Recurrence-based detection of the hyperchaos-chaos transition in an electronic circuit

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Some complex measures based on recurrence plots give evidence about hyperchaos-chaos transitions in coupled nonlinear systems [E. G. Souza *et al.*, "Using recurrences to characterize the hyperchaos-chaos transition," Phys. Rev. E **78**, 066206 (2008)]. In this paper, these measures are combined with a significance test based on twin surrogates to identify such a transition in a fourth-order Lorenz-like system, which is able to pass from a hyperchaotic to a chaotic behavior for increasing values of a single parameter. A circuit analog of the mathematical model has been designed and implemented and the robustness of the recurrence-based method on experimental data has been tested. In both the numerical and experimental cases, the combination of the recurrence measures and the significance test allows to clearly identify the hyperchaos-chaos transition.
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Dynamical systems show a wide range of complex behavior depending on certain order parameters. When the mathematical model of a dynamical system is known, the Lyapunov spectrum can be calculated in a quite accurate way, allowing the extraction of information needed to characterize the system's behavior. In fact, one positive Lyapunov exponent is related to a chaotic behavior, while two or more positive exponents are a signature of hyperchaos.[1] However, in many applications the mathematical model is not available, but only time series are observable. In these cases, the analysis of recurrences can give important insights for the detection of transitions in the dynamical behavior. The aim of this paper is to detect a hyperchaos-chaos transition numerically and experimentally in a fourth-order Lorenz-like system, which, by varying the value of a single parameter, can exhibit periodic, chaotic, and even hyperchaotic oscillations. Starting from recently introduced recurrence-based measures[2] and using the numerical equations, the hyperchaos-chaos transition is identified and confirmed by a statistical test. Moreover, the same recurrence method is applied to a suitably designed and implemented electronic circuit able to mimic the behavior of the considered dynamical model.

I. INTRODUCTION

The analysis of recurrences of a trajectory of nonlinear systems in the phase space can often give substantial insights for understanding their dynamical properties. A recurrence occurs when the trajectory recurs to a neighborhood of formerly visited states in the phase-space. Recurrence plots (RPs) (Ref. 3) are efficient graphical tools able to unveil whether or not the system visits similar states.

It has been shown that the characterization of lines in RPs allows to determine whether the system behaves periodically, chaotically, or stochastically without any knowledge on the mathematical model.[4] Typical parameters as correlation dimension and entropy can also be evaluated through a recurrence quantification analysis.[5]

Some studies have shown that recurrence-based analyses are able to identify bifurcation points, especially chaos-order transitions.[6] Moreover, measures of complexity based on the recurrence time have been recently introduced even for a quantitative analysis of the synchronization of trajectories on strange nonchaotic attractors (SNAs).[7,8] These measures, able to detect transitions, through different routes, from regular to chaotic motion via SNAs have been also tested on experimental data acquired from suitably designed nonlinear circuits. The analyzed circuits, in fact, show transitions to SNAs for increasing values of a single system parameter.[8]

The hyperchaos-chaos transition can be encountered in many domains of life science,[9] such as, for example, in fluid dynamics,[10] dynamics on complex networks,[11] transition from phase to lag synchronization,[12,13] and so on. This hyperchaos-chaos transition can also be characterized by a recurrence-based analysis. In Ref. 2 a recurrence quantification analysis is proposed in order to detect such a transition starting from the time series generated by a system consisting of two coupled subunits. This system behaves hyperchaotically when the coupling strength is not sufficiently high for achieving synchronization; when synchronization occurs, a transition to simple chaotic behavior occurs. In Ref. 2 through numerical simulations it has been demonstrated that the approach proposed is able to detect such a transition. Moreover, the authors of Ref. 2 claim that the proposed recurrence-based measures are sensitive enough to detect the

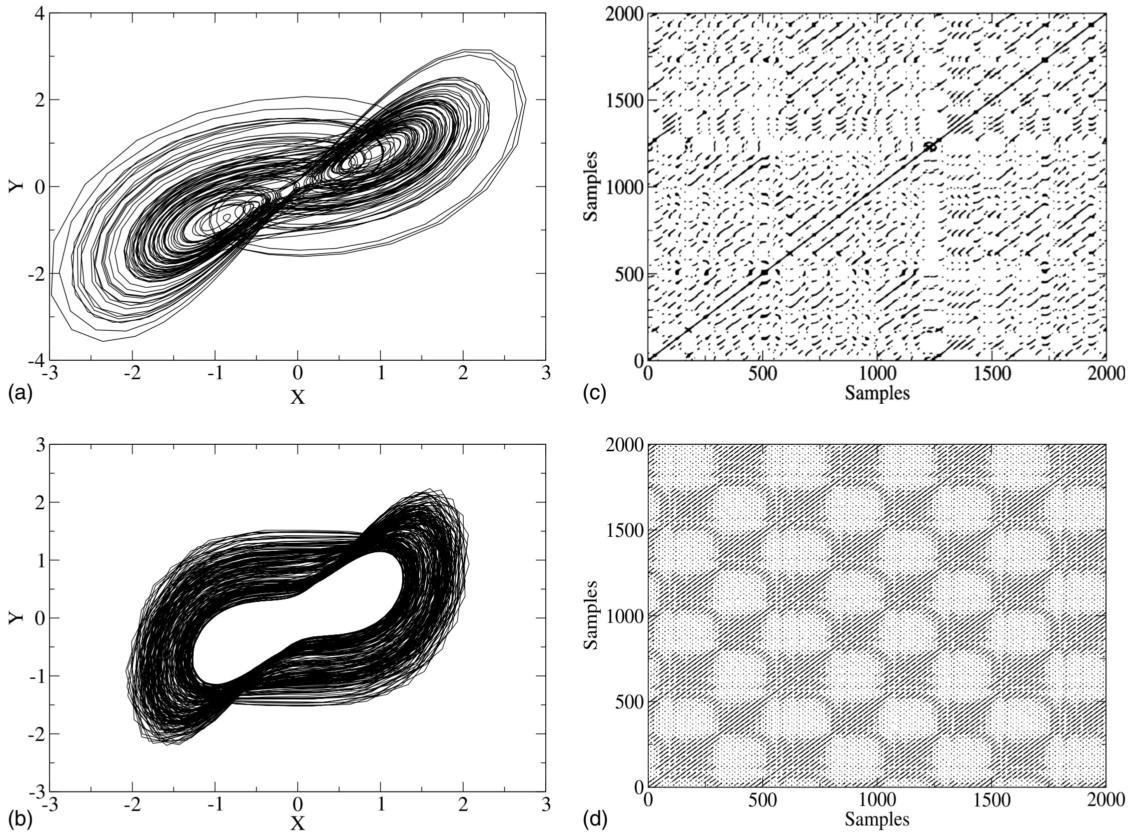

FIG. 1. Phase-plane X-Y and recurrence plots for the model [Eq. (6)]. The plots represent the different regimes shown by the system at two different values of the parameter c. [(a) and (c)] $c=90$ (hyperchaos); [(b) and (d)] $c=270$ (chaos).

dimension reduction that characterizes such a hyperchaos-chaos transition even in dynamical systems not necessarily formed by only two coupled units.

In this paper the detection of the hyperchaos-chaos transition through the application of the recurrence quantification analysis is studied in a Lorenz-like system, which exhibits, in dependence on a control parameter, either hyperchaotic or relatively simple chaotic regimes.[14] Both the mathematical model and an experimental implementation of the considered hyperchaotic system are investigated. The analysis of the time series obtained by integrating the model allows to test the capability of disclosing the hyperchaos-chaos transition in a system that does not exhibit a synchronization subspace. Furthermore, the problem of a statistical validation of this transition is dealt with by using an approach based on a twin surrogate significance test. The results allow to assess that the transition is effectively detected by the recurrence measures. Finally, using the data acquired from a real circuit, the robustness of the proposed technique is investigated.

The paper is organized as follows. In Sec. II, the recurrence-based approach for the characterization of a hyperchaos-chaos transition is discussed. In Sec. III the application of the approach on the mathematical model of the considered hyperchaotic system is discussed. In Sec. IV the experimental realization of the system under study is given and the results obtained from experimental data are shown. Section V draws some concluding remarks.

II. CHARACTERIZATION OF THE HYPERCHAOS-CHAOS TRANSITION

The approach, discussed in this paper, to detect the hyperchaos-chaos transition is based on RPs and twin surrogates. RPs allow to visualize recurrences to a certain state performed by a given trajectory $\{\vec{\mathbf{x}}_i\}_{i=1}^{N}$ of a dynamical system. To achieve this, a $N\times N$ matrix R is calculated. The elements of R are defined as

$$R_{i,j} = \Theta(\delta - \|\vec{\mathbf{x}}_i - \vec{\mathbf{x}}_j\|), \quad i,j = 1, \ldots ,N, \tag{1}$$

where $\vec{\mathbf{x}}_i \in \mathbb{R}^n$, δ is a threshold value, $\Theta(\cdot)$ is the Heaviside function, and $\|\cdot\|$ denotes a norm. Nonzero elements of the matrix R identify similar (δ closer) states, while zero elements represent rather different states. The matrix R can be graphically represented in a two dimensional plot identifying "1" and "0" elements, respectively, as black and white pixels. In this study, we use a fixed value of the threshold $\delta=0.2$ with normalized data and the maximum norm.

In order to have a quantitative definition of the distribution of points in a RP, the recurrence rate R_r is defined as the probability that a recurrence occurs,

$$R_r = \frac{1}{N^2} \sum_{i,j=1;i\neq j}^{N} R_{i,j}. \tag{2}$$

Diagonal lines in the RP occur when a whole segment of the trajectory runs δ-near parallel to another segment. The exis-

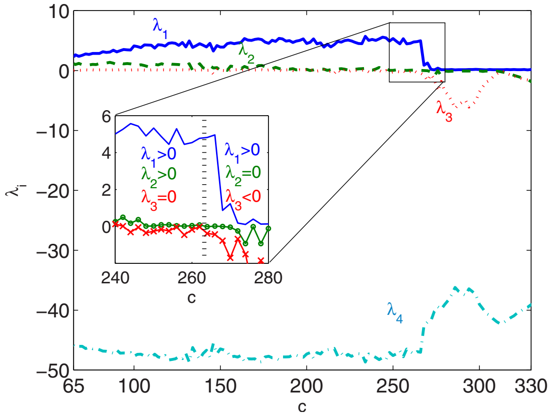

FIG. 2. (Color online) Lyapunov spectrum of the system [Eq. (6)] for different values of the parameter c.

tence of such structures corresponds to the case in which segments of the trajectory evolve visiting the same region of the phase space at different times. The measure referred as *determinism* (DET) given by diagonal lines can be expressed in relationship to their length. The longer diagonal lines, in fact, correspond to wider time intervals during which the trajectory runs parallel to previous segments. Defining with $P(l)$ the frequency distribution of the lengths l of diagonal lines in the RP, the measure determinism is defined as

$$\mathrm{DET} = \frac{\sum_{l=l_{\min}}^{N} lP(l)}{\sum_{l=1}^{N} lP(l)}. \tag{3}$$

In this work, $l_{\min}=2$ is chosen as the minimum length allowed for a diagonal line. The maximum length $L_{\max}$ of the existing diagonal lines, except the main diagonal line, will be also considered in the present analysis.

The last indicator to detect a hyperchaos-chaos transition is the average diagonal length L defined as

$$L = \frac{\sum_{l=l_{\min}}^{N} lP(l)}{\sum_{l=l_{\min}}^{N} P(l)}. \tag{4}$$

The transition from a hyperchaotic to a chaotic behavior is characterized by a decrement of the system's complexity, since the Lyapunov spectrum passes from at least two to one positive exponent. In order to identify this, the four previously introduced recurrence measures have to be evaluated while varying a bifurcation parameter. An abrupt change in their values has to be observed at the critical value of the parameter at which the behavior of the system changes. However, other peaks may appear, so that the transition needs to be validated with a statistical test. In order to validate that the detected transition actually occurs at the

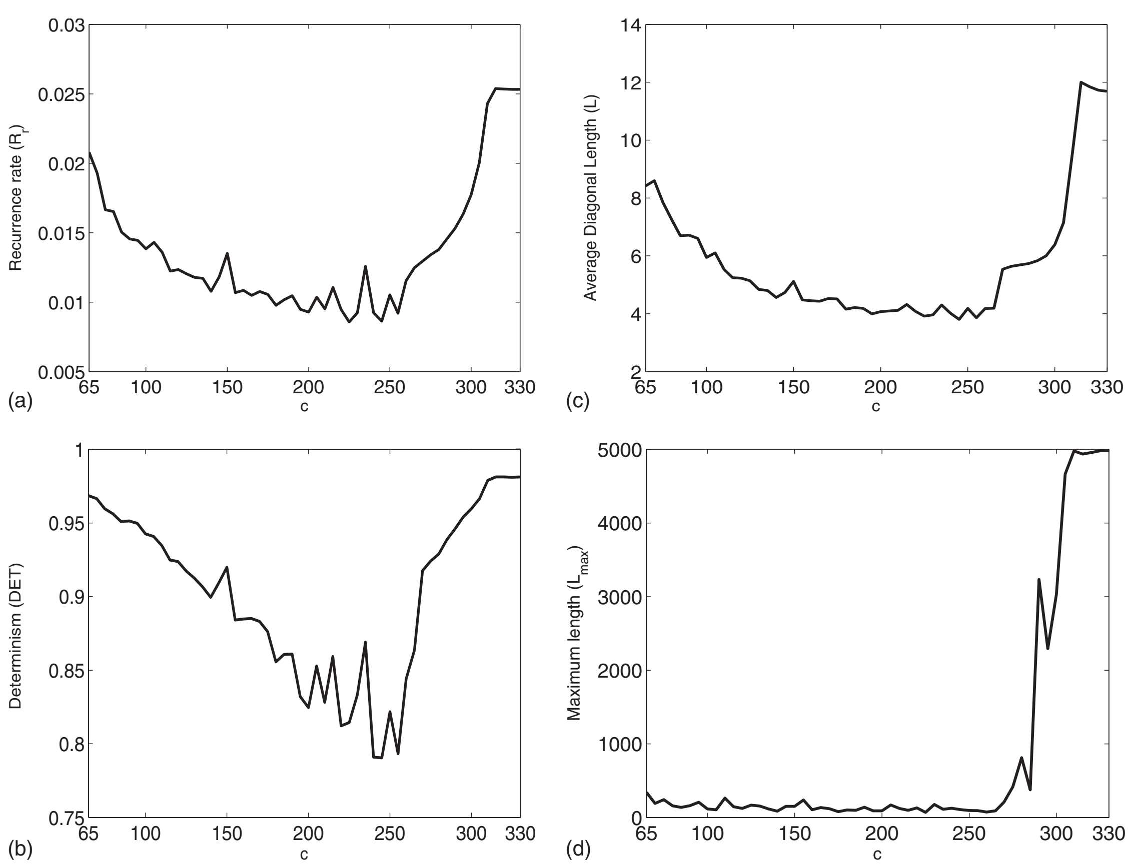

FIG. 3. Recurrence measures for the characterization of the hyperchaos-chaos transition as a function of the system parameter c. (a) Recurrence rate R_r, (b) determinism DET, (c) average diagonal length L, and (d) maximum diagonal length $L_{\max}$. All the four diagnostics abruptly change at $c \approx 265$, the value at which the system begins to exhibit only one positive Lyapunov exponent.

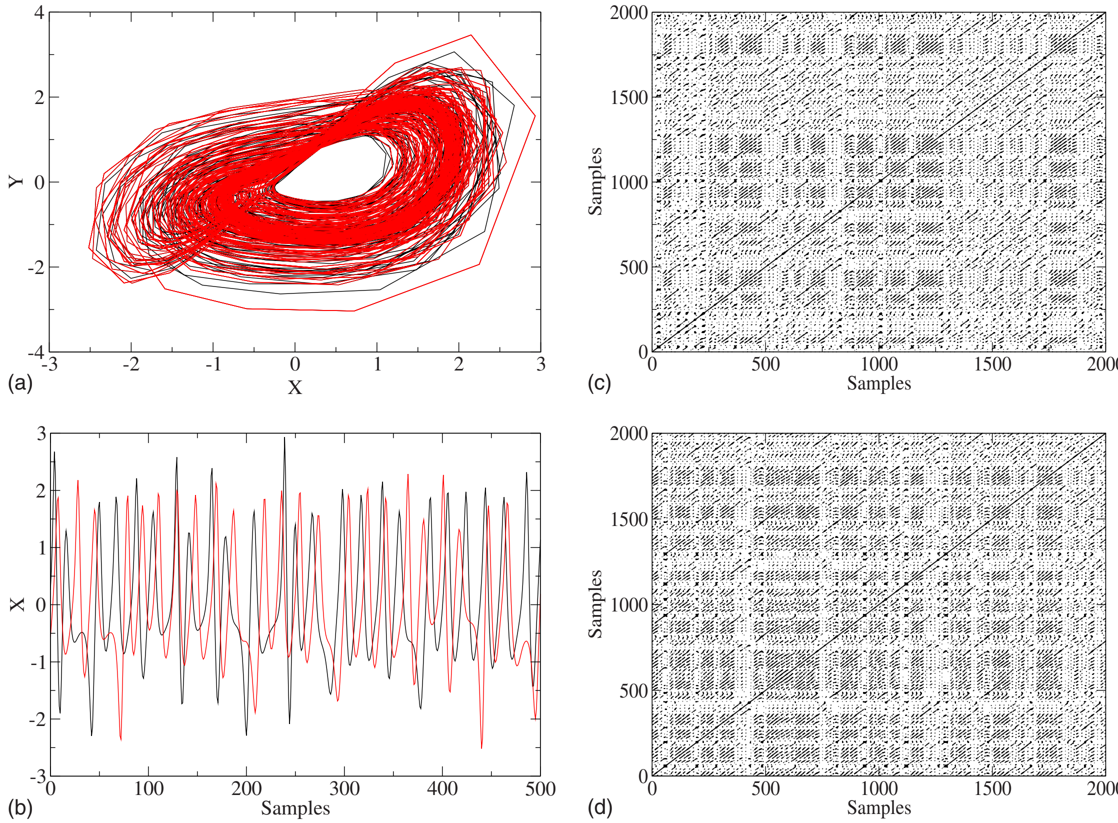

FIG. 4. (Color online) An attractor of the system for the parameter value $c=260$ and a twin surrogate. (a) Phase space; (b) 500 samples of the time series of the original trajectory and the twin surrogate. [(c) and (d)] Recurrence plots respectively of the original trajectory and his twin surrogate.

predicted value, the recurrence-based measures should be repeated for trajectories generated starting from different initial conditions. In this paper we develop a significance test which is effective also when only a single measured data series is available. To achieve this purpose, an approach based on twin surrogates (TSs), which are trajectories corresponding to the same underlying system but starting at different initial conditions, is used. These surrogates are generated from recurrence properties as in Ref. 15, where they have been applied to test the effectiveness of synchronization in data series.

The first step to generate such surrogates is to find in the RP of the trajectory of the underlying system those points which are not only neighbors but which also share the same neighborhood in phase space. Those points are called twins and they typically do exist, because in the RP it is possible to find identical columns. Once the twins have been localized, an arbitrary starting point for the surrogate trajectory is chosen. The surrogate trajectory is then generated by substituting randomly the next step in the trajectory by either its own future or the one of its twin. The surrogates mimic closely the basic dynamical properties of the underlying system, as will be shown in Sec. III with a numerical example.

Let us indicate with $\bar{c}$ the critical value of the bifurcation parameter, derived by the analysis of the RP measures. In order to validate that the transition occurs at this point, let us consider the behavior at two different values of the bifurcation parameter, before and after the hypothetical transition. We indicate as $c_m=\bar{c}-\Delta c$ and $c_M=\bar{c}+\Delta c$ (where Δc is the step size adopted for the bifurcation parameter variations) the values of the bifurcation parameter one step before the transition and one step after the transition, respectively. At this point, N_{TS} twin surrogates are generated for the trajectory of the system with $c=c_m$, and for each surrogate, the four recurrence measures are computed. These recurrence measures are also computed for the original trajectory for $c=c_M$, where the term original trajectory refers in the numerical case to a trajectory directly extracted from the integration of the model under examination and in the experimental case to the data acquired at this value. Our null hypothesis is that the values of the recurrence measures remain unchanged for different parameter values, especially before and after the critical value, such that, if we compute the recurrence measures for different trials enabled by the twin surrogates and we always find that the values of the recurrence measures before and after the critical value are different, then we can reject the null hypothesis and conclude that there is a transition at the value $c=\bar{c}$.

In order to quantify how far are the values of the recurrence measures obtained for the original data from the distribution of values obtained for the twin surrogates, the following parameter has been calculated:

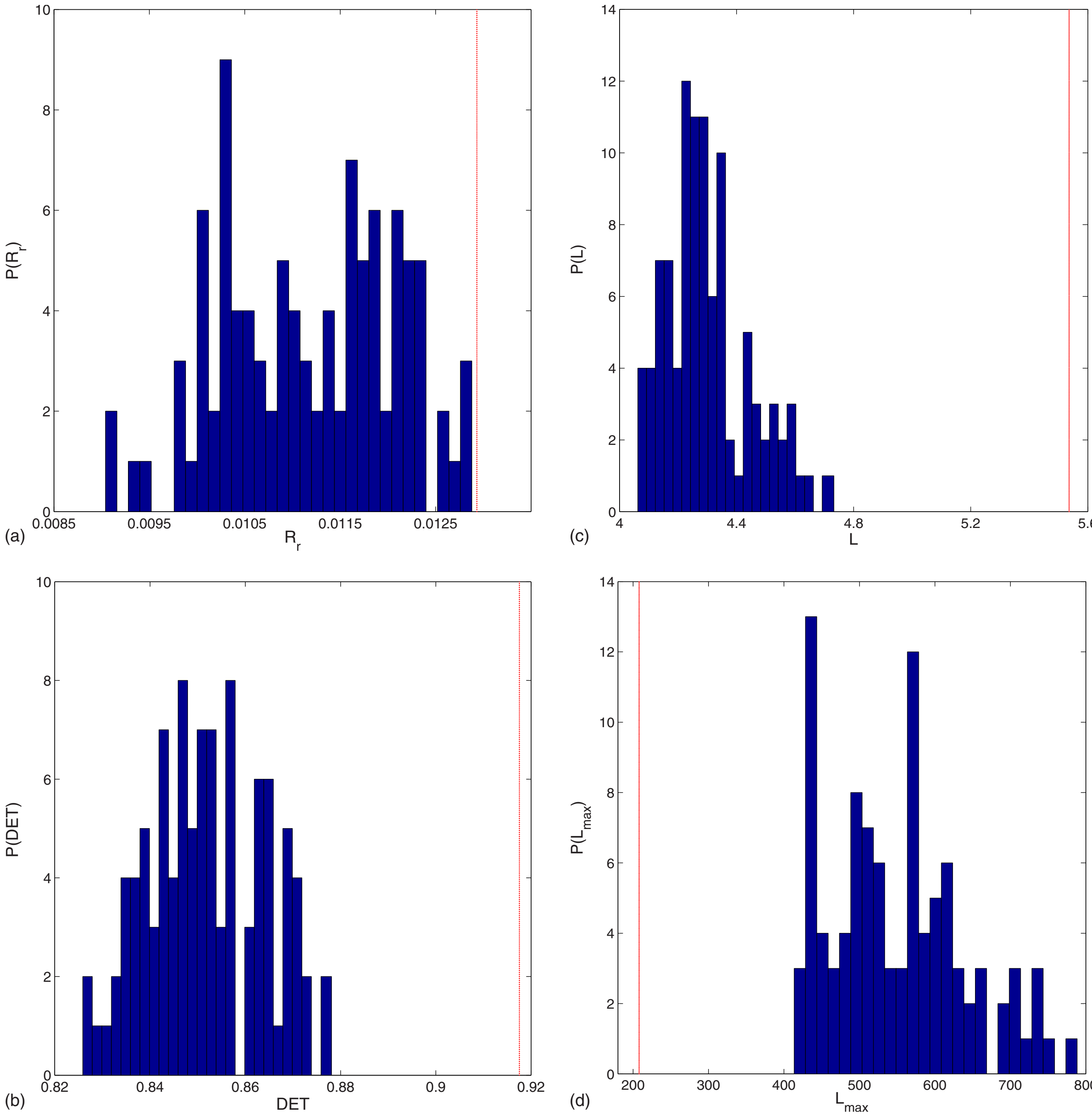

FIG. 5. (Color online) Histogram of the values of the recurrence measures obtained for 100 twin surrogates of the trajectory for the parameter $c=260$. The vertical line indicates the values of the recurrence measures obtained for the original trajectory for the parameter $c=270$.

$$\alpha = \frac{|\tilde{m} - V_{\mathrm{od}}|}{\tilde{\sigma}}, \tag{5}$$

where $\tilde{m}$ and $\tilde{\sigma}$ are, respectively, the mean value and the standard deviation of the distribution of values obtained for the twin surrogates, and V_{od} is the value of the recurrence measures obtained for the original data.

III. NUMERICAL ANALYSIS OF THE HYPERCHAOS-CHAOS TRANSITION IN A LORENZ-LIKE HYPERCHAOTIC SYSTEM

We consider the fourth-order dynamical system described by the following dimensionless equations:[14]

$$\begin{aligned}
\dot{x} &= a(y - x) + yz, \\
\dot{y} &= cx - xz - y - \tfrac{1}{2}w, \\
\dot{z} &= xy - 3z, \\
\dot{w} &= \tfrac{1}{2}xz - bw.
\end{aligned} \tag{6}$$

These equations, each characterized by a cross-product term, represent a generalization of the Lorenz system. System (6) is able to show a hyperchaotic but also a simply chaotic behavior. The numerical bifurcation analysis of Eq.

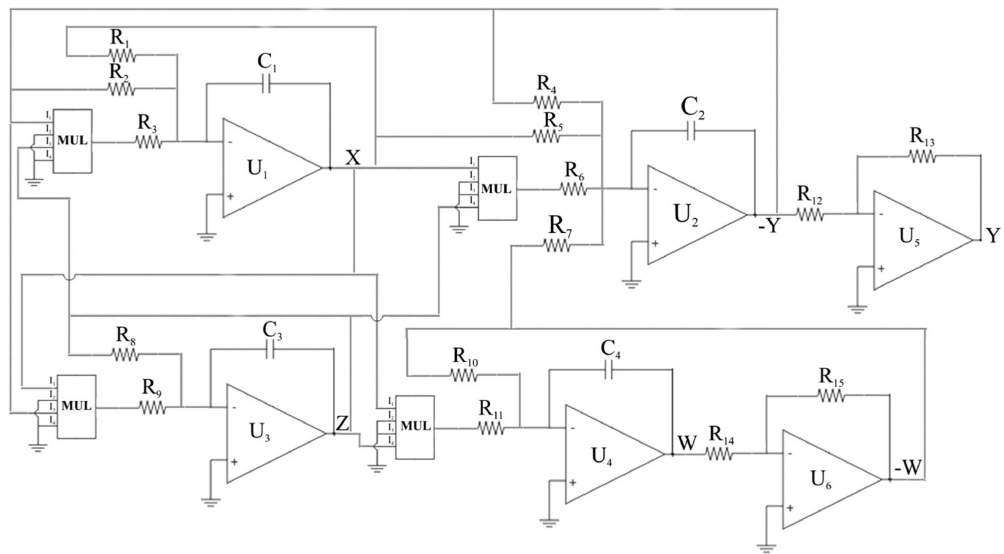

FIG. 6. Schematic representation of the considered hyperchaotic circuit. The values of the components are the following: $R_1=R_2=2.5\ \mathrm{k}\Omega$, $R_3=R_6=R_9=200\ \Omega$, $R_4=100\ \mathrm{k}\Omega$, $R_5=1\ \mathrm{k}\Omega$, $R_7=25\ \mathrm{k}\Omega$, $R_8=33.2\ \mathrm{k}\Omega$, $R_{10}=66.9\ \mathrm{k}\Omega$, $R_{11}=1.6\ \mathrm{k}\Omega$, $R_{12}=R_{13}=5.6\ \mathrm{k}\Omega$, $R_{14}=R_{15}=560\ \Omega$, and $C_1=C_2=C_3=C_4=100\ \mathrm{nF}$.

(6) has been recently performed in Ref. 14, disclosing the regions of the parameter space in which it evolves along periodic, chaotic, or hyperchaotic trajectories. In particular, when $a=40$, $b=-1.5$, and $c=90$, the system exhibits two positive Lyapunov exponents. This leads to the hyperchaotic behavior shown in Fig. 1(a). However, the system is also able to show chaotic behavior (i.e., with exactly one positive Lyapunov exponent) for $a=40$, $b=-1.5$, and $c=270$, as shown in Fig. 1(b). The two types of behavior are characterized by qualitatively and quantitatively strongly different recurrence plots shown in Figs. 1(c) and 1(d).

The hyperchaos-chaos transition is a codimension-1 bifurcation, and it can be demonstrated that the parameter c in Eq. (6) acts as a bifurcation parameter. To show this, the

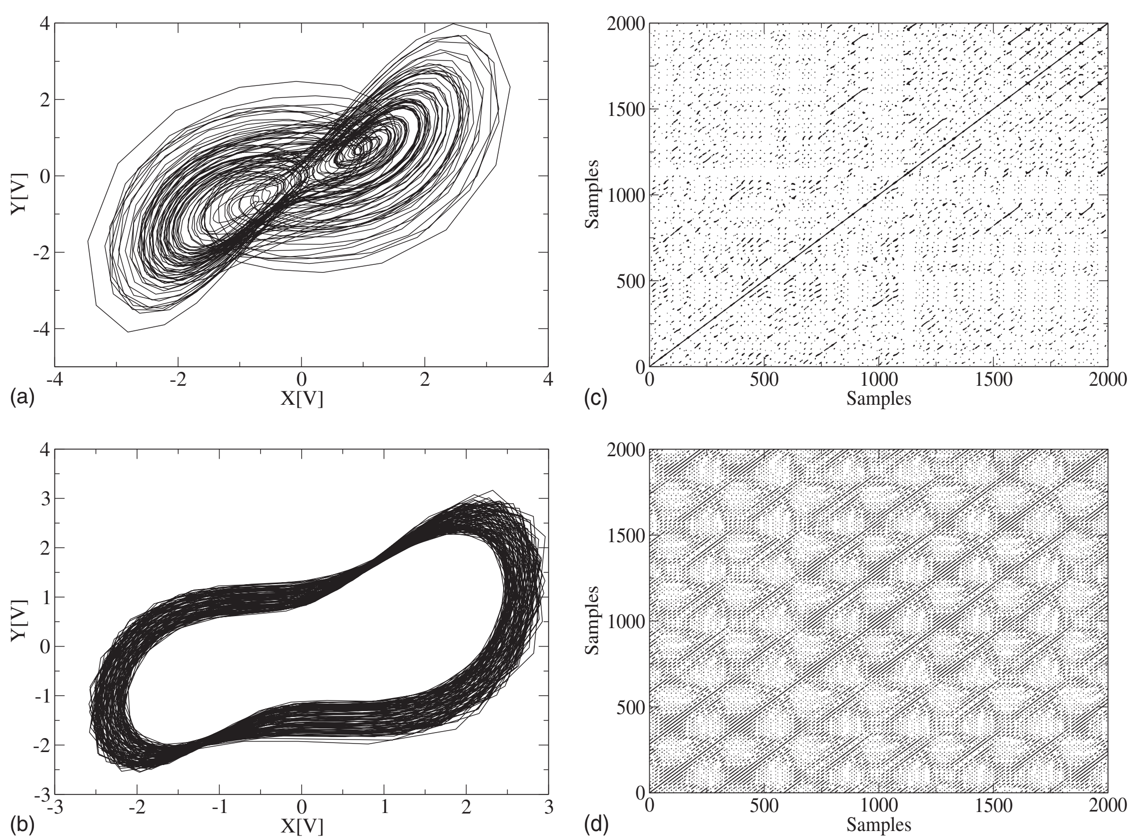

FIG. 7. Phase-plane X-Y and recurrence plots calculated with time series of 2000 samples acquired from the experimental circuit. The plots represent the different behavior shown by the circuit at two different values of the system parameter c: [(a) and (c)] $c=90$ (hyperchaos), [(b) and (d)] $c=270$ (chaos).

Lyapunov spectrum has been calculated with respect to different values of c. The algorithm of Wolf *et al.*[16] has been used. A trajectory of 800 000 samples has been integrated with a fixed step-size integration routine (step size $h=0.001$). The Lyapunov spectrum calculated for values of the parameter c in the range [65,330] shows a transition from hyperchaotic to chaotic behavior at $c \approx 265$ (Fig. 2). For $c<265$, the Lyapunov exponents λ_1 and λ_2 are positive, $\lambda_3=0$, and λ_4 is negative; when c approaches the critical value, λ_1 decreases but remains positive while λ_2 tends to zero and λ_3 becomes negative.

The four RP measures defined in Sec. II, namely, the recurrence rate R_r, the determinism DET, the average diagonal length L, and the maximum diagonal length $L_{\max}$, have been evaluated varying the parameter c in the same range in which the Lyapunov spectrum has been calculated. The plots reported in Fig. 3 allow to identify the critical value of c for which a sudden change occurs, indicating a decrement of the system's complexity in correspondence of the hyperchaos-chaos transition occurring approximately at $\bar{c} \approx 265$.

This value has been confirmed by a significance test based on twin surrogates and described in Sec. II. In particular, in our case $\Delta c=5$, $c_m=260$, and $c_M=270$. $N_{\mathrm{TS}}=100$ surrogates have been generated for $c=c_m$. First of all, we show that these surrogates closely mimic the basic dynamical properties of the underlying system. This is illustrated in Fig. 4, where an attractor of the system (6) for the parameter value $c=c_m$ is shown with his twin surrogate. The attractor is obtained from a time series of length $N=5000$ points calculated with a fourth-order Runge–Kutta integrator with fixed step width $h=0.001$. The sampling time is $\Delta t=0.005$.

Then, the four recurrence measures have been computed for each surrogate as well as for the original trajectory ($c=c_M$). We recall that the null hypothesis of our significance test is that the values of the recurrence measures remain unchanged for different parameter values, especially before and after the critical value. In Fig. 5, the histograms of the values of the recurrence measures, obtained for the N_{TS} twin surrogates, are compared with their values obtained for the original trajectory for $c=c_M$ (vertical line). It can be seen that, in all the four cases, the values of the recurrence measures obtained for the original data are outside the distribution of values obtained for the twin surrogates.

Finally, the parameter α in Eq. (5) has been calculated for each of the RP measures. We have obtained $\alpha=2$, $\alpha=6$, $\alpha=9$, and $\alpha=4$ for the recurrence measures R_r, DET, L, and $L_{\max}$, respectively. This clearly indicates that the null hypothesis can be rejected.

It is important to mention that, contrary to the long trajectories (800 000 samples) required for computing the Lyapunov spectrum (Fig. 2), the RP method has the advantage that it can be also applied when rather short data are available. In this work, in fact, the whole recurrence analysis has been performed with 5000 samples only.

In the rest of the paper, the implementation of a circuital analog of the model under study and the use of the described procedure on related experimental data are discussed in order to show the robustness of the RP-based approach in detecting the hyperchaos-chaos transition from experimental data acquired from this circuit.

IV. EXPERIMENTAL ANALYSIS OF THE HYPERCHAOS-CHAOS TRANSITION IN THE LORENZ-LIKE HYPERCHAOTIC CIRCUIT

The experimental data analyzed in this section have been acquired from an electronic circuit characterized by the same dynamics of the mathematical model described in Sec. III. The electronic circuit reproducing Eq. (6) is reported in Fig. 6. It has been designed following an approach based on the assumption that each state variable is associated with the voltage of a capacitor.[17] The circuit makes use of six operational amplifiers, four of which are connected in a Miller integrator configuration, and four *AD*633 multipliers implementing the nonlinearities of the system. In order to obtain state variables oscillating within suitable voltage supply limits (i.e. ± 15 V), the state variables x, y, and z have been rescaled by a factor $\frac{1}{50}$, while w has been rescaled by a factor $\frac{1}{200}$. The time variable has been rescaled by a factor $k=100$, which also allows a faster observation of the system behavior. The rescaled system reads as follows:

$$\begin{aligned}
\dot{X} &= k(a(Y-X)+50YZ),\\
\dot{Y} &= k(cX-50XZ-Y-4W),\\
\dot{Z} &= k(50XY-3Z),\\
\dot{W} &= k(3.125XZ-bW),
\end{aligned} \tag{7}$$

where $X=x/50$, $Y=y/50$, $Z=z/50$, and $W=w/200$ are the new state variables implemented in the circuit.

The circuit equations that can be easily derived from Fig. 6 are as follows:

$$\begin{aligned}
\frac{dX}{d\tau} &= k\left(\frac{1}{R_2C_1}Y-\frac{1}{R_1C_1}X+\frac{1}{10R_3C_1}YZ\right),\\
\frac{dY}{d\tau} &= k\frac{R_{13}}{R_{12}}\left(\frac{1}{R_5C_2}X-\frac{1}{10R_6C_2}XZ-\frac{1}{R_4C_2}Y\right.\\
&\qquad \left.-\frac{R_{15}}{R_{14}}\frac{1}{R_7C_2}W\right),\\
\frac{dZ}{d\tau} &= k\left(\frac{1}{10R_9C_3}XY-\frac{1}{R_8C_3}Z\right),\\
\frac{dW}{d\tau} &= k\left(\frac{1}{10R_{11}C_4}XZ+\frac{R_{15}}{R_{14}}\frac{1}{R_{10}C_4}W\right),
\end{aligned} \tag{8}$$

where $k\tau=t$. The components of the circuit have been chosen in order to match Eq. (7). The values of the components are given in the caption of Fig. 6.

Operational amplifiers $U1$, $U2$, $U3$, and $U4$ act as algebraic adders and integrators, while $U5$ and $U6$ are inverting buffers. As shown in Fig. 6, the multipliers are driven by four inputs I_1, I_2, I_3, and I_4 with their output given by $V_{\mathrm{mul}}=(I_1-I_2)\cdot(I_3-I_4)/10V$.

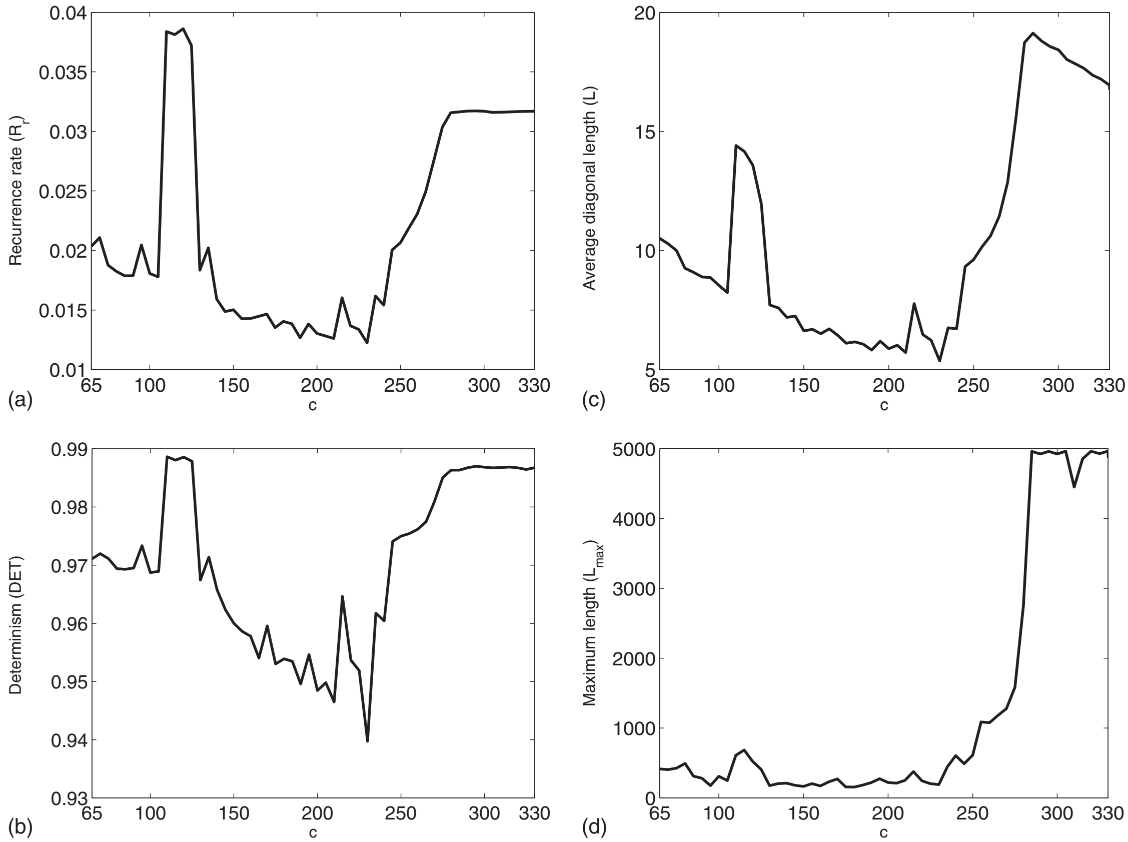

FIG. 8. Recurrence measures applied on the data acquired from the circuit implementation of the system [Eq. (5)] for the characterization of the hyperchaos-chaos transition as a function of the parameter c: (a) recurrence rate, (b) determinism, (c) average diagonal length, and (d) maximum diagonal length.

The hyperchaos-chaos transition can be characterized by recording the trends of the state variables of the circuit for different values of the parameter c and applying the analysis described in Sec. II. In particular we let c vary from $c=65$ to $c=330$ at steps of five.

From Eq. (8) it can be noticed that the parameter c is equal to $1/R_5C_2$. This ratio can be varied either by choosing R_5 as a trimmer or, in order to increase the resolution of our investigation, using an additional multiplier driven by the state variable X and by a constant digitally fixed voltage V_r. The constant voltage V_r and the parameter c are then related through the expression $kc=V_r/R_5C_2$.

All the data have been acquired by using a data acquisition board (National Instruments AT-MIO 1620E) with a sampling frequency $f_s=200$ kHz for $T=2$ s (i.e., 400 000 samples for each time series). In all the acquisitions the other parameters have been chosen, as discussed in Sec. III. The circuit implemented is able to reproduce the dynamical behavior of the mathematical model (6), as shown in Fig. 7(a) illustrating an example of the hyperchaotic behavior experimentally observed, and in Fig. 7(b) illustrating an example of the attractor obtained in the chaotic range of parameters.

The recurrence-based diagnostics defined in Sec. II have then been applied to analyze the experimental data. Only 5000 samples from the 400 000 samples were used. In Figs. 7(c) and 7(d) the recurrence plots computed from the experimental data for $c=90$ and $c=270$ are shown.

From the analysis of the four measures defined in Sec. II, obtained from experimental data and whose trends with respect to increasing values of the parameter c are reported in Fig. 8, the same conclusion found in the mathematical model can be derived. The four recurrence measures undergo a drastic change at $\bar{c}\approx 250$, which is quite close to that found in the numerical analysis. We have thus performed the significance test based on twin surrogates to validate the bifurcation value. In particular, $\Delta c=5$, $c_m=245$, and $c_M=255$. $N_{\mathrm{TS}}=100$ twin surrogates of data at $c=c_m$ have been generated. Then, for each surrogate, the four recurrence measures have been computed and compared with those computed for the experimental data recorded with $c=c_M$. In Fig. 9, the histograms of the values of the recurrence measures, obtained for the N_{TS} twin surrogates, are compared with their values obtained for the data at $c=c_M$ (red vertical line). The comparison shows how the vertical line (obtained for $c=c_M$) lies outside the values obtained for $c=c_m$. The calculation of the parameter α in Eq. (5) confirms that the null hypothesis can be rejected. In fact, we have obtained $\alpha=15$, $\alpha=8$, $\alpha=9$, and $\alpha=3$ for the recurrence measures R_r, DET, L, and $L_{\max}$, respectively.

In Fig. 8 it could be noticed that several peaks appear. In particular, R_r, DET, and L show two major peaks at $c\approx 120$ and $c\approx 210$ and some minor peaks. The peak around $c\approx 120$ indicates a transition from hyperchaos to a more regular behavior. For this parameter value, the attractor exhibited by the circuit is shown in Fig. 10(a). This behavior, not shown by the mathematical model as reported in Fig. 10(b), may be due to tolerances on circuit components. However, the recurrence method is effective in detecting the ex-

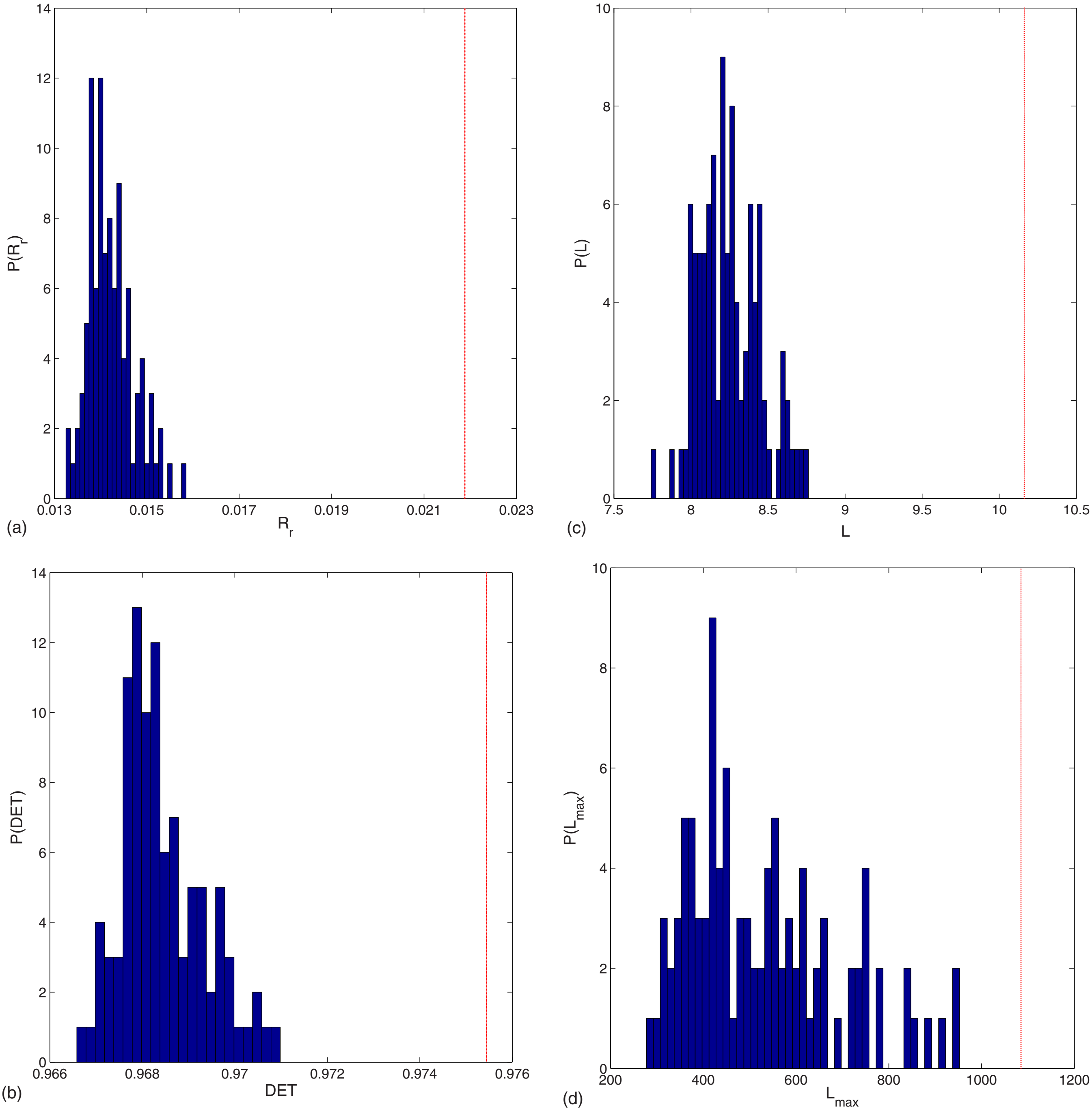

FIG. 9. (Color online) Histogram of the values of the recurrence measures computed for 100 twin surrogates of the data acquired at $c=245$. The vertical line indicates the values of the recurrence measures computed for the data acquired at $c=255$.

istence of this transition. Another peak is visible in Figs. 8(a) and 8(b) for $c \approx 210$. This peak corresponds to an intermittent behavior observed also in the mathematical model for $c \approx 215$, as shown by Figs. 11(a) and 11(b) where the x state variable for both cases is reported. It can be concluded that these peaks are not artifacts of the analysis method, but represent other dynamical behaviors appearing in the circuit. The presence of these major and minor peaks therefore does not constitute a particular problem for the method which is able to deal with them, since major peaks correspond to effective local changes in the dynamics of the system, and minor peaks can be discarded according to the significance test based on twin surrogates.

The existence of parameter mismatches due to tolerances on circuit components does not significantly affect the detection of the transition. Furthermore, the method reveals its robustness to measurement errors allowing the identification of the critical value of the parameter at which the dynamical behavior of the observed system changes from hyperchaos to chaos.

We have then calculated the Lyapunov spectrum from the experimental data and compared the results obtained with

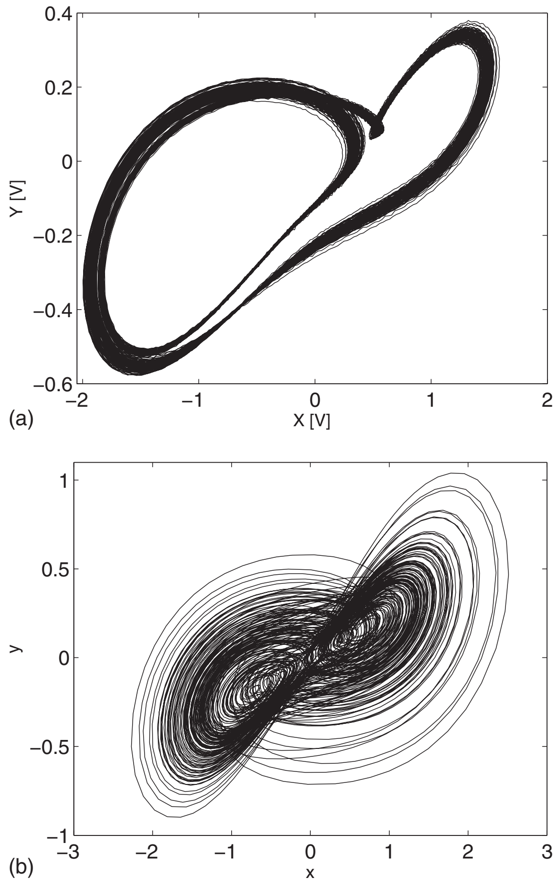

FIG. 10. Phase-plane X-Y for $c=120$. Behavior exhibited by (a) the circuit of Fig. 6 and (b) the mathematical model [Eq. (6)].

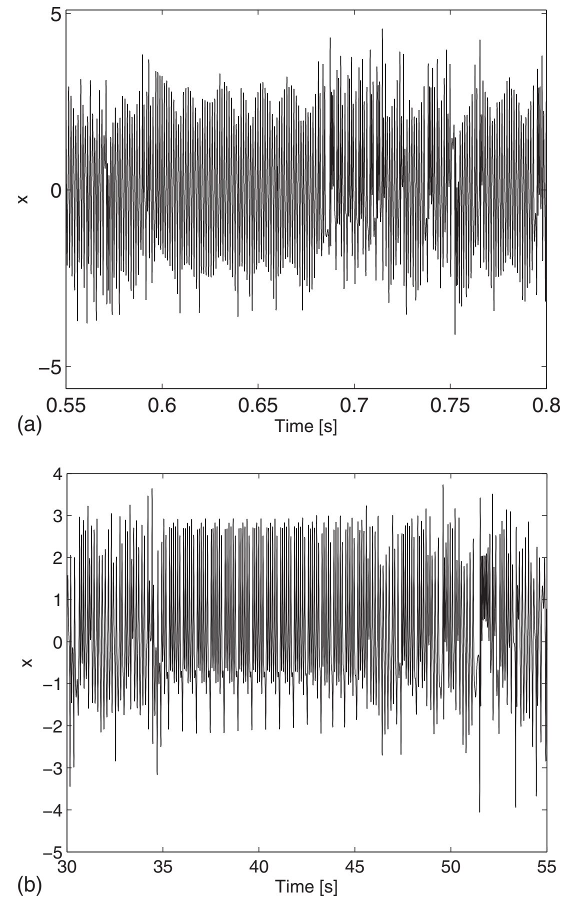

FIG. 11. Intermittent behavior of the state variable x. Time series related to (a) the circuit of Fig. 6 for $c=210$ and (b) the mathematical model [Eq. (6)] for $c=215$.

the analysis based on recurrence measures. The Lyapunov spectrum has been calculated by taking into account the same number of points ($N=5000$) of the RP-based analysis and by using the measurements of all the four state variables of the circuit. The TISEAN package[18] has been used for this purpose. The Lyapunov spectrum calculated on experimental data is shown in Fig. 12. We have insight in the system dynamics by monitoring mainly the exponent λ_2, since λ_1 remains positive while λ_3 and λ_4 remain negative during the transition: ideally $\lambda_2>0$ in the hyperchaotic region and $\lambda_2=0$ in the chaotic region. In the practice, the value of λ_2 is quite small in the whole parameter region and identifying the transition is quite difficult. The value of λ_2 becomes small and negative for $c\geq 230$. Therefore, if the same number of points in the trajectory is used, the Lyapunov spectrum is much less reliable than the RP method. Longer trajectory may be required for a more accurate identification of the transition when using the Lyapunov spectrum.

V. CONCLUSIONS

In this paper, the four recurrence-based measures introduced in Ref. 2 have been used in combination with a statistical method based on twin surrogates generated from recurrence properties in order to detect the hyperchaos-chaos transition in nonlinear systems. The approach has been applied numerically and experimentally, and in both cases the method was efficient in detecting the transition.

The considered recurrence measures exhibit a sudden change when the Lyapunov spectrum of the system passes from two to one positive Lyapunov exponent. A significance test has been then applied to validate the transition point and to discard minor peaks which may appear in the recurrence measures. The effectiveness of the detection method has been proven through the analysis of the mathematical model of a fourth-order generalization of the classic Lorenz system. In this case a drastic jump is observed in each of the recur-

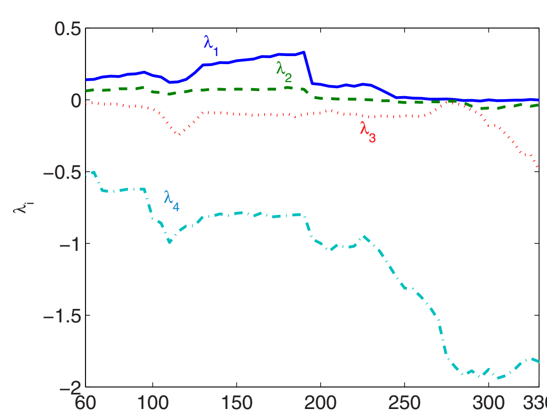

FIG. 12. (Color online) Lyapunov spectrum computed using the data acquired from the circuit of Fig. 6 for different values of the parameter c.

rence measures, at the value at which the system begins to show a chaotic behavior, and the statistical test validates the estimated transition point.

The experimental part of the paper has been achieved through a circuit realization referring to the considered mathematical model. The circuit, based on operational amplifiers, has been designed and implemented in order to obtain an electrical analog of this mathematical model. The waveforms generated by the circuit have been acquired and the experimental data have been analyzed using the same recurrence-based measures. Even if the tolerance on electrical components introduces some parameter mismatches, the detection method proves its robustness. The hyperchaos-chaos transition can be identified with a good matching between the recurrence measures computed from the numerical and experimental data.

This result confirms that the recurrence-based approach is able to correctly identify transitions between different complex behaviors not only when numerically generated data are available, but also when data are obtained from observations from real systems, whose model is not accessible as, for example, it may happen in fluid dynamics, biological systems, and so on. The proposed approach has the advantage to be still applicable when only rather short data are available.

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