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Spectral universality of phase synchronization in non-identical oscillator networks

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Abstract. We employ a spectral decomposition method to analyze synchronization of a non-identical oscillator network. We study the case that a small parameter mismatch of oscillators is characterized by one parameter and phase synchronization is observed. We derive a linearized equation for each eigenmode of the coupling matrix. The parameter mismatch is reflected on inhomogeneous term in the linearized equation. We find that the oscillation of each mode is essentially characterized only by the eigenvalue of the coupling matrix with a suitable normalization. We refer to this property as spectral universality, because it is observed irrespective of network topology. Numerical results in various network topologies show good agreement with those based on linearized equation. This universality is also observed in a system driven by additive independent Gaussian noise.

PACS. 05.45.Xt Synchronization; coupled oscillators – 89.75.Hc Networks and genealogical trees

1 Introduction

One of the most important subject in recent network science [1–3] is to understand the interplay between its functionality and network topology such as small-world [4,5], scale-free [6], etc. The eigenvalue decomposition of the matrix defining the network topology provides much information about the network structure [7]. For example, universality in spectral properties of adjacency and coupling matrices has been reported [8–10].

Another important topic in this field is to study synchronization of oscillators embedded on network, because synchronizability is deeply related to network topology [3]. A sufficient condition for frequency synchronization in the Kuramoto model network has been investigated [11]. Moreover, mechanisms of phase synchronization [12–17] as well as complete synchronization [18–20] in chaotic oscillator networks have been investigated [21–23].

The spectral decomposition method is applicable to synchronization analysis. The relationship between frequency synchronization and spectral properties of the coupling matrix in the Kuramoto model has been reported [24,25]. Another important application is the estimation of the transition to complete synchronization of a coupled identical chaotic oscillator network using the master stability function, the largest Lyapunov exponent for each eigenmode of the coupling matrix [26–28]. However, it is unclear whether such a mode decomposition is applicable to phase synchronization of non-identical chaotic systems. Such an extension is required, e.g., to extend the discussion of the effect of parameter mismatch or noise on on-off intermittency [29–31] to complex networks.

Noise can have constructive influence on complex systems. A prominent phenomenon is stochastic resonance [32]. Synchronization can be influenced also by noise [33]. One result is that common noise can induce complete synchronization [34]. Synchronization on oscillator networks driven by noise should be studied more. Note that one has to be careful in estimating the phases in case of stochastic systems [35,36].

In the present work, we apply the spectral decomposition technique to non-identical oscillator networks and show a universal property. This is characteristics of phase synchronization close to complete synchronization irrespective of the network topology. We show that this phenomenon is also observed in a network when independent Gaussian noise is imposed in each oscillator.

The present paper is organized as follows. In Section 2, we describe the spectral decomposition method and apply it to a Rössler oscillator network. Numerical results are approximated by the linearized equation derived in Section 3. Section 4 is devoted to show a universal property of phase synchronization close to complete synchronization independent of the network structure. Our method is applicable to a stochastic system driven by additive Gaussian noise (Sect. 5). Finally, our results are summarized and discussed in the last section.

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2 Spectral decomposition analysis

In this section, we describe how to decompose time series into eigenmodes of the coupling matrix. In Sections 2–4, we consider a network of N oscillators governed by the deterministic evolution equation

$$\dot{\boldsymbol{X}}_i = c_i \boldsymbol{f}(\boldsymbol{X}_i) + \boldsymbol{F}(\boldsymbol{X}_i) - \sigma \sum_{j=1}^{N} G_{ij} \boldsymbol{H}(\boldsymbol{X}_j). \quad (1)$$

Here, $\boldsymbol{X}_i$ denotes the M dimensional dynamical variable at ith node. Common part of the local dynamics is denoted by $\boldsymbol{F}(\boldsymbol{X})$, and difference between the oscillators is represented by $c_i \boldsymbol{f}(\boldsymbol{X})$. The difference is characterized by one parameter c_i. The interactions between the oscillators are represented by the coupling matrix G_{ij} with zero eigenvalue. The coupling strength is characterized by σ, and $\boldsymbol{H}(\boldsymbol{X})$ is an arbitrary function. The right eigenvector corresponding to the zero eigenvalue has the form $\eta_i^0 = (1, 1, \ldots, 1)^T/\sqrt{N}$ in all examples in the present paper. Note that this form of the zero eigenvector is not necessary in this case. This point will be briefly discussed in the next section.

We denote left and right eigenvectors of G_{ij} corresponding to the eigenvalue λ^α $(0 = \lambda^0 < \lambda^1 < \cdots < \lambda^{N-1})$ as $\overline{\eta}_i^\alpha$ and η_i^α, which satisfy

$$\sum_i \overline{\eta}_i^\alpha G_{ij} = \lambda^\alpha \overline{\eta}_j^\alpha, \quad \sum_j G_{ij} \eta_j^\alpha = \lambda^\alpha \eta_i^\alpha. \quad (2)$$

Throughout this paper, subscripts with Latin symbols and superscripts with Greek symbols denote the label for node number and eigenvalue of the coupling matrix, respectively. Since $\sigma\lambda^\alpha$ plays an important role, it is also called the eigenvalue in this paper. The norm of η_i^α is set to unity, and that of $\overline{\eta}_i^\alpha$ is determined to satisfy the orthonormality $\sum_i \overline{\eta}_i^\alpha \eta_i^{\alpha'} = \delta_{\alpha\alpha'}$, where $\delta_{\alpha\alpha'}$ denotes the Kronecker delta. Note that normalization of eigenvectors is irrelevant to our results. We can take c_i satisfying $\sum_i \overline{\eta}_i^0 c_i = 0$ without loss of generality. Due to the orthonormality, we can expand $\boldsymbol{X}_i$ as

$$\boldsymbol{X}_i(t) = \sum_{\alpha=0}^{N-1} \boldsymbol{\zeta}^\alpha(t) \eta_i^\alpha, \quad \boldsymbol{\zeta}^\alpha(t) = \sum_{i=1}^{N} \overline{\eta}_i^\alpha \boldsymbol{X}_i(t). \quad (3)$$

As a numerical example, we consider the Rössler oscillator network

$$\begin{aligned} \dot{X}_i &= -\omega Y_i - Z_i - \sigma \sum_{j=1}^{N} G_{ij} X_j \\ \dot{Y}_i &= \omega X_i + a Y_i \\ \dot{Z}_i &= 0.2 + c_i + Z_i (X_i - 8.5) \end{aligned} \quad (4)$$

with $a = 0.16$ and $\omega = 1.0$. The constant c_i is uniformly distributed $c_i \in [-\delta, \delta]$. The parameter δ is a measure of parameter mismatch. We numerically integrate equation (4) with the fourth order Runge-Kutta method

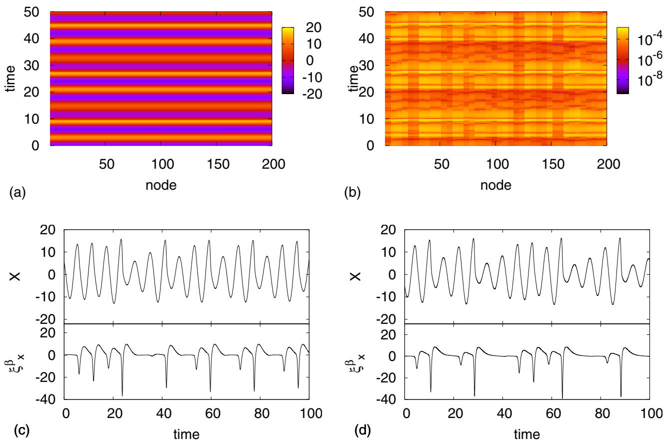

Fig. 1. (Color online) Time series of the Rössler model (4) with $\delta = 10^{-4}$ in a small-world network. (a) Spatio-temporal plot. (b) Deviation from the mean field $|X_i(t) - X(t)|$, where $X(t)$ represents the mean field. The horizontal axis in (a) and (b) is the index of node i. Time series of X and $\xi_x^\beta = \zeta_x^\beta / c^\beta$ corresponding to eigenvalue $\sigma\lambda^\beta \approx 2$ for (c) the nonlinear system (4), and for (d) the linearized equation (9).

with the time step $\Delta t = 0.005$. We construct small-world [5], scale-free [6], and Erdös-Rényi [37,38] networks with $N = 200$. We take the diagonal entries of the coupling matrix as $G_{ii} = 1$. Off-diagonal entries are $G_{ij} = -1/d_i$ (d_i is the degree of the ith node) if nodes i and j are connected, and $G_{ij} = 0$ otherwise. Although G_{ij} is asymmetric in this case, we can show that all eigenvalues of G_{ij} are real [22,23].

The obtained numerical results are shown in Figure 1. Figure 1a is the spatio-temporal plot. In this parameter value, mean field $X = \sum_i X_i/N$ shows chaotic oscillation. All oscillators are almost completely synchronized but slightly fluctuate because of small parameter mismatch as shown in Figure 1b. It is clear from Figure 1a that phase of each oscillator $\phi_i \equiv \tan^{-1}(Y_i/X_i)$ is locked. Thus, this is not a complete synchronization but a phase synchronization. We will clarify the statistical property of the fluctuation in the following sections. Figures 1c and 1d represent the time series of X and $\xi_x^\beta = \zeta_x^\beta / c^\beta$ corresponding to eigenvalue $\sigma\lambda^\beta \approx 2$ in equations (4) and (9), respectively. Comparison between these figures will be discussed in Section 4.

3 Derivation of linearized equation

In order to understand the deviation from complete synchronization, we derive a linearized equation for a small parameter mismatch. In Section 4, the numerical solutions of the linearized equation will be compared with those of the original equation (4).

We introduce the reference state $\boldsymbol{X} \equiv (\sum_i \overline{\eta}_i^0 \boldsymbol{X}_i)/(\sum_i \overline{\eta}_i^0)$ which satisfies $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X})$. Note that $\boldsymbol{X}$ coincides with the mean field $\boldsymbol{X} = \sum_i \boldsymbol{X}_i/N$, if η_i^0 is uniform, i.e., $\eta_i^0 \propto (1, 1, \ldots, 1)^T$. The condition of uniform η_i^0 is required for the complete synchronization, but is not necessary in the present discussion. The assumption here is that the deviation of η_i^0 from the uniform one is small enough to perform the expansion approximation discussed below. Substituting $\boldsymbol{X}_i = \boldsymbol{X} + \boldsymbol{x}_i$ to equation (1) and assuming $c_i \sim \mathcal{O}(x_i)$, we get the time evolution equation for the perturbation $\boldsymbol{x}_i$ up to linear in x as

$$\dot{\boldsymbol{x}}_i = c_i \boldsymbol{f}(\boldsymbol{X}) + \partial \boldsymbol{F}(\boldsymbol{X})\boldsymbol{x}_i - \sigma \sum_{j=1}^{N} G_{ij} \partial \boldsymbol{H}(\boldsymbol{X})\boldsymbol{x}_j, \quad (5)$$

where $\partial \boldsymbol{F}$ and $\partial \boldsymbol{H}$ represent the Jacobian matrices of $\boldsymbol{F}$ and $\boldsymbol{H}$, respectively. We expand c_i and $\boldsymbol{x}_i(t)$ as $c_i = \sum_\alpha c^\alpha \eta_i^\alpha$ and $\boldsymbol{x}_i(t) = \sum_\alpha \boldsymbol{\zeta}^\alpha(t)\eta_i^\alpha$. The coefficients are determined as $c^\alpha = \sum_i \overline{\eta}_i^\alpha c_i$ and $\boldsymbol{\zeta}^\alpha = \sum_i \overline{\eta}_i^\alpha \boldsymbol{x}_i$. Definition of $\boldsymbol{\zeta}^\alpha$ is identical to equation (3) in the case of $\eta_i^0 \propto (1, 1, \ldots, 1)^T$.

Operating $\overline{\eta}_i^\alpha$ from left to equation (5) and taking the summation over i, we obtain the linearized equation for the αth eigenmode as

$$\dot{\boldsymbol{\zeta}}^\alpha = c^\alpha \boldsymbol{f}(\boldsymbol{X}) + [\partial \boldsymbol{F}(\boldsymbol{X}) - \sigma\lambda^\alpha \partial \boldsymbol{H}(\boldsymbol{X})]\boldsymbol{\zeta}^\alpha. \quad (6)$$

Since $\lambda^0 = 0$ and $c^0 = 0$ hold, the linearized equation for the zero eigenmode is

$$\dot{\boldsymbol{\zeta}}^0 = \partial \boldsymbol{F}(\boldsymbol{X})\boldsymbol{\zeta}^0. \quad (7)$$

We briefly comment on equation (6). First, the inhomogeneous term appears in this equation due to the parameter mismatch. When all oscillators are identical $c_i = c^\alpha = 0$ for all i and α, equation (6) coincides with the master stability equation [26–28] which determines the linear stability of chaotic complete synchronization. Second, equation (6) is invariant under the transformation $c^\alpha \to kc^\alpha$ and $\boldsymbol{\zeta}^\alpha \to k\boldsymbol{\zeta}^\alpha$, where k is an arbitrary real number. This property comes from that we can choose the norm of $\overline{\eta}_i^\alpha$ and η_i^α arbitrarily, and plays an important role for the spectral universality described in the next section.

4 Spectral universality of phase synchronization

We show here that statistical properties of each eigenmode are essentially determined only by $\sigma\lambda^\alpha$. In the case of $c^\alpha \neq 0$, we divide equation (6) by c^α and obtain the evolution equation for normalized vector $\boldsymbol{\xi}^\alpha = \boldsymbol{\zeta}^\alpha / c^\alpha$ as

$$\dot{\boldsymbol{\xi}}^\alpha = \boldsymbol{f}(\boldsymbol{X}) + [\partial \boldsymbol{F}(\boldsymbol{X}) - \sigma\lambda^\alpha \partial \boldsymbol{H}(\boldsymbol{X})]\boldsymbol{\xi}^\alpha. \quad (8)$$

The normalized equation (8) is independent of c^α, which implies that the statistical properties of $\boldsymbol{\xi}^\alpha$ depend only on $\sigma\lambda^\alpha$. We will see that this prediction is indeed observable in the rest of this section.

In the Rössler model, the linearized equation for $\boldsymbol{\zeta}^\alpha = (\zeta_x^\alpha, \zeta_y^\alpha, \zeta_z^\alpha)^T$ corresponding to equation (6) is written as

$$\begin{pmatrix} \dot{\zeta}_x^\alpha \\ \dot{\zeta}_y^\alpha \\ \dot{\zeta}_z^\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c^\alpha \end{pmatrix} + \begin{pmatrix} -\sigma\lambda^\alpha & -\omega & -1 \\ \omega & a & 0 \\ Z & 0 & X - 8.5 \end{pmatrix} \begin{pmatrix} \zeta_x^\alpha \\ \zeta_y^\alpha \\ \zeta_z^\alpha \end{pmatrix}, \quad (9)$$

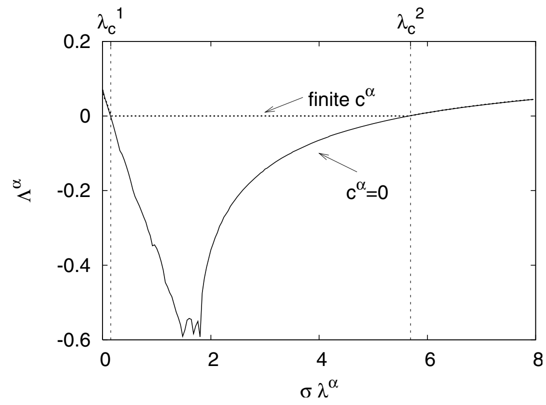

Fig. 2. Master stability function Λ^α of the Rössler model versus eigenvalue of the coupling matrix $\sigma\lambda^\alpha$ for $c^\alpha = 0$ (solid line) and $c^\alpha = 10^{-8}$ (dotted line). Λ^α changes its sign at $\lambda_c^1 \approx 0.15$ and $\lambda_c^2 \approx 5.66$. Note that Λ^α for $c^\alpha = 0$ and $c^\alpha \neq 0$ coincides when $\Lambda^\alpha > 0$.

and the normalized equation corresponding to equation (8) is

$$\begin{pmatrix} \dot{\xi}_x^\alpha \\ \dot{\xi}_y^\alpha \\ \dot{\xi}_z^\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\sigma\lambda^\alpha & -\omega & -1 \\ \omega & a & 0 \\ Z & 0 & X - 8.5 \end{pmatrix} \begin{pmatrix} \xi_x^\alpha \\ \xi_y^\alpha \\ \xi_z^\alpha \end{pmatrix}. \quad (10)$$

We numerically integrate equation (9) and compare the time series with that obtained with equation (4). In Figure 1d, time series of X and ξ_x^β corresponding to $\sigma\lambda^\beta \approx 2.0$ are shown. Their similarity to those in Figure 1c implies the validity of the linearization approximation.

Figure 2 shows the dependence of the largest Lyapunov exponent (master stability function) Λ^α on $\sigma\lambda^\alpha$. Linear stability of complete synchronization is determined by the sign of Λ^α [26–28]. For a network of identical oscillators $c^\alpha = 0$, Λ^α is positive for $\sigma\lambda^\alpha < \lambda_c^1$ and $\sigma\lambda^\alpha > \lambda_c^2$, and negative for $\lambda_c^1 < \sigma\lambda^\alpha < \lambda_c^2$. If all eigenmodes other than zero mode satisfy $\lambda_c^1 < \sigma\lambda^\alpha < \lambda_c^2$, complete synchronization is linearly stable. In the present study, we found that Λ^α vanishes for $\lambda_c^2 < \sigma\lambda^\alpha < \lambda_c^1$ for $c^\alpha \neq 0$. This implies that if complete synchronization is stable for $c^\alpha = 0$, then the observed state is linearly neutrally stable for small c^α. Since small fluctuations of each eigenmode oscillate with finite amplitude, the phase of each oscillator is locked. Thus, this state is a phase synchronized one which is close to complete synchronization. In this sense chaotic synchronization is robust against small parameter mismatch as far as the amplitude of deviation is so small that a linear description is valid.

In Figure 3a, the standard deviation $\Delta_\zeta^\alpha \equiv \sqrt{\langle(\zeta_x^\alpha - \langle\zeta_x^\alpha\rangle)^2\rangle}$ of equation (4) is depicted, where $\langle\ldots\rangle$ represents the time average. Although Δ_ζ^α does not show clear dependence on $\sigma\lambda$, the normalized standard deviation $\Delta_\xi^\alpha = \Delta_\zeta^\alpha / |c^\alpha|$ depends only on $\sigma\lambda^\alpha$ (Fig. 3b). As discussed at the beginning of this section, this result is expected from the normalized equation (8). This fact is observed irrespective of the details of the network topology, in small-world, scale-free, as well as random Erdös-Rényi networks. Thus, we refer to this phenomenon as the

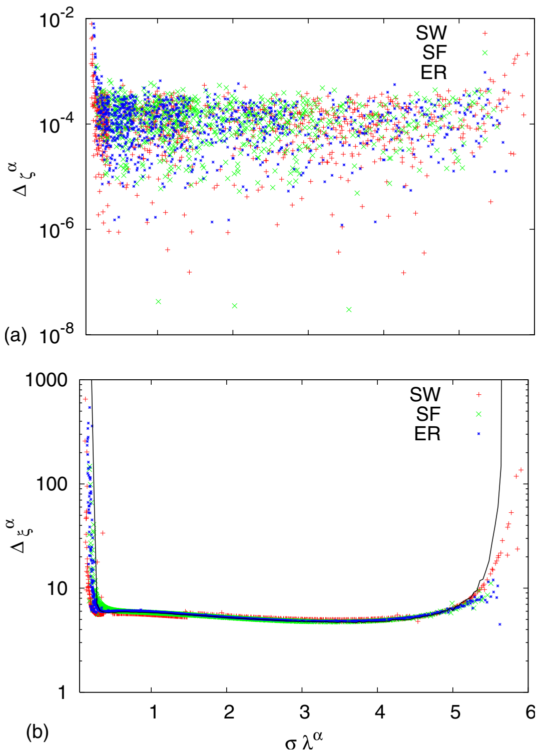

Fig. 3. (Color online) (a) Standard deviation Δ_ζ^α for equation (4) with $\delta = 10^{-4}$. (b) Normalized standard deviation $\Delta_\xi^\alpha = \Delta_\zeta^\alpha/|c^\alpha|$. Red, green, and blue points represent the eigenmodes for small-world, scale-free, and Erdös-Rényi networks, respectively. The black line in (b) is the result for linearized equation (9). Results for different coupling strength ($\sigma = 0.25,\ 1.0,\ 3.0$, and 4.0 for small-world network, $\sigma = 0.4,\ 1.0,\ 2.0$, and 3.5 for scale-free network, and $\sigma = 0.25,\ 0.5,\ 1.0,\ 2.0$, and 4.0 for Erdös-Rényi network) are plotted.

spectral universality. As discussed above, the linearized equation (10) well predicts this property. The spectral universality is a special property of the phase synchronization close to complete synchronization in the presence of a small parameter mismatch.

5 Spectral universality in a stochastic system

The present discussion is applicable to systems driven by independent Gaussian noise. We consider the stochastic equation

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}(\boldsymbol{X}_i) + D_i \boldsymbol{s}_i - \sigma \sum_{j=1}^{N} G_{ij} \boldsymbol{H}(\boldsymbol{X}_j). \tag{11}$$

Here, $\boldsymbol{s}_i = (s_{i1}, s_{i2}, \ldots, s_{iM})^T$ is an uncorrelated Normal Gaussian white noise at ith oscillator $\langle s_{ij}(t) s_{i'j'}(t')\rangle = \delta_{ii'}\delta_{jj'}\delta(t-t')$. We introduce $\boldsymbol{X} \equiv (\sum_i \overline{\eta}_i^0 \boldsymbol{X}_i)/(\sum_i \overline{\eta}_i^0)$ satisfying

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}) + D^0 \boldsymbol{s}^0, \tag{12}$$

$$D^0 \boldsymbol{s}^0 = \frac{\sum_i \overline{\eta}_i^0 D_i \boldsymbol{s}_i}{\sum_i \overline{\eta}_i^0}. \tag{13}$$

Note that $\boldsymbol{s}^0$ is normal Gaussian white noise, because it is sum of $\boldsymbol{s}_i$, and the intensity is $D^0 = \sqrt{\sum_i (\overline{\eta}_i^0 D_i)^2}/(\sum_i \overline{\eta}_i^0)$. We assume that equation (12) has only one attractor, and it is globally stable. This assumption implies that trajectories do not move to another attractor due to small noise.

Substituting $\boldsymbol{X}_i = \boldsymbol{X} + \boldsymbol{x}_i$ to equation (11), we obtain

$$\dot{\boldsymbol{X}} + \dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{X}) + \partial \boldsymbol{F}(\boldsymbol{X})\boldsymbol{x}_i + D_i \boldsymbol{s}_i - \sigma \sum_j G_{ij} \partial \boldsymbol{H}(\boldsymbol{X}) \boldsymbol{x}_j. \tag{14}$$

Operating $\overline{\eta}_i^0$ from left and applying equations (12) and (13), we get

$$\dot{\boldsymbol{\zeta}}^0 = \partial \boldsymbol{F}(\boldsymbol{X}) \boldsymbol{\zeta}^0. \tag{15}$$

Furthermore, we operate $\overline{\eta}_i^\alpha$ ($\alpha \neq 0$) from left to obtain the linearized equation for $\boldsymbol{\zeta}^\alpha$ as

$$\dot{\boldsymbol{\zeta}}^\alpha = [\partial \boldsymbol{F}(\boldsymbol{X}) - \sigma\lambda^\alpha]\boldsymbol{\zeta}^\alpha + D^\alpha \boldsymbol{s}^\alpha, \tag{16}$$

$$D^\alpha \boldsymbol{s}^\alpha = \sum_i \overline{\eta}_i^\alpha D_i \boldsymbol{s}_i. \tag{17}$$

The noise intensity is $D^\alpha = \sqrt{\sum_i (\overline{\eta}_i^\alpha D_i)^2}$. Equations (15) and (16) are linearized equations for the eigenmodes driven by Gaussian white noise. Although $\boldsymbol{s}^0$ and $\boldsymbol{s}^\alpha$ are correlated (see Eqs. (13) and (17)), the correlation is neglected in the following numerical integration of the linearized equation.

As an example, we consider a network of N FitzHugh-Nagumo oscillators (Refs. [39,40])

$$\epsilon \dot{X}_i = X_i - X_i^3/3 - Y_i - \sigma \sum_{j=1}^{N} G_{ij} X_j$$

$$\dot{Y}_i = X_i + a + D_i s_i, \tag{18}$$

where $a = 0.95$, $\epsilon = 0.01$, and the noise intensity is uniformly distributed $D_i \in [-\delta, \delta]$ with $\delta = 10^{-4}$, and s_i represents independent Gaussian noise. We get the linearized equation

$$\begin{pmatrix} \epsilon \dot{\zeta}_x^\alpha \\ \dot{\zeta}_y^\alpha \end{pmatrix} = \begin{pmatrix} 1 - X^2 - \sigma\lambda^\alpha & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta_x^\alpha \\ \zeta_y^\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ D^\alpha s^\alpha \end{pmatrix}. \tag{19}$$

As before, equation (19) can be normalized with dividing by D^α. Numerical integration of equations (18) and (19) was performed with the Euler-Maruyama method ($\Delta t = 0.005$).

Figures 4a and 4b show time series of X and $\xi_x^\beta = \zeta_x^\beta / D^\beta$ corresponding to $\sigma\lambda^\beta \approx 0.5$ for equations (18) and (19), respectively. In both figures, ξ_x^β spikes when X spikes, but the amplitudes of the spike are slightly different. Figure 5 shows that the normalized standard deviation $\Delta_\xi^\alpha = \Delta_\zeta^\alpha / D^\alpha$ is a function of $\sigma\lambda^\alpha$ irrespective of the network topology. Moreover, the linearized equation fits this curve for $\sigma\lambda^\alpha \lesssim 1$. However, the linearized equation cannot mimic the numerical results for $\sigma\lambda^\alpha \gtrsim 1$. Nevertheless, spectral universality holds even in this region.

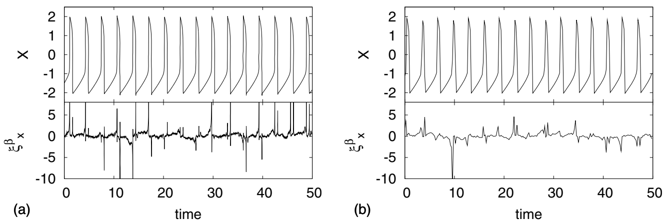

Fig. 4. Mean field X and normalized eigenmode ξ_x^β corresponding to $\sigma\lambda^\beta \approx 0.5$ for (a) the FitzHugh-Nagumo equation (18) and (b) the linearized equation (19).

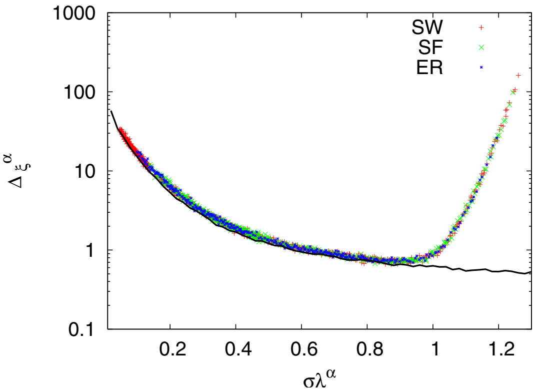

Fig. 5. (Color online) Normalized standard deviation $\Delta_\xi^\alpha = \Delta_\zeta^\alpha / D^\alpha$ versus eigenvalues of the coupling matrix $\sigma\lambda^\alpha$ for the FitzHugh-Nagumo oscillator network. Red, green, and blue points represent the results for small-world, scale-free, and Erdös-Rényi networks, respectively. The black line represents the prediction of the linearized equation (19).

6 Summary and discussion

In this paper, we have discussed properties of phase synchronization close to complete synchronization in the presence of a small parameter mismatch or additive Gaussian noise in complex networks. The derived linearized equation describes the statistical property of the fluctuation of each eigenmode well.

The observed state is phase synchronization close to complete synchronization. Our result also implies that the breakdown of phase synchronization occurs due to the nonlinear mode coupling between the eigenmodes. This point should be studied in future by taking nonlinear effect into consideration.

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