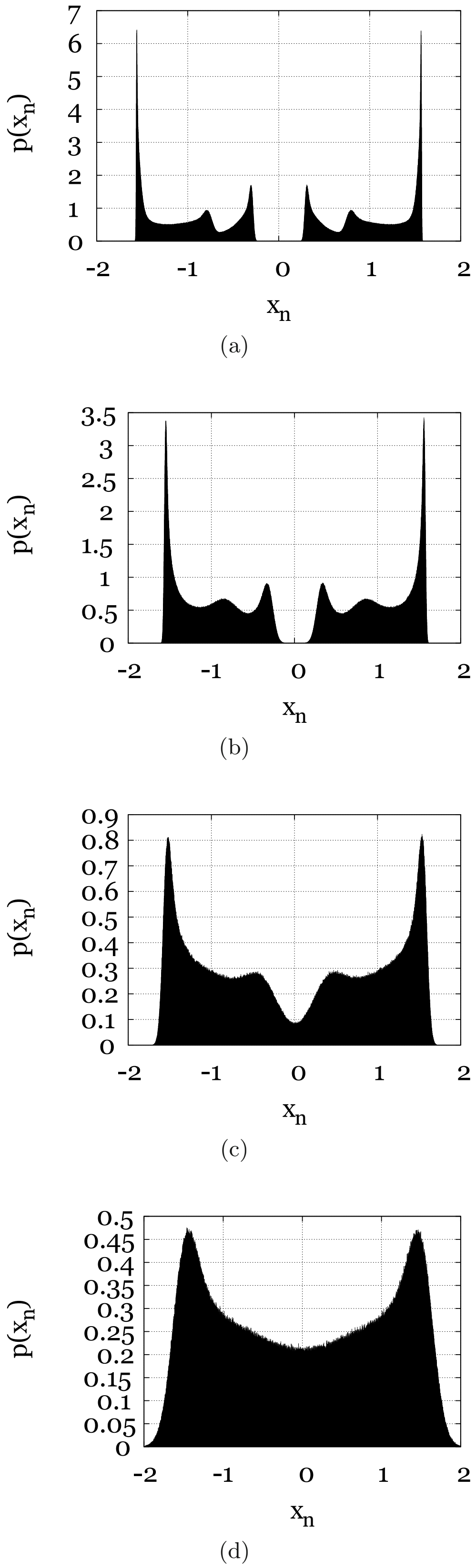

Fig. 1. Distribution density $p(x_n)$ for different values of the noise intensity D: 10^{-5} (a); 10^{-4} (b); 10^{-3} (c), and 10^{-2} (d). Plots (a, b) correspond to two symmetric attractors and plots (c, d) to the single (merged) one.

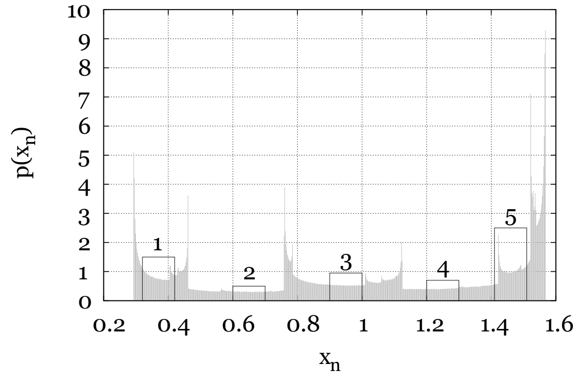

Fig. 2. Distribution density $p(x_n)$ on the one of the symmetric attractors of system (7) for $\alpha = 2.7$ and $D = 0$. This figure also illustrates the choice of vicinities of the five points.

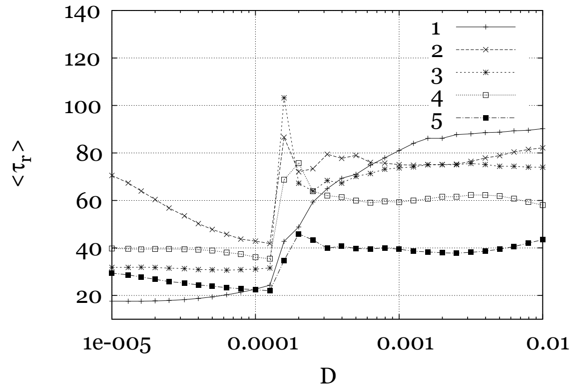

Fig. 3. Mean return times $\langle \tau_r \rangle$ as functions of the noise intensity for the five chosen points on the attractor of system (7).

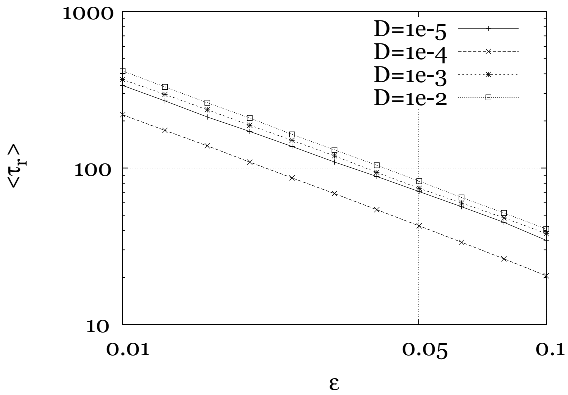

Fig. 4. Dependence of the mean return time $\langle \tau_r \rangle$ on ε in the ε-vicinity of a point $x_0^2 = 0.65$ for different values of noise intensity D.

Now we intend to find out the character of the dependence $\tau_r(\varepsilon)$ and to compare the obtained numerical results with theoretical data. With this purpose we consider the fixed point $x_0^2 = 0.65$ and calculate the mean return times in its ε-vicinity for ε values from 10^{-2} till 10^{-1}. The calculation results are presented in Figure 4.

Approximating the obtained data by the least square method, one can establish that the plots in Figure 4 in the double logarithmic scale correspond to the following straight lines:

$$\lg\langle\tau_r(\varepsilon)\rangle = \lg C(D) - k\lg\varepsilon. \tag{9}$$

Here $k = 1.00 \pm 0.02$, and only the coefficients C depend on the noise intensity

$$\begin{aligned} &C_1 = 3.756\ (D = 10^{-5}),\ C_2 = 1.958\ (D = 10^{-4}),\\ &C_3 = 3.900\ (D = 10^{-3}),\ C_4 = 4.012\ (D = 10^{-2}). \end{aligned} \tag{10}$$

The coefficients $C(D)$ (10) and $k = 1.00 \pm 0.02$ were directly obtained by approximating the data of Figure 4. If Kac's theorem (1) is also valid for the system (7) in the presence of noise, then the coefficients $C(D)$ (10) must correspond to the quantities $1/p(x_0)$. The latter are defined by the corresponding distributions in Figure 1 that were computed for the noise intensity D from the range $10^{-5} \div 10^{-2}$. Our calculations have shown that the coefficients obtained by using the probability densities $p(x_0, D)$ coincide with the approximation data (10) with an accuracy of $\pm 5\%$.

Indeed, expression (1) can be rewritten as follows:

$$\langle\tau_r(\varepsilon)\rangle = C\varepsilon^{-1},\ \text{where}\ C = \frac{1}{p(x_0)}. \tag{11}$$

Thus, the experimental results in Figure 4 that are approximated by expressions (9) and (11) fully correspond to formula (1) both without and in the presence of additive noise in system (7).

The peculiarity of the local approach consists in the fact that the return times depend on the vicinity of a chosen point on an attractor $\langle\tau(x_0)\rangle$. This fact is very important in some applied tasks such as controlling chaos (for example, [7,8]). Figure 5 illustrates the calculation of $\langle\tau_r\rangle$ depending on point x on the attractor of system (7). Let us choose the regime of merged attractor for $\alpha = 2.84$ without noise. The distribution density $p(x_n)$ for $0 \le x_n \le 1.7$ is shown in Figure 5a. The mean return times (Fig. 5b) fully correspond to the $p(x_n)$ changes (Fig. 5a) and can be calculated by using Kac's theorem. Hence the correspondence to Kac's theorem remains in the presence of noise.

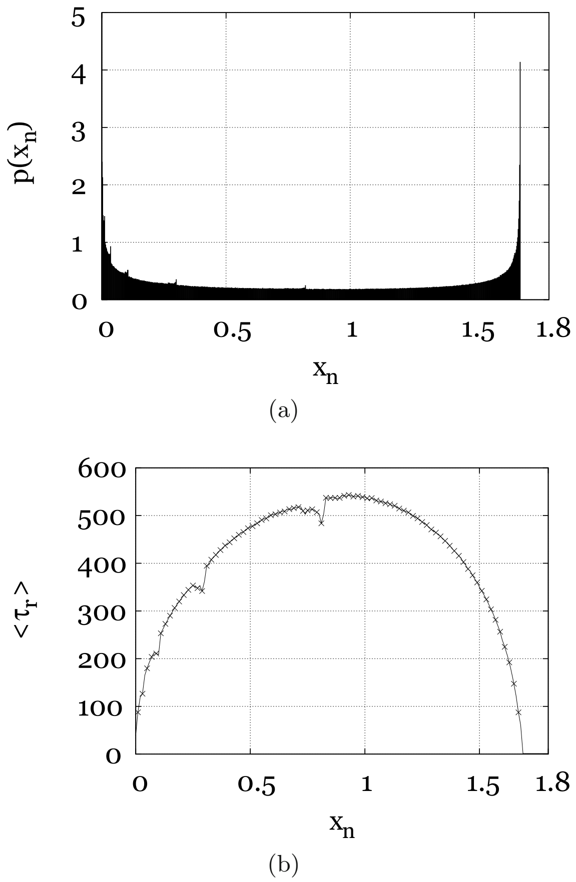

Fig. 5. Distribution density $p(x_n)$ on the system (7) attractor for $\alpha = 2.84$ and $D = 0$ (a) and the mean return time in the ε-vicinity of point x_n for $\varepsilon = 0.01$ (b). The results of direct calculations of $\langle\tau_r\rangle$ are marked by symbols $\times$ and the calculations according to Kac's theorem (1) are shown by the solid line.

3 Global approach. The AP-dimension of the Poincaré recurrences

In contrast to the Poincaré return times described in the framework of the local approach, the global approach is based on a different definition [3,4]. The attractor of system (7) must be covered by segments (pieces) ε_i. A minimal time of the first return $\inf\tau_r(\varepsilon_i)$ is calculated for each of the segments ε_i $(i = 1, 2, \ldots, N)$, and then all the obtained quantities are averaged over all ε_i:

$$\langle\tau_{r\inf}(\varepsilon)\rangle = \frac{1}{N}\sum_{i=1}^{N}\tau_{r\inf}(\varepsilon_i). \tag{12}$$

When the cover interval ε changes, all computations are repeated [9].

Since the topological entropy of system (7) is positive in a chaotic regime, we will use expression (6) written in the following form:

$$\langle\tau_{r\inf}(\varepsilon)\rangle = C - \frac{d}{\alpha_c}\ln\varepsilon. \tag{13}$$

The dependence $\langle\tau_{r\inf}(\varepsilon)\rangle$ on $\ln\varepsilon$ is assumed to be a straight line with slope $-d/\alpha_c$, where $d = 1$, and α_c is the AP-dimension [9]. The coefficient C defines the shift of the straight line with respect to the origin of coordinates.

We are interested in the straight line slope (13) that is determined by the relation d/α_c and enables to estimate the dimension of Poincare recurrences α_c. The numerical results are presented in Figure 6 for two different regimes in system (7) without noise ($D = 0$): before the attractor crisis ($\alpha = 2.7$, Fig. 6, line 1) and after the crisis ($\alpha = 2.84$, Fig. 6, line 2). Two conclusions can be drawn from the obtained plots. Firstly, the calculations verify that the return times linearly depend on $\ln\varepsilon$ (13). Secondly, the slopes of

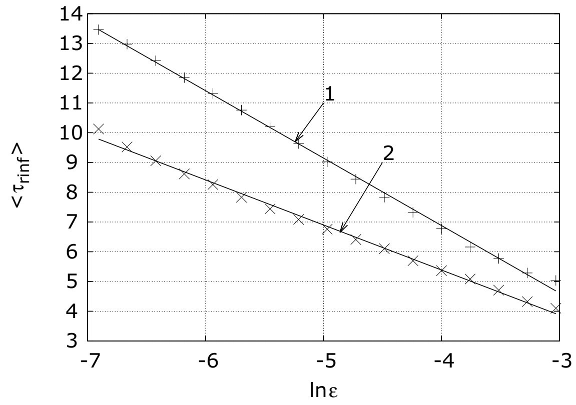

Fig. 6. Dependence of the return times $\langle\tau_{r\mathrm{inf}}(\varepsilon)\rangle$ on the interval size ε for $\alpha = 2.7$ (1) and $\alpha = 2.84$ (2) without noise. The straight line slope defines α_c being equal to 0.428 (1) and 0.671 (2).

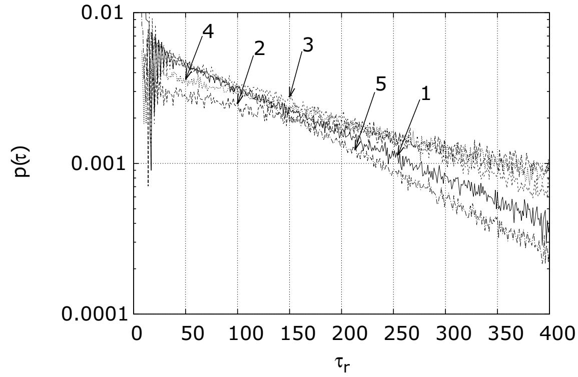

Fig. 7. Distribution densities of the Poincare return times for the chosen 5 points on the attractor (Fig. 2), calculated for $\alpha = 2.7$, $D = 0$ and $\varepsilon = 10^{-2}$

Table 1. Comparison of $P(\Delta_i)$ with the approximation data (15) for the chosen points on the attractor x_0^i

i	x_0^i	$P(\Delta_i)$	γ_i	C_i	$\delta_1, \%$	$\delta_2, \%$
1	0.37	7.5×10^{-3}	6.9×10^{-3}	6.3×10^{-3}	7.9	8.9
2	0.65	3.1×10^{-3}	3.2×10^{-3}	3.2×10^{-3}	3.3	2.0
3	0.95	5.2×10^{-3}	5.5×10^{-3}	5.9×10^{-3}	5.8	6.1
4	1.25	4.1×10^{-3}	4.2×10^{-3}	4.4×10^{-3}	3.6	3.7
5	1.46	9.7×10^{-3}	8.4×10^{-3}	7.1×10^{-3}	13.9	14.7

the straight lines are different before the crisis and after that. For $\alpha = 2.7$ the slope $\frac{d}{\alpha_c} = 2.333$ that corresponds to the AP-dimension $\alpha_c = 0.428$. At $\alpha = 2.84$ the results are different, i.e., $\frac{d}{\alpha_c} = 1.490$ and the AP-dimension is $\alpha_c = 0.671$. Thus, the merging of chaotic attractors due to a crisis results in the growth of the return time dimension α_c. Our numerical analysis of the influence of noise on the AP-dimension has shown that as the noise intensity increases, the straight line slope in Figure 6 decreases and the AP-dimension α_c grows. It has been established that this effect may depend on the noise source statistics and requires special detailed studies.

4 Return time distribution density

For hyperbolic systems with intermixing properties the first Poincaré return times τ_r in a given region of an attractor $\Delta = x_0 \pm \varepsilon/2$ satisfy an exponential law in the limit $\varepsilon \to 0$ and for $\tau_r \geq \tau_r^*$, where τ_r^* is a small value of the return time [2,13,14]:

$$p(\tau_r) = C\exp(-\gamma\tau_r). \tag{14}$$

The coefficients C and γ depend on x_0 and ε. In the limit $\varepsilon \to 0$, $P(\Delta)$ also tends to zero and the relation $\gamma = P(\Delta) = 1/\langle\tau_r\rangle$ is valid, where $\langle\tau_r\rangle$ is the mean value of the return times [13,14]. In numerical experiments for small but finite $\varepsilon \ll 1$, that equality may be violated and this depends on particular values of ε and a system under study. The coefficient γ can be defined by considering (14) in a logarithmic scale. The slope of the linear dependence

$$\ln p(\tau_r) = \ln C - \gamma\tau_r \tag{15}$$

provides the values of γ, and its shift with respect to the origin of coordinates – the values of coefficient C in (14).

Consider the calculation results of $p(\tau_r)$ for the map (7). We choose the same 5 points x_0^i on the attractor of system (7) as it was done in Section 1, and compute the corresponding distribution densities $p_i(\tau_r)$ for vicinities $\Delta_i = x_0^i \pm \varepsilon/2$ of the chosen points for $\varepsilon = 10^{-2}$. At the beginning we perform calculations for the noise-free system (7) with $\alpha = 2.7$ (before the attractor crisis). The obtained results are shown in Figure 7. For $\tau > 25$ all the graphs can be approximated by straight lines and we can estimate their slopes γ_i and coefficients C_i. Our calculations have demonstrated that γ_i correspond to probabilities $P(\Delta_i)$ reasonably well for $\varepsilon = 10^{-2}$. The results are compared in Table 1. Here $\delta_1 = |P(\Delta_i) - \gamma_i|/P(\Delta_i)$, $\delta_2 = |\gamma_i - C_i|/\gamma_i$ are respective deviations of the corresponding quantities.

As seen from Table 1, the calculation results of probabilities $P(\Delta_i)$ as well as the findings for γ and C obtained by approximating (15) the results in Figure 7 coincide with each other within the deviation $\delta < 15\%$. Our special computations have shown that as the interval Δ decreases, this deviation also decreases. One can assume that in the limit $\Delta \to 0$ ($\varepsilon \to 0$) the equality $\gamma = C = P(\Delta)$ can be achieved. In such a case the expression (14) can be rewritten as follows:

$$p(\tau_r) = \gamma\exp(-\gamma\tau_r) = \frac{1}{\langle\tau_r\rangle}\exp\left(-\frac{\tau_r}{\langle\tau_r\rangle}\right). \tag{16}$$

This means that the distribution $p(\tau_r)$ satisfies the Poisson law.

Now we analyze the influence of noise on the distribution $p(\tau_r)$. Our calculations performed for $\alpha = 2.7$ and the noise intensity $D < 10^{-4}$ (before the attractor crisis) have shown that $p(\tau_r)$ obeys the Poisson distribution (16). In the presence of noise the probability measure becomes more smooth. This effect results in the fact that for the

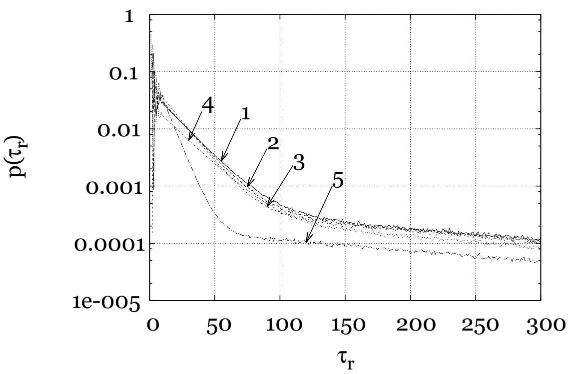

Fig. 8. Distribution densities $p(\tau_r)$ for system (7) in the presence of noise with intensity $D = 10^{-3}$ (after the attractor crisis) for the chosen attractor points (Fig. 2) and for $\alpha = 2.7$ and $\varepsilon = 0.1$.

same ε the return time distribution density $p(\tau_r)$ corresponds to the Poisson law (16) in the presence of noise to a greater extent than without it. The coefficients γ and C (15) are close to the probability values $P(\Delta)$ in the presence of noise. For $\Delta = 10^{-2}$ the deviation does not exceed $\pm 5\%$. Since the influence of noise causes the probability measure to change, the values of $P(\Delta)$ are slightly different from the data in Table 1 for the case $D = 0$. As the noise intensity grows up to $D = 10^{-3}$ when the attractor crisis is realized, the obtained results qualitatively change. The attractor crisis switches the system (7) dynamics into a bistable regime. The system trajectory $\{x_n\}$ spends some period of time in the region of one of the symmetric attractors and then switches to the other one [11]. The system now possesses two characteristic time scales: a local and a global one. The presence of two characteristic time scales can explain the character of the graphs in Figure 8 where the regions of local dynamics ($\tau_r < 100$) and global dynamics ($\tau_r > 100$) are clearly seen. Each of the graphs can be accordingly approximated by two straight lines, and the corresponding coefficients $\gamma_i^{1,2}(x_0^i)$ can be defined.

Our calculations have demonstrated that for $\varepsilon = 0.1$ in the vicinities of all the five points x_0^i the following inequality is fulfilled

$$\gamma^1 > P(\Delta) > \gamma^2. \qquad (17)$$

For example, for $x_0^4 = 1.25$ and $\varepsilon = 0.1$ we have $\gamma^1 = 0.457$, $\gamma^2 = 0.004$, and $P(\Delta) = 0.035$. The fact that the dependences $p(\tau_r)$ (Fig. 8) do not fit the regularity (16) can be explained by the finite value $\varepsilon = 10^{-1}$.

Now we consider the calculation results of $p(\tau_r)$ for the particular point $x_0 = 1.25$ ($i = 4$) and for different values of ε. These findings are presented in Figure 9. As can be seen from the graphs, as ε decreases, the influence of bistability noticeably diminishes and practically disappears for $\varepsilon = 10^{-3}$. With this, the values of γ^1 and γ^2 also decrease and approach $P(\Delta)$ in magnitude. For instance, for $\varepsilon = 10^{-3}$ we have got $\gamma^1 = 0.0043$, $\gamma^2 = 0.00033$, and $P(\Delta) = 0.00035$. It can be assumed that in the limit $\varepsilon \to 0$ the equality $\gamma^1 \simeq \gamma^2 \simeq P(\Delta)$ will take place. Note that

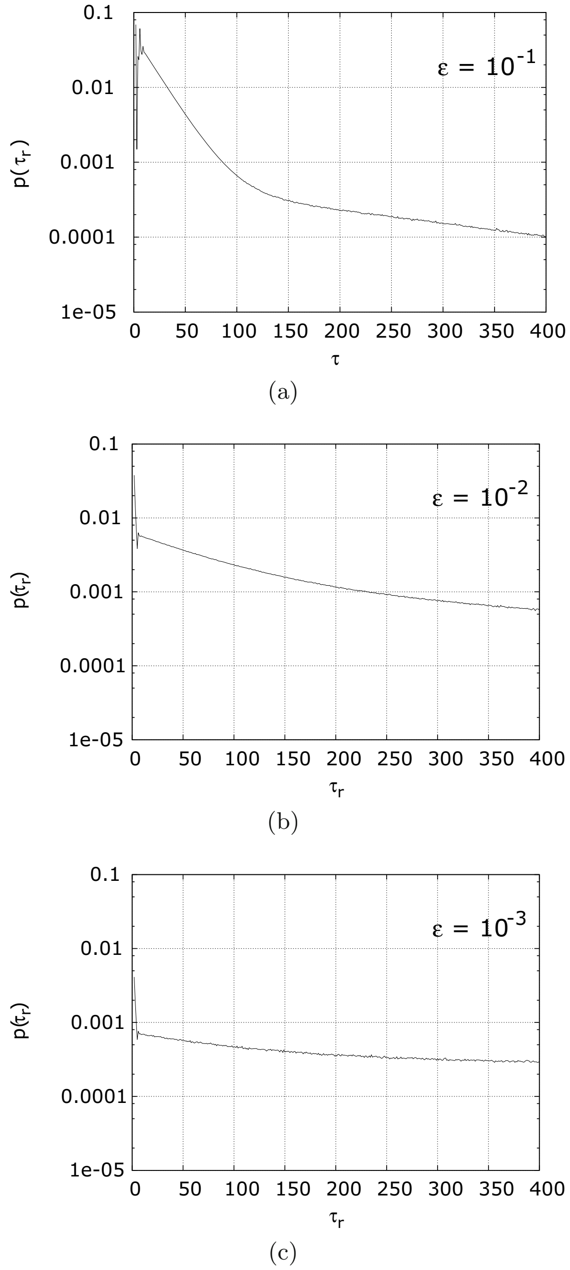

Fig. 9. Evolution of probability density $p(\tau_r)$ for different values of ε for the attractor point $x_0^4 = 1.25$ at $\alpha = 2.7$ and $D = 10^{-3}$.

in this case $P(\Delta) \to 0$ and the return times dramatically increase.

5 Conclusion

The following conclusions can be made as a result of our studies.

1. The studies of return times into an ε-vicinity of a given point for nonhyperbolic map (7) in a chaotic regime when the system control parameter and the noise intensity are varied have shown the full correspondence with Kac's theorem: for sufficiently small $\varepsilon \ll 1$ the

mean Poincaré return time is defined by the probability measure (1) for both the deterministic and noisy system.
2. The influence of noise can lead to nonlinear changes in the probability measure and thus, the mean Poincaré return times can both decrease and increase. This effect depends on the noise intensity and on the chosen neighborhood of an attractor point.
3. The distribution of Poincaré recurrences $p(\tau_r)$ both without and in the presence of noise corresponds to the Poisson distribution (16) before the attractor crisis. For $\varepsilon = 10^{-2}$ deviations do not exceed 5–15% and decrease if $\varepsilon < 10^{-2}$.
4. The mean minimal return time $\langle \tau_{r\inf} \rangle$ for the chaotic system under study corresponds to the dependence (6) being typical for systems with nonzero topological entropy. The transition to a bistable regime is accompanied by the increase of the AP-dimension.
5. In the bistable regime for small but finite $\varepsilon \simeq 10^{-1}$ the dependence $\ln p(\tau_r)$ demonstrates two slopes that correspond to small and large return times (see Fig. 9). This effect is related with the presence of two characteristic time scales in the system. However, this phenomenon dissapears when ε decreases ($\varepsilon \leq 10^{-3}$), and the distribution $p(\tau_r)$ approximates to the Poisson law (16).

The first three conclusions can be considered as a detailed numerical confirmation of the well-known mathematical results that are also valid for systems with a higher dimension $N \geq 2$. In this sense their detailed and consistent description has a methodical and educational importance.

The fact that relation (6) is fulfilled for the system under study is not obvious and requires to be experimentally verified. The obtained result has enabled us to formulate the fourth conclusion. One can give examples of chaotic systems for that relation (6) is not fulfilled. The AP-dimension being the dimension of a return time sequence has its independent value and is not directly related with the local approach on the base of Kac's theorem. This is a global characteristic of the set and can be treated as the complexity measure of a random sequence of Poincaré recurrences. This conclusion is verified by the result that the AP-dimension increases after the attractor crisis when the system dynamics becomes more complicated.

Let us discuss the fifth conclusion. The presence of two slopes in the distribution $p(\tau_r)$ for finite $\varepsilon < 1$ (Fig. 8) has a fundamental applied importance in studing the effect of stochastic resonance in bistable systems. Calculations of $p(\tau_r)$ in bistable systems for finite values of ε enable us to analyse and control the charactetistic switching time by changing a control parameter as well to diagnose the regime of stochastic resonance. In this case no contradictions with the theory arise because the distribution tends to the Poisson law in the limit $\varepsilon \to 0$.

The second conclusion and result is also important from the applied point of view. In paper [7] the effect and mechanism of the return time decrease are discussed in the Henon map in the presence of noise. No results are presented there that would be connected with the analysis of probability measure. Our studies on the base of the probability measure analysis show that the probability measure changes in the presence of noise and as a consequence, the return times can both decrease and increase.

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