

# Inverse synchronizations in coupled time-delay systems with inhibitory coupling

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(Received 26 January 2009; accepted 5 April 2009; published online 6 May 2009)

Transitions between inverse anticipatory, inverse complete, and inverse lag synchronizations are shown to occur as a function of the coupling delay in unidirectionally coupled time-delay systems with inhibitory coupling. We have also shown that the same general asymptotic stability condition obtained using the Krasovskii–Lyapunov functional theory can be valid for the cases where (i) both the coefficients of the  $\Delta(t)$  (error variable) and  $\Delta_\tau = \Delta(t - \tau)$  (error variable with delay) terms in the error equation corresponding to the synchronization manifold are time independent and (ii) the coefficient of the  $\Delta$  term is time independent, while that of the  $\Delta_\tau$  term is time dependent. The existence of different kinds of synchronization is corroborated using similarity function, probability of synchronization, and also from changes in the spectrum of Lyapunov exponents of the coupled time-delay systems. © 2009 American Institute of Physics. [DOI: 10.1063/1.3125721]

**Synchronization of chaotic oscillations is one of the most fundamental phenomena exhibited by coupled chaotic oscillators. Since the identification of chaotic synchronization in identical systems, different kinds of synchronizations, such as generalized, phase, lag, anticipatory and intermittent synchronizations, and their variants, have been reported in literature. Recent studies on synchronization have also been focused on the existence of inverse synchronization. In this connection, we have identified different kinds of inverse synchronization and transitions among them as a function of a single parameter in coupled time-delay systems with inhibitory coupling. It is now recognized that inhibitory couplings are important and widespread in nature, specifically in biological systems, neurophysiology, and in many natural networks. We have also obtained a suitable stability condition for asymptotic stability of the synchronized state using Krasovskii–Lyapunov functional theory. The results are corroborated using similarity function, probability of synchronization, and from the changes in the spectrum of Lyapunov exponents of specific coupled time-delay systems.**

## I. INTRODUCTION

Synchronization is an interesting dynamical phenomenon exhibited by interacting oscillators in diverse areas of science and technology.<sup>1,2</sup> It has become an area of active research since the identification of synchronization in chaotic oscillators.<sup>3</sup> In recent years, different types of synchronization and generalizations have been reported both experimen-

tally and theoretically.<sup>1–29</sup> Among them, inverse or antisynchronization<sup>20–29</sup> constitutes an important class of synchronization, which is a typical feature of dynamical systems interacting through inhibitory coupling.

By definition, inverse synchronization is the phenomenon<sup>20–29</sup> where the state vectors, say the drive  $x(t)$  and the response  $y(t)$ , of the synchronized systems have the same absolute values but of opposite signs, that is,  $x(t) = -y(t)$ . While this is inverse complete synchronization (ICS), one may also identify inverse anticipatory synchronization (IAS) where the response anticipates the drive, that is,  $y(t) = -x(t + \tau)$ , where  $\tau > 0$  is a constant. Similarly, when the response system lags the drive, one has inverse lag synchronization (ILS), that is,  $y(t) = -x(t - \tau)$ ,  $\tau > 0$ . Just as in the case of regular synchronization, especially in nonlinear dynamical systems with inhibitory coupling, one may encounter all the above types of inverse synchronization for appropriate ranges of the parameters.

The importance of inhibitory or repulsive coupling is well acknowledged in biological systems. It is a well established fact that couplings between neurons are both excitatory and inhibitory.<sup>30</sup> Ecological webs typically have both positive and negative connections between their components.<sup>31</sup> Coupled lasers with negative couplings have also been widely studied.<sup>32</sup> The well-known Swift–Hohenberg and Kuramoto–Sivashinsky equations include such a term.<sup>33</sup> Currently, it has also been realized that a large class of natural networks also have inhibitory interactions among the nodes.<sup>34,35</sup>

The first experimental observation of inverse synchronization was demonstrated in coupled semiconductor laser diodes,<sup>20</sup> in which it was established that inverse synchronization was caused by nonresonant coupling between the drive and the response lasers. It was also shown that switch-

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ing between synchronization and inverse synchronization was possible by slightly changing the pump current of the drive laser.<sup>21</sup> Experimental observations and numerical simulations of synchronization and inverse synchronization of low frequency power dropouts and jump-ups of chaotic semiconductor lasers were carried out in Ref. 23. Inverse synchronization was also observed both experimentally and numerically in unidirectionally coupled laser systems with optical feedback,<sup>24,27</sup> as well as in a class of chaotic delayed neural networks<sup>28</sup> and in coupled Ikeda systems with multi-feedback and multiple time delays.<sup>25</sup> Inverse anticipating synchronization was demonstrated in coupled Ikeda systems.<sup>22</sup> Further inverse retarded/lag synchronization and the role of parameter mismatch were discussed in Ref. 26. However it may be noted that in none of the above studies the role of inhibitory coupling was investigated.

Despite the fact that a vast amount of literature is now available on the phenomenon of synchronization, inverse synchronizations have not been studied adequately, in particular, with inhibitory couplings between interacting dynamical systems. In this paper, we report inverse synchronizations (inverse anticipatory, inverse complete, and ILSs) in unidirectionally coupled time-delay systems with inhibitory coupling. We also present a sufficient stability condition for asymptotic stability of the synchronized state following the Krasovskii–Lyapunov functional approach for the cases where (i) the coefficients of the  $\Delta(t)$  (error variable) and  $\Delta_\tau = \Delta(t - \tau)$  (error variable with delay) terms of the error equation corresponding to the synchronization manifold are constant and (ii) the coefficient of the  $\Delta$  term is constant and that of the  $\Delta_\tau$  term is time dependent. We show that there is a transition from inverse anticipatory to ILS through complete inverse synchronization as a function of the delay time in the coupling. The tools to get these results are similarity function, probability of synchronization, and largest Lyapunov exponents of the coupled time-delay systems.

The plan of the paper is as follows. In Sec. II, we deduce a sufficient condition for the asymptotic stability of the synchronized state for a system of unidirectionally coupled scalar delay differential equations with inhibitory coupling. We consider a piecewise linear delay differential equation as an example for the case where the coefficients of both the  $\Delta$  and  $\Delta_\tau$  terms in the error equation are constant and demonstrate the existence of different types of inverse synchronization as a function of the coupling delay in Sec. III. Using the paradigmatic Ikeda system in Sec. IV for the case where the coefficient of the  $\Delta_\tau$  term is time dependent while that of the other is time independent, we show that the same general stability condition is valid for the asymptotic stability of different types of inverse synchronizations again as a function of the coupling delay. We also demonstrate these dynamical transitions through numerical analysis. Finally, in Sec. V, we summarize our results.

**II. COUPLED SYSTEM AND THE STABILITY CONDITION**

Consider the following unidirectionally coupled drive,  $x(t)$ , and response,  $y(t)$ , systems with inhibitory coupling of the form

$$\dot{x}(t) = -ax(t) + b_1 f(x(t - \tau_1)), \tag{1a}$$

$$\dot{y}(t) = -ay(t) + b_2 f(y(t - \tau_1)) - b_3 f(x(t - \tau_2)), \tag{1b}$$

where  $b_1, b_2,$  and  $b_3$  are positive parameters,  $a > 0,$  and  $\tau_1$  and  $\tau_2$  are the feedback and the coupling delays, respectively. The nonlinear function  $f(x)$  is chosen to be a piecewise linear function which has been studied in detail recently,<sup>36–38</sup>

$$f(x) = \begin{cases} 0, & x \leq -4/3 \\ -1.5x - 2, & -4/3 < x \leq -0.8 \\ x, & -0.8 < x \leq 0.8 \\ -1.5x + 2, & 0.8 < x \leq 4/3 \\ 0, & x > 4/3, \end{cases} \tag{2}$$

as the first example, and

$$f(x) = \sin(x_\tau) \equiv \sin(x(t - \tau)), \tag{3}$$

which is the well-known Ikeda system,<sup>39</sup> as the second example.

Now the stability condition for the synchronization of the coupled time-delay systems [Eq. (1)] with the inhibitory delay coupling,  $-b_3 f(x(t - \tau_2)),$  can be obtained as follows. The time evolution of the difference system (error function), associated with inverse synchronization, with the state variable  $\Delta(t) = x_{\tau_2 - \tau_1} + y(t),$  where  $x_{\tau_2 - \tau_1} = x(t - (\tau_2 - \tau_1)),$  can be written for small values of  $\Delta,$  by using the evolution Eq. (1), as

$$\dot{\Delta} = \dot{x}_{\tau_2 - \tau_1} + \dot{y}(t) \tag{4}$$

$$= -a\Delta + (b_1 - b_2 - b_3)f(x(t - \tau_2)) + b_2 f'(x(t - \tau_2))\Delta_{\tau_1}. \tag{5}$$

The above evolution equation (5) corresponding to the error function of the inverse synchronization manifold is inhomogeneous and so it is difficult to analyze the system analytically. Nevertheless, the evolution equation can be written as a homogeneous equation

$$\dot{\Delta} = -a\Delta + b_2 f'(x(t - \tau_2))\Delta_{\tau_1} \tag{6}$$

for the specific choice of the parameters

$$b_1 = b_2 + b_3. \tag{7}$$

Therefore, we will concentrate on this parametric choice. The inverse synchronization manifold  $\Delta = x_{\tau_2 - \tau_1} + y = 0$  corresponds to the following distinct cases:

- (1) IAS occurs when  $\tau_2 < \tau_1$  with  $y(t) = -x(t - \hat{\tau}); \hat{\tau} = \tau_2 - \tau_1 < 0,$  where the state of the response system anticipates the inverse state of the drive system in a synchronized manner with the anticipating time  $\hat{\tau}$  [whereas in the case of direct anticipatory synchronization, the state of the response system anticipates exactly the state of the drive system, that is,  $y(t) = x(t - \hat{\tau})$ ].
- (2) ICS results when  $\tau_2 = \tau_1$  with  $y(t) = -x(t); \hat{\tau} = \tau_2 - \tau_1 = 0,$  where the state of the response system evolves in a synchronized manner with the inverse state of the drive system [whereas in the case of complete synchronization, the state of the response system evolves exactly identical to the state of the drive system, that is,  $y(t) = x(t)$ ].

(3) ILS occurs when  $\tau_2 > \tau_1$  with  $y(t) = -x(t - \hat{\tau})$ ;  $\hat{\tau} = \tau_2 - \tau_1 > 0$ , where the state of the response system lags the inverse state of the drive system in a synchronized manner with the lag time  $\hat{\tau}$  [whereas in the case of direct lag synchronization, the state of the response system lags exactly the state of the drive system, that is,  $y(t) = x(t - \hat{\tau})$ ].

The synchronization manifold corresponding to Eq. (6) is locally attracting if the origin of the above error equation is stable. Following the Krasovskii–Lyapunov functional approach,<sup>37,40</sup> we define a positive definite Lyapunov functional of the form

$$V(t) = \frac{1}{2} \Delta^2 + \mu \int_{-\tau_1}^0 \Delta^2(t + \theta) d\theta, \tag{8}$$

where  $\mu$  is an arbitrary positive parameter,  $\mu > 0$ .

It is to be noted that for the above error equation (6), the coefficient of the  $\Delta$  term is always constant, while that of the  $\Delta_\tau$  term can be time dependent. Hence one can obtain two cases depending on the choice of the nonlinear functional form  $f(x)$ . If the derivative of the function turns out to be a constant as in the case of the piecewise linear function [Eq. (2)], one obtains a constant coefficient for the  $\Delta_\tau$  term. On the other hand, if the derivative still depends on time as in the case of the Ikeda system [Eq. (3)], then the  $\Delta_\tau$  term always has a time dependent coefficient. In the following, we will show that the same general stability condition derived from the Krasovskii–Lyapunov approach can be valid for both the cases for the asymptotic stability of the different types of inverse synchronizations. However, there also arises an even more general situation where the coefficients of both the  $\Delta$  and  $\Delta_\tau$  terms are time dependent. In this case the arbitrary positive parameter  $\mu$  in the Lyapunov functional is no longer a positive constant. We have also designed a suitable coupling form for this rather general situation, for which we can show that the same general stability condition arrived from the Krasovskii–Lyapunov approach is still valid for the asymptotic stability of the synchronized state; these results will be published elsewhere.

Note that from Eq. (8),  $V(t)$  approaches zero as  $\Delta \rightarrow 0$ . Hence, the required solution  $\Delta = 0$  to the error equation [Eq. (6)] is stable only when the derivative of the Lyapunov functional  $V(t)$  along the trajectory of Eq. (6) is negative. This requirement results in the condition for stability as

$$\Gamma(\mu) = 4\mu(a - \mu) > b_2^2 f'(x_1(t - \tau_2))^2. \tag{9}$$

Again  $\Gamma(\mu)$  as a function of  $\mu$  for a given  $f'(x)$  has an absolute minimum at  $\mu = a/2$  with  $\Gamma_{\min} = a^2$ . Since  $\Gamma \geq \Gamma_{\min} = a^2$ , from inequality (9), it turns out that a sufficient condition for asymptotic stability is

$$a > |b_2 f'(x(t - \tau_2))|. \tag{10}$$

This general stability condition indeed corresponds to the stability condition for inverse anticipatory, complete inverse, and ILSs for suitable values of the coupling delay  $\tau_2$  corresponding to a fixed value of the feedback delay  $\tau_1$  for both the piecewise linear and Ikeda time-delay systems corresponding to the cases where the coefficient of the  $\Delta_\tau$  term in

the error equation is time independent and time dependent, respectively.

Further, it is interesting to note that if one substitutes  $y \rightarrow \hat{y} = -y$  in Eq. (1b), then the coupling becomes excitatory for the choice of functional forms we have chosen. This is exactly the case we have studied in Ref. 37 where direct anticipatory, complete, and lag synchronizations exist as a function of the coupling delay. However, one cannot obtain inverse (anticipatory, complete, and lag) synchronization with excitatory coupling or direct (anticipatory, complete, and lag) synchronization with inhibitory coupling for the chosen form of the unidirectional nonlinear coupling because of the nature of the parametric relation between  $b_1$ ,  $b_2$ , and  $b_3$  and the stability condition (10).

Now from the form of the function  $f(x)$  in Eq. (2) for the piecewise linear time-delay system, one can obtain a less stringent stability condition as<sup>37</sup>

$$a > b_2, \tag{11}$$

while

$$a > 1.5b_2 \tag{12}$$

is the most general condition specified by Eq. (10) for asymptotic stability of the synchronized state  $\Delta = 0$ .

Correspondingly, one can obtain the stability condition for the coupled Ikeda systems as

$$a > |b_2 \cos(x(t - \tau_2))|. \tag{13}$$

### III. INVERSE SYNCHRONIZATIONS IN THE COUPLED PIECEWISE LINEAR TIME-DELAY SYSTEMS [EQS. (1) AND (2)]

Here, we will show the existence of inverse anticipatory, inverse complete, and ILSs as a function of the coupling delay in the coupled piecewise linear time-delay systems [Eqs. (1) and (2)] with inhibitory coupling as an illustration for the case where the coefficients of both the  $\Delta$  and  $\Delta_\tau$  terms in the error equation (6) are constant.

#### A. IAS

In this section, we present the existence of IAS for the values of the coupling delay  $\tau_2 < \tau_1$  and for fixed values of the other system parameters. In particular, we have fixed the parameters as  $a = 1.0$ ,  $b_1 = 1.2$ ,  $\tau_1 = 8.0$ ,  $\tau_2 = 6.0$ , and the other two parameters  $b_2$  and  $b_3$  are fixed according to the parametric condition (7). The first ten maximal Lyapunov exponents  $\lambda_{\max}$  of the uncoupled piecewise linear time-delay system in the range of delay time  $\tau \in (2, 29)$  for the above choice of parameters are shown in Fig. 1(a). It is clear from Fig. 1(a) that for  $\tau > 0.5$ , at least two of the Lyapunov exponents are positive and that the system is hyperchaotic. As an illustration, the hyperchaotic attractor with two positive Lyapunov exponents for the value of the delay time  $\tau_1 = \tau = 8.0$  is plotted in Fig. 1(b). The time trajectories of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  of the coupled piecewise linear time-delay systems [Eqs. (1) and (2)] are plotted in Fig. 2, depicting the existence of IAS for  $b_2 = 0.6$ , for which the more general stability condition,  $a > 1.5b_2$ , is satisfied. It is evident from

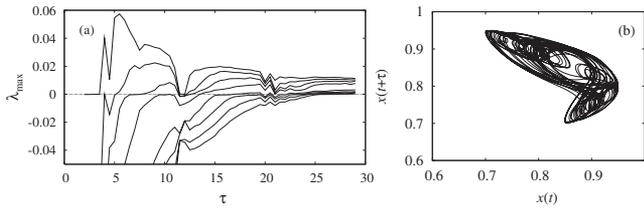


FIG. 1. (a) The first ten maximal Lyapunov exponents  $\lambda_{\max}$  of the piecewise linear time-delay systems [Eqs. (1a) and (2)] for the parameter values  $a = 1.0, b = 1.2$ , and  $\tau \in (2, 29)$ . (b) Hyperchaotic attractor for the delay time  $\tau = 8.0$  with two positive Lyapunov exponents for the above values of the other parameters. Here  $\tau_1 = \tau$ .

the figure that the state of the response system  $y(t)$  anticipates the inverse state of the drive system  $x(t)$ . This is made visually clear by plotting the inverse state of the response, that is,  $-y(t)$ . From Fig. 2, it is clear that the inverse state of the response ( $-y(t)$ ) anticipates the state of the drive  $x(t)$ , thereby illustrating the existence of IAS between the drive and the response systems.

The existence of IAS for the above choice of parameters is further confirmed using the notion of similarity function  $S(b_2)$ , probability of synchronization  $\Phi(b_2)$ , and also from the spectrum of Lyapunov exponents of the coupled time-delay systems. Now we use the notion of similarity function, introduced in Ref. 11 for characterizing the lag synchronization, to characterize the existence of IAS. The similarity function,  $S_a(b_2)$ , is defined as a time averaged difference between the variables  $x(t)$  and  $-y(t)$  (with mean values being subtracted) taken with the time shift  $\tau$ ,

$$S_a^2(b_2) = \frac{\langle [y(t - \tau) + x(t)]^2 \rangle}{[\langle x^2(t) \rangle \langle y^2(t) \rangle]^{1/2}}, \tag{14}$$

where  $\langle x \rangle$  means time average over the variable  $x$ . If the signals  $x(t)$  and  $-y(t)$  are independent, the difference between them is of the same order as the signals themselves. If  $x(t) = y(t)$ , as in the case of complete synchronization, the similarity function reaches a minimum,  $S_a(b_2) = 0$ , for  $\tau = 0$ . On the other hand, if  $S_a(b_2) = 0$  for the case  $\tau \neq 0$ , there exists a time shift  $\tau$  between the two signals  $x(t)$  and  $-y(t)$  such that  $y(t - \tau) = -x(t)$ , demonstrating IAS. The similarity function,  $S_a(b_2)$ , as a function of the parameter  $b_2$  is shown in Fig. 3(a), which clearly indicates that the value of  $S_a(b_2)$  oscillates with finite amplitude ( $S_a(b_2) > 0$ ) above the value of the control parameter  $b_2 \approx 0.97$ , indicating the desynchronized evolution between the response,  $y(t)$ , and the drive,

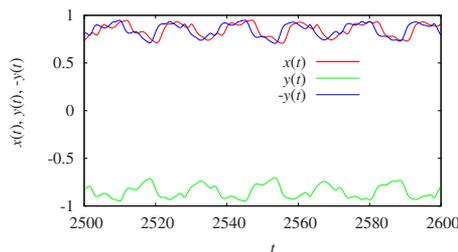


FIG. 2. (Color online) The time trajectory of the variables  $x(t), y(t)$ , and  $-y(t)$  for  $a = 1.0, b_1 = 1.2, b_2 = 0.6, b_3 = 0.6, \tau_1 = 8.0$ , and  $\tau_2 = 6.0$  of the coupled piecewise linear time-delay systems, indicating IAS.

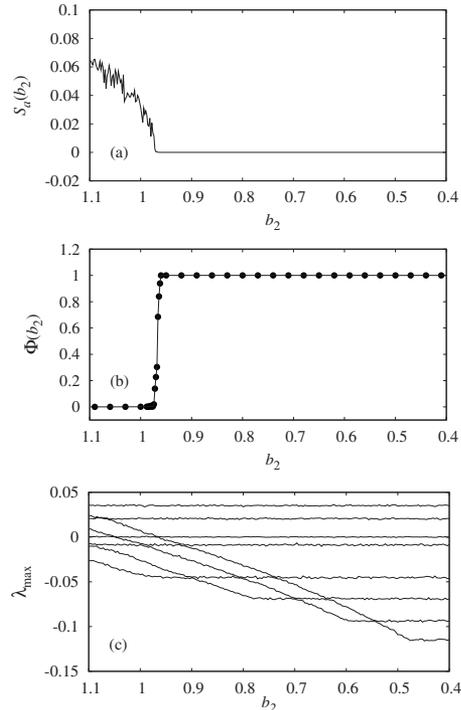


FIG. 3. (a) Similarity function,  $S_a(b_2)$ , (b) probability of synchronization,  $\Phi(b_2)$ , and (c) first eight maximal Lyapunov exponents,  $\lambda_{\max}$ , of the coupled piecewise linear time-delay systems as a function of the control parameter  $b_2$ , indicating the existence of IAS for  $b_2 < 0.97$ .

$x(t)$ , systems. For the value of the control parameter  $b_2 < 0.97$ , the similarity function become zero ( $S_a(b_2) = 0$ ), indicating that the response system anticipates the inverse state of the drive system confirming the existence of IAS.

It is to be noted that one can observe exact IAS in the present study even for the choice of the parameter  $b_2$  for which only the less stringent stability condition (11) is satisfied in contrast to our earlier studies,<sup>37</sup> where one can observe only approximate direct anticipatory synchronization under this condition, while exact synchronizations are observed for the values of the parameters satisfying the general stability condition for the coupled piecewise linear time-delay systems. This is due to the fact that in the present study, we have chosen the value of the delay time in the drive system,  $x(t)$ , as  $\tau = 8$ , for which the system has only two positive Lyapunov exponents (as may be observed from the Fig. 1). This implies that the number of transversely unstable manifolds is less and hence they are all stabilized even for the least values of the parameters satisfying even the less stringent stability condition, and so one can observe exact inverse synchronization. This is in contrast to our earlier studies on direct anticipatory synchronization,<sup>37</sup> where we have chosen  $\tau = 25$ , for which the drive system exhibits more than seven positive Lyapunov exponents. Correspondingly, there exist a large number of transversely unstable manifolds, and hence, the general stability condition is required to be satisfied in order to stabilize all the transversely unstable manifolds to obtain exact synchronization.

As IAS is a special case of generalized synchronization, the existence of IAS can also be further confirmed using the auxiliary system approach by augmenting the coupled piece-

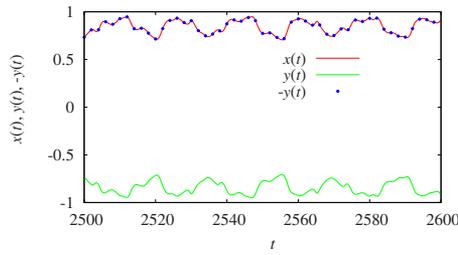


FIG. 4. (Color online) The time trajectory of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  of the coupled piecewise linear time-delay systems, indicating ICS for the coupling delay  $\tau_2 = \tau_1 = 8.0$ , while the other parameter values are the same as in Fig. 2.

wise linear time-delay systems [Eqs. (1) and (2)] with an additional auxiliary system for the variable  $z(t)$  identical to the response system, satisfying the equation

$$\dot{z}(t) = -az(t) + b_2f(z(t - \tau_1)) - b_3f(x(t - \tau_2)). \quad (15)$$

Now, the existence of IAS can be characterized by using the probability of synchronization,  $\Phi(b_2)$ , calculated between the response,  $y(t)$ , and the auxiliary systems,  $z(t)$ , which can be defined as the fraction of time during which  $|y(t) - z(t)| < \epsilon$  occurs, where  $\epsilon$  is a small but arbitrary threshold. The probability of synchronization,  $\Phi(b_2)$ , remains zero for the value of the control parameter  $b_2 \geq 0.97$ , as shown in Fig. 3(b), where there is no synchronization between the response,  $y(t)$ , and the auxiliary systems,  $z(t)$ . On the other hand, for  $b_2 < 0.97$ , the probability of synchronization attains the value of unity for the chosen threshold value for  $\epsilon$ , clearly indicating the existence of complete synchronization between the response,  $y(t)$ , and the auxiliary,  $z(t)$  systems (we have fixed the threshold value at  $\epsilon = 10^{-10}$  throughout the manuscript). Correspondingly, there exists IAS between the coupled drive,  $x(t)$ , and the response,  $y(t)$ , systems with inhibitory coupling.

Further, the existence of IAS can also be characterized by the changes in the spectrum of the Lyapunov exponents of the coupled systems [Eqs. (1) and (2)]. The first eight largest Lyapunov exponents of the coupled piecewise linear time-delay system is shown in Fig. 3(c). The two largest Lyapunov exponents of the drive system,  $x(t)$ , remain positive, while the ones corresponding to the response system,  $y(t)$ , decrease in their value as a function of the parameter  $b_2$ . The least positive Lyapunov exponent of the response system becomes negative at  $b_2 = 1.06$ , while the largest positive Lyapunov exponent of the response system becomes negative at the value of  $b_2 = 0.97$ , confirming the onset of IAS at  $b_2 = 0.97$ .

**B. ICS**

The synchronization manifold,  $\Delta = x_{\tau_2 - \tau_1} + y$ , becomes an ICS manifold for  $\tau_1 = \tau_2 = 8.0$ . The time trajectory of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  are plotted in Fig. 4, depicting the existence of ICS for the same values of the other parameters as in Fig. 2. It is clear from Fig. 4 that the state of the response system,  $y(t)$ , evolves in synchrony with the inverse state of the drive system  $x(t)$ . For the ease of clear visualization of this phenomenon, the inverse of the response  $-y(t)$  is

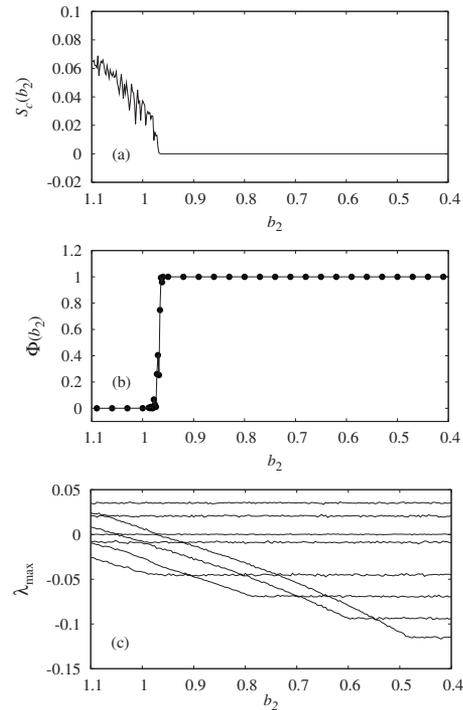


FIG. 5. (a) Similarity function,  $S_c(b_2)$ , (b) probability of synchronization,  $\Phi(b_2)$ , and (c) first eight maximal Lyapunov exponents,  $\lambda_{\max}$ , of the coupled piecewise linear time-delay systems as a function of the control parameter  $b_2$ , indicating the existence of ICS for  $b_2 < 0.97$  and for the coupling delay  $\tau_2 = \tau_1 = 8.0$ .

shown as filled circles in Fig. 2 and hence the inverse state of the response  $-y(t)$  clearly evolves in synchrony with the state of the drive  $x(t)$ , thereby illustrating the existence of ICS between the drive,  $y(t)$ , and the response,  $x(t)$ , systems.

We have also calculated the similarity function for the ICS defined as

$$S_c^2(b_2) = \frac{\langle [y(t) + x(t)]^2 \rangle}{[\langle x^2(t) \rangle \langle y^2(t) \rangle]^{1/2}}, \quad (16)$$

as a function of the parameter  $b_2$  in Fig. 5(a). The similarity function,  $S_c(b_2)$ , oscillates with a finite value above zero for the value of the control parameter  $b_2 \geq 0.97$  as shown in Fig. 5(a), where there does not exist any correlation between the interacting systems. However, it is evident from Fig. 5(a) that the similarity function becomes zero for the values of  $b_2 < 0.97$  indicating the existence of exact ICS in the coupled piecewise linear time-delay system even for the value of the parameter satisfying the less stringent stability condition as discussed in Sec. III A. We have also characterized the existence of ICS using the probability of synchronization ( $\Phi(b_2)$ ), as shown in Fig. 5(b), which clearly shows that the probability of synchronization becomes unity for  $b_2 < 0.97$ , depicting the existence of ICS between the coupled drive,  $x(t)$ , and the response,  $y(t)$ , systems with inhibitory coupling. However, for  $b_2 \geq 0.97$ , the value of the probability of synchronization becomes zero,  $\Phi(b_2) = 0$ , indicating that there exists no correlation between the coupled systems for this range of parameters. The eight largest Lyapunov exponents of the coupled time-delay systems is shown in Fig. 5(c). It shows that the two largest Lyapunov exponents of the drive

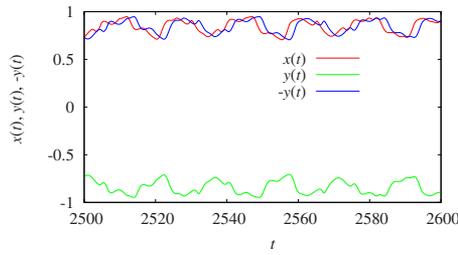


FIG. 6. (Color online) The time trajectory of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  of the coupled piecewise linear time-delay systems, indicating ILS for the value of the coupling delay  $\tau_2=10.0$ , while the other parameter values are the same as in Fig. 2.

system remain positive, while the largest positive Lyapunov exponent of the response system attains a negative value at  $b_2=0.97$ , confirming the existence of ICS for  $b_2 < 0.97$ .

**C. ILS**

Now, we will demonstrate the existence of ILS in the coupled piecewise linear time-delay systems [Eqs. (1) and (2)] for the value of coupling delay  $\tau_2 > \tau_1$  and for the same values of the other parameters as in Sec. III A. The time series of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  of the coupled systems are shown in Fig. 6 for  $b_2=0.6$  and  $\tau_2=10.0$ , depicting the existence of ILS in the coupled systems. It is evident from the figure that the state of the response system  $y(t)$  lags the inverse state of the drive system  $x(t)$ . In order to made this clear visually, the inverse state of the response  $-y(t)$  is plotted in Fig. 6 and hence the inverse state of the response  $-y(t)$  lags the state of the drive  $x(t)$ , thereby illustrating the existence of ILS between the drive and the response systems.

The similarity function for ILS defined as

$$S_1^2(b_2) = \frac{\langle [y(t + \tau) + x(t)]^2 \rangle}{[\langle x^2(t) \rangle \langle y^2(t) \rangle]^{1/2}} \tag{17}$$

is plotted in Fig. 7(a) as a function of  $b_2$ . The similarity function oscillates with a finite amplitude for  $b_2 \geq 0.97$  as in the other cases, as there is no synchronous evolution among the coupled systems. However, the similarity function,  $S_1(b_2)$ , reaches zero for  $b_2 < 0.97$ , confirming the existence of ILS in the coupled time-delay systems. Similarly, the value of the probability of synchronization,  $\Phi(b_2)$ , calculated between the response,  $y(t)$ , and the auxiliary,  $z(t)$ , systems attains unity for  $b_2 < 0.97$ , confirming the existence of complete synchronization between them as shown in Fig. 7(b). Correspondingly, there exists ILS between the drive,  $x(t)$ , and the response,  $y(t)$ , systems. The two largest Lyapunov exponents of the drive system remain positive, as shown in Fig. 7(c), while the largest positive Lyapunov exponent of the response system becomes negative at  $b_2=0.97$ , confirming the existence of ILS between the drive and the response systems.

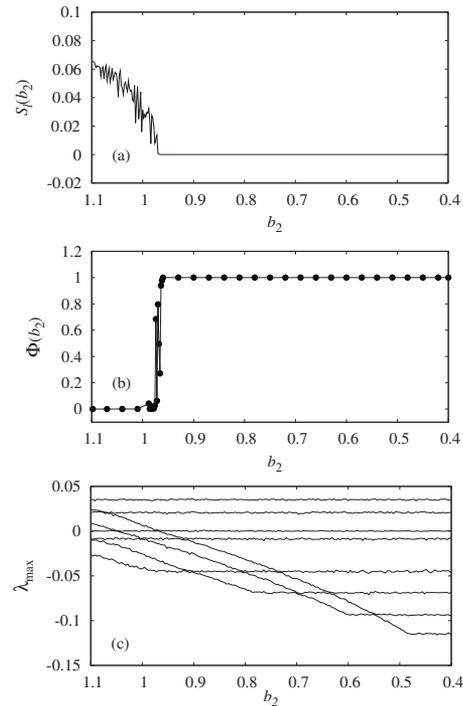


FIG. 7. (a) Similarity function,  $S_1(b_2)$ , (b) probability of synchronization,  $\Phi(b_2)$ , and (c) first eight maximal Lyapunov exponents,  $\lambda_{\max}$ , of the coupled piecewise linear time-delay systems as a function of the control parameter  $b_2$ , indicating the existence of ILS for  $b_2 < 0.97$  and for the coupling delay  $\tau_2=10.0$ .

**IV. INVERSE SYNCHRONIZATIONS IN THE COUPLED IKEDA TIME-DELAY SYSTEMS [EQS. (1) AND (3)]**

In this section, as an illustration for the case where the coefficient of the  $\Delta$  term in the error equation (6) is constant, while that of the  $\Delta_\tau$  term is time dependent, we will demonstrate the existence of inverse anticipatory, inverse complete, and ILSs as a function of the coupling delay in the coupled Ikeda time-delay systems (1) and (3) with inhibitory coupling.

**A. IAS**

We have fixed the values of the parameters of the coupled Ikeda time-delay systems [Eqs. (1) and (3)] as  $a=1.0$ ,  $b_1=5$ ,  $\tau_1=4.0$ , and  $\tau_2=3.0$ , while the other two parameters  $b_2$  and  $b_3$  are fixed according to the parametric condition  $b_1=b_2+b_3$ . To appreciate the chaotic and hyperchaotic nature of the uncoupled system, we present in Fig. 8(a) the first 11 largest Lyapunov exponents for the above values of the parameters in the range of delay time  $\tau \in (2, 25)$  where several of them take positive values. As an illustration, the hyperchaotic attractor with three positive Lyapunov exponents of the uncoupled system for  $\tau_1=\tau=4.0$  is depicted in Fig. 8(b).

Now, we will demonstrate the existence of IAS for the value of the coupling delay  $\tau_2$  less than that of the feedback delay  $\tau_1$  and for the values of the other parameters satisfying the stability condition (13). It is evident from Fig. 8(b) that

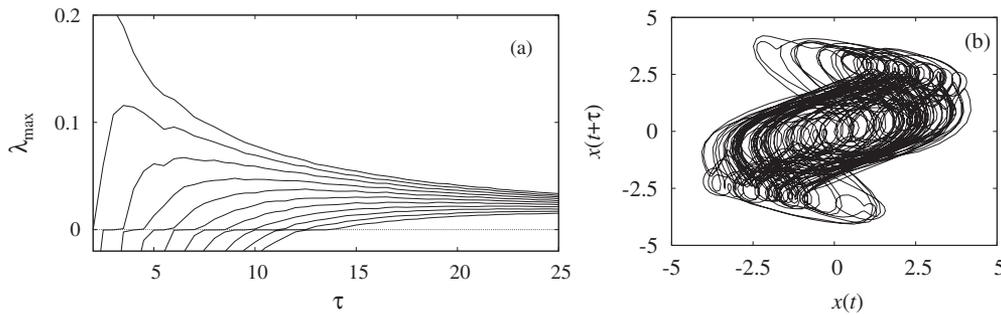


FIG. 8. (a) The first 11 maximal Lyapunov exponents  $\lambda_{\max}$  of the Ikeda time-delay systems [Eqs. (1a) and (3)] for  $a=1.0$ ,  $b=5$ , and  $\tau \in (2, 25)$ . (b) Hyperchaotic attractor for the delay time  $\tau_1 = \tau = 4.0$  with three positive Lyapunov exponents for the above values of the other parameters.

the maximum value of  $x(t)$  does not exceed  $x_{\max} = 5$ . Correspondingly,  $x(t - \tau_2)_{\max} = 5$ . As a consequence, the stability condition (13) can be written as

$$a > b_2 \cos(5) = 0.284b_2. \tag{18}$$

Then, one can obtain an asymptotically stable synchronized state for the values of parameters satisfying the above stability condition. The time trajectories of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  depicting the existence of IAS are plotted in Fig. 9 for the value of the parameter  $b_2 = 2.0$  satisfying the stability condition (18). The minimum of the similarity function,  $S_a(b_2)$ , defined by Eq. (14) becomes zero for  $b_2 < 2.88$  as shown in Fig. 10(a), indicating the existence of IAS in the coupled Ikeda time-delay system.

The existence of IAS is further characterized by the probability of synchronization,  $\Phi(b_2)$ , for complete synchronization between the response,  $y(t)$ , and the auxiliary,  $z(t)$ , systems by augmenting the coupled Ikeda time-delay systems [Eqs. (1) and (3)] with an additional auxiliary system for the variable  $z(t)$  identical to the response system as in Eq. (15). Figure 10(b) indicates that the value of the probability of synchronization becomes  $\Phi(b_2) = 1.0$  for  $b_2 < 2.88$ , confirming the existence of IAS between the coupled drive,  $x(t)$ , and the response,  $y(t)$ , systems. The existence of IAS for  $b_2 < 2.88$  is further confirmed from the changes in the spectrum of the largest Lyapunov exponents of the coupled Ikeda time-delay systems [Eqs. (1) and (3)]. The nine largest Lyapunov exponents,  $\lambda_{\max}$ , is shown in Fig. 10(c) as a function of  $b_2$ . While the largest three positive Lyapunov exponents of the drive system remain positive, the largest positive exponents of the response system gradually become negative

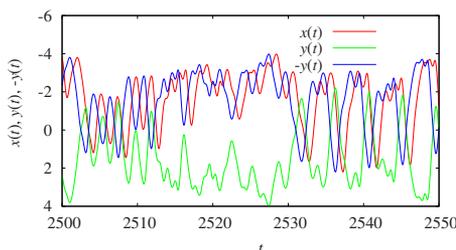


FIG. 9. (Color online) The time trajectory of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  for  $a=1.0$ ,  $b_1=5.0$ ,  $b_2=2.0$ ,  $b_3=3.0$ ,  $\tau_1=4.0$ , and  $\tau_2=3.0$  of the coupled Ikeda time-delay systems, indicating IAS.

as a function of  $b_2$  and finally they become negative at  $b_2 = 2.88$ , confirming the existence of IAS for  $b_2 < 2.88$  satisfying the stability condition (18).

### B. ICS

The time series of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  indicating the existence of ICS, for the value of the coupling delay  $\tau_2 = \tau_1 = 4.0$ , and for the same values of the other parameters satisfying the stability condition (18) as in Sec. IV A, are shown in Fig. 11. The similarity function,  $S_c(b_2)$ , given by Eq. (16), for ICS shown in Fig. 12(a) indicates that the minimum of  $S_c(b_2) = 0$  for  $b_2 < 2.88$ , depicting the existence of ICS in the corresponding range of  $b_2$ . It is also evident from the probability of synchronization [Fig. 12(b)] that  $\Phi(b_2) = 1$  for  $b_2 < 2.88$ , confirming the existence of ICS between the drive and the response systems. This transition

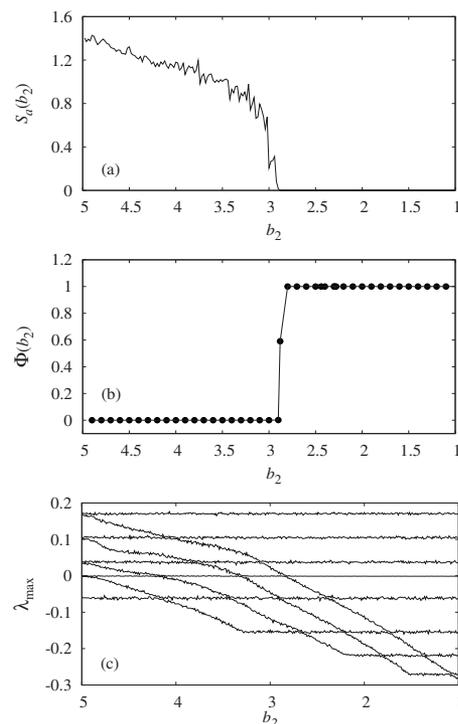


FIG. 10. (a) Similarity function,  $S_f(b_2)$ , (b) probability of synchronization,  $\Phi(b_2)$ , and (c) first nine maximal Lyapunov exponents,  $\lambda_{\max}$ , of the coupled Ikeda time-delay systems as a function of the control parameter  $b_2$ , indicating the existence of IAS for  $b_2 < 2.88$  and for the coupling delay  $\tau_2 = 3.0$ .

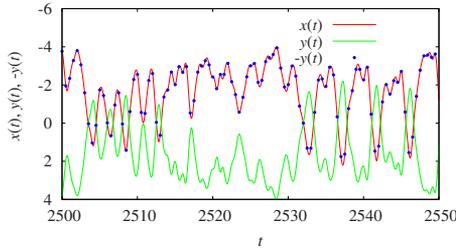


FIG. 11. (Color online) The time trajectory of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  for  $a=1.0$ ,  $b_1=5.0$ ,  $b_2=2.0$ ,  $b_3=3.0$ ,  $\tau_1=4.0$ , and  $\tau_2=4.0$  of the coupled Ikeda time-delay systems, indicating ICS.

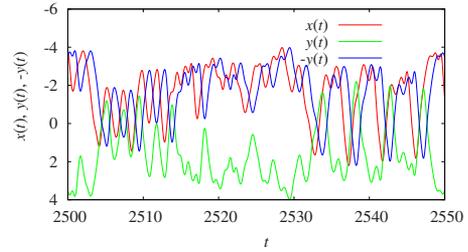


FIG. 13. (Color online) The time trajectory of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  for  $a=1.0$ ,  $b_1=5.0$ ,  $b_2=2.0$ ,  $b_3=3.0$ ,  $\tau_1=4.0$ , and  $\tau_2=5.0$  of the coupled Ikeda time-delay systems, indicating ILS.

from a desynchronized state to ICS for the values of parameters satisfying the stability condition (18) is also confirmed from the changes in the spectrum of the Lyapunov exponents of the coupled Ikeda systems, as shown in Fig. 12(c). The largest positive Lyapunov exponents of the response system become negative at  $b_2=2.88$ , confirming the existence of ICS for  $b_2 < 2.88$ , while the largest three positive Lyapunov exponents of the drive system remain positive.

**C. ILS**

For the coupling delay  $\tau_2=5.0$  greater than the feedback delay  $\tau_1=4.0$ , the synchronization manifold  $\Delta=0$  corresponds to the ILS manifold. The time trajectories of the variables  $x(t)$ ,  $y(t)$ , and  $-y(t)$  of the coupled Ikeda time-delay systems (Fig. 13) clearly depict the ILS for the above choice of the coupling delay [the other parameters are fixed as in Sec. IV A satisfying the stability condition (18)]. The mini-

mum of the similarity function for ILS turns out to be  $S_I(b_2)=0$  as shown in Fig. 14(a) for  $b_2 < 2.88$ , indicating the existence of ILS for  $b_2 < 2.88$ . Similarly, the value of the probability of synchronization [Fig. 14(b)] becomes unity in the corresponding range of  $b_2$ , confirming the existence of ILS between the coupled Ikeda time-delay systems. The existence of ILS for  $b_2 < 2.88$  is also further confirmed by the changes in the spectrum of the Lyapunov exponents of the coupled Ikeda time-delay systems, as shown in Fig. 14(c). The largest three positive Lyapunov exponents of the drive system remain unchanged, while those of the response system decrease in their values as a function of  $b_2$  and they become negative at  $b_2=2.88$ , confirming the emergence of ILS in coupled Ikeda time-delay systems.

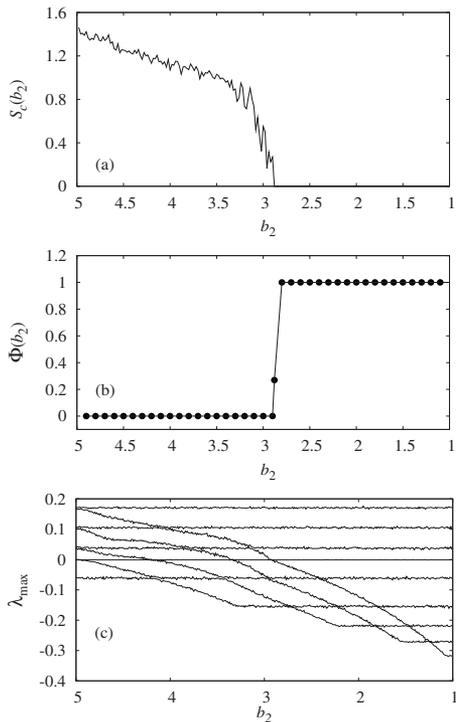


FIG. 12. (a) Similarity function,  $S_I(b_2)$ , (b) probability of synchronization,  $\Phi(b_2)$ , and (c) first nine maximal Lyapunov exponents,  $\lambda_{max}$ , of the coupled Ikeda time-delay systems as a function of the control parameter  $b_2$ , indicating the existence of ICS for  $b_2 < 2.88$  and for the coupling delay  $\tau_2=\tau_1=4.0$ .

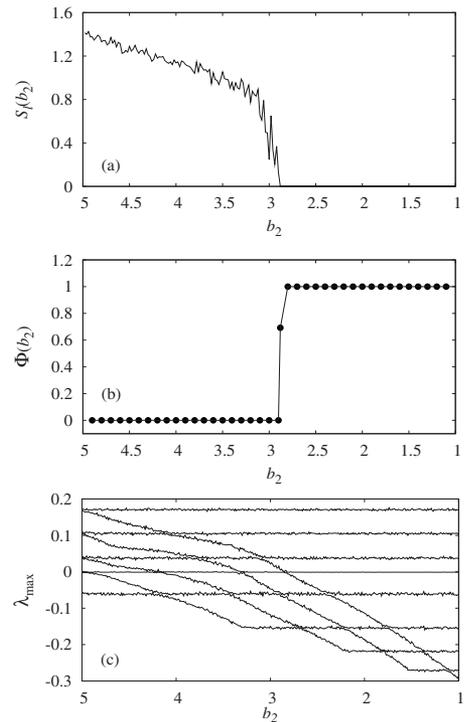


FIG. 14. (a) Similarity function,  $S_I(b_2)$ , (b) probability of synchronization,  $\Phi(b_2)$ , and (c) first nine maximal Lyapunov exponents,  $\lambda_{max}$ , of the coupled Ikeda time-delay systems as a function of the control parameter  $b_2$ , indicating the existence of ILS for  $b_2 < 2.88$  and for the coupling delay  $\tau_2=5.0$ .

## V. SUMMARY AND CONCLUSION

In this paper, we have shown the transition from inverse anticipatory to inverse lag via ICS as a function of the coupling delay  $\tau_2$  for fixed value of the feedback delay  $\tau_1$  in unidirectionally coupled time-delay systems with inhibitory coupling. We have also arrived at a suitable stability condition for the asymptotic stability of the synchronized states using the Krasovskii–Lyapunov functional theory. We have demonstrated that the same general stability condition resulting from Krasovskii–Lyapunov functional approach can be valid for two different cases, where (i) both the coefficients of the  $\Delta$  and  $\Delta_\tau$  terms of the error equation corresponding to synchronization manifold are constants and (ii) the coefficient of the  $\Delta_\tau$  term is time dependent, while that of the other is time independent using suitable examples. The existence of different types of inverse synchronizations is corroborated using similarity function, probability of synchronization, and from the changes in the spectrum of the largest Lyapunov exponents of the coupled time-delay systems. We have also designed suitable couplings for the case where both coefficients of the  $\Delta$  and  $\Delta_\tau$  terms are time dependent to show the validity of the same general stability condition (10) resulting from the Krasovskii–Lyapunov functional theory, the results of that will be published in a forthcoming paper.

## ACKNOWLEDGMENTS

The work of D.V.S. has been supported by Alexander von Humboldt Foundation. J.K. has been supported by his Humboldt-CSIR research award and NoE BIOSIM (EU) Contract No. LSHB-CT-2004-005137. M.L. acknowledges the support from Department of Science and Technology (DST), Government of India sponsored IRHPA research project and DST Raja Ramanna Program.

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