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The effect of time-delay on anomalous phase synchronization

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ABSTRACT

Anomalous phase synchronization in nonidentical interacting oscillators is manifest as the increase of frequency disorder prior to synchronization. We show that this effect can be enhanced when a time-delay is included in the coupling. In systems of limit-cycle and chaotic oscillators we find that the regions of phase disorder and phase synchronization can be interwoven in the parameter space such that as a function of coupling or time-delay the system shows transitions from phase ordering to disorder and back.

1. Introduction

A population of nonidentical nonlinear oscillators will eventually synchronize when sufficiently strongly coupled in a suitable topology. In some cases it has been observed that the degree of disorder in the oscillator frequencies—as characterized by their variance, say—first increases with coupling before eventually decreasing. This phenomenon has been termed *anomalous* phase synchronization (APS) [1–4], which has a broad range of applications.

The emergence of synchrony in groups of interacting nonidentical oscillators is a phenomenon of considerable interest [5], with applications in a variety of contexts ranging from epidemiology to cellular biology. Phase synchronization (PS) [6]—as opposed to complete synchronization—arises naturally in many areas of physical sciences since the subsystems that are coupled can have different amplitudes and a range of internal time-scales.

How does global phase synchronization come about in such a population? The manner in which all the oscillators in a mutually interacting group eventually adopt a common frequency of oscillation is of considerable importance, and one which has been explored to some extent in earlier work [1–3]. A natural expectation might be that the approach to global synchronization is monotonic: namely that two of the oscillators synchronize, then three, and then gradually increasing numbers of oscillators mutually synchronize.

This expectation does not hold in systems that show the anomalous phase synchronization. The interaction among the different systems acts to first drive systems out of synchrony before the strength of the interaction eventually forces them to a common dynamics. The intermediate disorder can arise from a number of different sources—non-isochronicity [7], shear [8] or differences in other internal parameters [5]. The full generality of this phenomenon is not known, and thus the APS is of interest both from a conceptual as well as an applications point of view.

In the present work we study the process of phase-synchronization in oscillators with time-delayed coupling. Time-delay is both natural and inevitable when considering interactions among systems that are spatially separated. From a mathematical point of view, time-delay makes the dynamical system effectively infinite-dimensional: this can open up a range of time-scales, interactions, and novel dynamical behaviour such as amplitude-death [9–12] and the phase-flip bifurcation [12,13]. In addition, delay offers an additional parameter that can be varied, and if exogenous, can provide a suitable means of effecting control. The implications of APS in such systems is therefore of considerable interest.

Our main result here is that APS can be enhanced with time-delay: the degree of initial disorder may be significantly larger than for the zero time-delay (or instantaneous coupling) case [1,2]. We also find that in situations when there is no APS, delay coupling can cause APS to occur. Using the delay or the coupling

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strength as a parameter, systems which are in phase synchrony can be driven out of synchrony and back again into phase synchrony. The regions of parameter space corresponding to desynchronized dynamics are interspersed among those corresponding to the synchronized phase, and thus the anomalous behaviour can be manifest as a transition from PS to phase disorder, and back to PS.

Some insight can be obtained from a study of the simplest situation, namely of two nonidentical delay coupled oscillators and we treat this case in the next section. The specific models we examine are the Landau–Stuart system, where the dynamics can be on limit cycles, and a somewhat more complex food-web model studied in [1,2], where APS was observed. Globally coupled oscillators are studied in Section 3. In all these examples, our results indicate that upon inclusion of time-delay, the region of APS can be enlarged, and the degree of disorder can be enhanced. The Letter concludes with a brief summary in Section 4.

2. Anomalous synchronization in two nonidentical oscillators

2.1. The Landau–Stuart system

Consider the case of two delay-coupled limit cycle Landau–Stuart oscillators [11,12]:

$$\dot{Z}_1(t) = \left(1 + i\Omega_1 - \left|Z_1(t)\right|^2\right)Z_1(t) + \epsilon\left[Z_2(t-\tau) - Z_1(t)\right],$$
$$\dot{Z}_2(t) = \left(1 + i\Omega_2 - \left|Z_2(t)\right|^2\right)Z_2(t) + \epsilon\left[Z_1(t-\tau) - Z_2(t)\right]. \tag{1}$$

The variables $Z_j(t)$ are the complex amplitudes of the oscillators, $|Z_j| = 1$ is the attracting limit cycle, Ω_j are the corresponding frequencies in absence of coupling, and ϵ is the coupling strength. We consider here strongly mismatched oscillators with $\Omega_1 = 4$ and $\Omega_2 = 7$ ($\Delta\Omega = 3$). For the case of instantaneous coupling, $\tau = 0$, the effective averaged frequencies $\omega_{1,2}$ of the two oscillators are plotted in Fig. 1(a) as a function of the coupling strength ϵ. The frequency difference $\Delta\omega$ decreases monotonically as shown in Fig. 1(b) and PS results above a critical coupling, $\epsilon \sim 1.5$. At these parameter values, APS does not occur: see Refs. [1,2] for details.

Introduction of a finite delay ($\tau \neq 0$) gives the phase diagram shown in Fig. 2; the oscillators are not in synchrony in the shaded region in the τ–ϵ plane and therefore there is the possibility of APS. Along a line of fixed coupling strength $\epsilon = 1$, say, examination of the largest few Lyapunov exponents [14] and the difference in frequency of the two oscillators as a function of τ reveals APS, as shown in Fig. 3(a), (b). As can be seen there is a finite range of time-delay for which the individual oscillator frequencies are different, and the difference $\Delta\omega$ can be larger than the initial $\Delta\Omega = 3$. Note that this region of anomaly is in effect caused by time-delay, since when $\tau = 0$, there is no region of APS (Fig. 1). Similar results have been also reported for phase only oscillators in Ref. [15] and subsequently in Ref. [16–19]. However, unlike the situation in Ref. [15], we do not find any evidence for hysteresis here: the curve in Fig. 3(b) is a composite of 50 simulations, each starting from different initial conditions (ruling out the possibility of multistability). Over a broad range of coupling parameters (results not shown here) there does not appear to be any hysteresis.

As is also evident in Fig. 2, APS can occur at fixed time-delay by variation of the coupling parameter ϵ. Note that the introduction of delay can reduce the onset of synchronization (see for small τ in Fig. 2). Results are shown in Fig. 3(c) and (d) for the Lyapunov exponentand frequency difference respectively at $\tau = 0.75$. Since this occurs with variation of the coupling strength for a fixed nonzero time-delay as well, the process is an example of APS arising from the time-delay interaction.

For the symmetric system when $\Omega_1 = \Omega_2$, there is no anomalous synchronization, but there is evidence for a phase-flip bifurcation [13] when the frequency increases in a manner similar to the

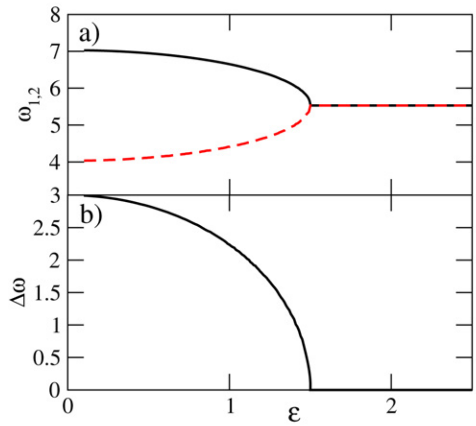

Fig. 1. In the Landau–Stuart system, Eq. (1) for the case of instantaneous coupling, $\tau = 0$, the variation of (a) the individual frequencies, $\omega_{1,2}$ of the two oscillators, and (b) their frequency difference, $\Delta\omega$ as a function of the coupling strength, ϵ.

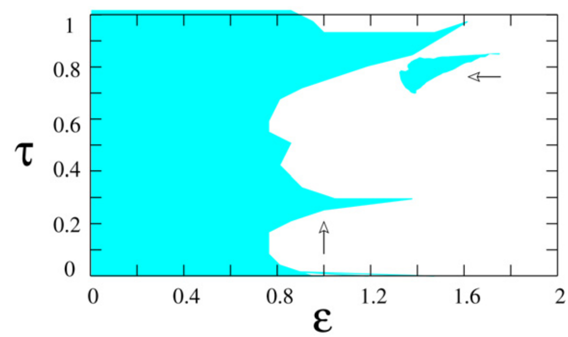

Fig. 2. Schematic phase diagram in the τ–ϵ plane for nonidentical Landau–Stuart oscillators, Eq. (1). The shaded region corresponds to desynchronized motion, and therefore APS can occur when parameter variation includes a path that traverses this region.

frequency increase shown in Fig. 3(b). This suggests that the phenomenon of anomalous synchronization may be the counterpart of the phase-flip in nonidentical systems. Since experimental verification of the phase-flip has been carried out in recent work [20], a systematic exploration of anomalous phase synchronization in a similar dynamical system (delay-coupled Chua oscillators) should be feasible.

Because of the nature of the boundary of the synchronized region, the anomalous effect can be evidenced as a transition from a synchronized state to a desynchronized state, and back to synchrony, as in Fig. 3(b), (d); see the parameter range marked by arrow A. In this system APS occurs when the dynamics is quasiperiodic ($\lambda_1 = \lambda_2 = 0$) but in general, the motion can even be chaotic, as in the example below.

2.2. Chaotic oscillators

We next analyze a model that has been studied earlier [1,2],

$$\dot{x}_{1,2}(t) = x_{1,2} - 1.5 - 0.1x_{1,2}y_{1,2},$$
$$\dot{y}_{1,2}(t) = -\beta_{1,2}y_{1,2} + 0.1x_{1,2}y_{1,2} - 0.6y_{1,2}$$
$$+ \epsilon\left[y_{2,1}(t-\tau) - y_{1,2}(t)\right],$$
$$\dot{z}_{1,2}(t) = -10z_{1,2} + 0.1 + 0.6y_{1,2}z_{1,2}. \tag{2}$$

This is a system of two coupled food-webs, each of which (differentiated by subscripts 1 or 2) describes a three level "vertical" food chain. The variables x correspond to the vegetation, which is

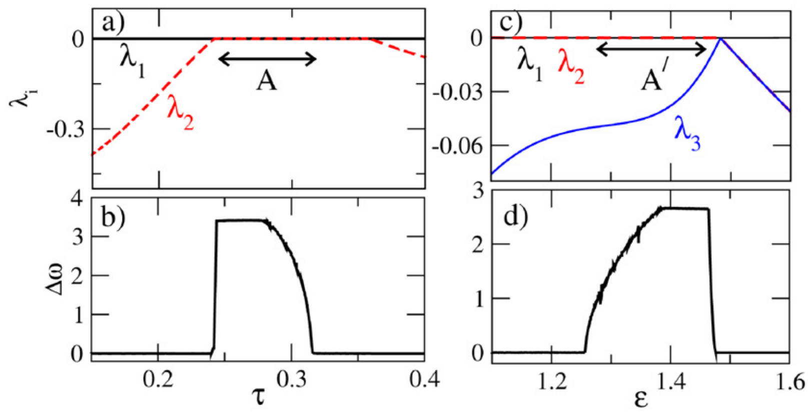

Fig. 3. (a), (c) The largest few Lyapunov exponents and the (b), (d) frequency difference between two coupled nonidentical Landau–Stuart oscillators, Eq. (1). The left panel shows the variation with τ at fixed $\epsilon = 1$ (vertical arrow in Fig. 2), while the right panel is for varying ϵ at fixed time-delay $\tau = 0.75$ (horizontal arrow in Fig. 2). APS occurs in the range marked A and A′. In (b) and (d) the frequency difference curve shown is a composite of 50 different simulations.

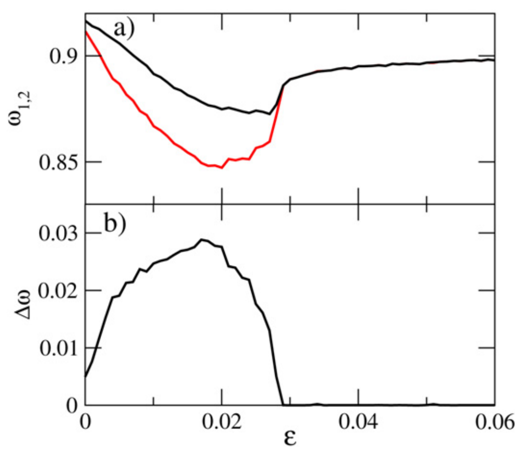

Fig. 4. (a) Individual frequencies and (b) frequency difference for $\tau = 0$ as a function of the coupling strength in the food-web system, Eq. (2).

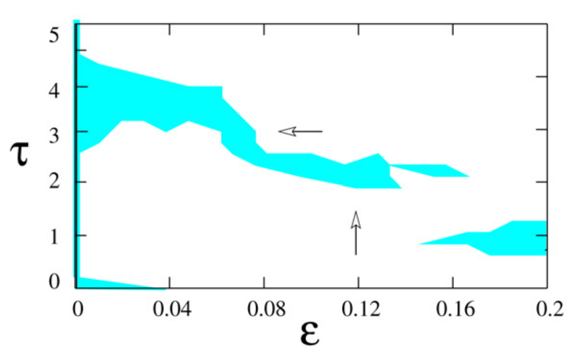

Fig. 5. Schematic phase diagram for the food-web oscillator system, Eq. (2) as a function of the parameters (ϵ, τ). The individual systems do not synchronize in the shaded region, and this indicates the region where anomalous phase synchronization can potentially be observed. Note that, near $\epsilon \sim 0$, there is a thin shaded region along delay.

consumed by herbivores (y) which themselves are preyed upon by the top predator (z). The individual uncoupled systems can show chaotic dynamics, and therefore in the coupled system the synchronized dynamics can also be chaotic.

To make contact with earlier studies [1,2] we use the same parameters as before, $\beta_1 = 0.945$ and $\beta_2 = 0.995$. As can be seen in Fig. 4, for instantaneous coupling i.e. $\tau = 0$ [1,2], there is a clear maximum in the frequency difference $\Delta\omega$ indicative of the anomalous synchronization region. A schematic phase diagram is shown in Fig. 5 where the shaded region indicates the lack of synchronization and thus the possibility of APS which can be observed by varying either of the parameters, coupling strength or time-delay.

For fixed $\epsilon = 0.12$, the largest few Lyapunov exponents are shown as a function of τ in Fig. 6(a). From the difference in the frequencies (Fig. 6(b)) it can be seen that the extent of anomaly appears to be significantly larger than in the case without delay. Thus time-delay can enhance the level of intrinsic frequency disorder. Also note that delay can reduce the onset of synchronization (at least near zero delay).

Similar results are obtained when the coupling parameter is varied for fixed time-delay, as shown in Fig. 6(c) for $\tau = 3$ where the largest few Lyapunov exponents are plotted. The frequency difference in Fig. 6(d) clearly indicates that there is APS on variation of coupling strength. The dynamics is multistable in the region marked R in Fig. 6(a). Note that the motion can be periodic ($\lambda_2 < 0$), quasiperiodic ($\lambda_1 = \lambda_2 = 0$) or even chaotic ($\lambda_1 > 0$) (see Figs. 6(a) and 6(c)) in the region of the anomaly.

3. Anomalous synchronization in an ensemble of nonidentical oscillators

Having considered the case of two coupled oscillators with time-delay, we now study the case of N globally (all to all) coupled systems. Shown in Fig. 7 are the variance in the frequencies of individual oscillators in $N = 100$ coupled Landau–Stuart systems with variation of time-delay and coupling strength. The internal frequencies of the uncoupled systems were taken uniformly in an interval, $\Omega_i \in [5, 10]$ and globally diffusive coupling is effected by adding the term

$$\epsilon \sum_{j=1, j\neq i}^{N} \big[z_j(t-\tau) - z_i(t)\big]/(N-1) \tag{3}$$

in the dynamical equations (cf. Eq. (1)). The variance in individual frequencies are shown in Fig. 7(a) as a function of τ for fixed coupling strength $\epsilon = 1.4$, and in (b) as a function of the coupling parameter ϵ for fixed time-delay $\tau = 1.6$. These clearly indicate that APS occurs over a wide range of parameters either in τ or ϵ. Similar results have been obtained when N nonidentical food-web systems are globally coupled, with external (Gaussian) noise, and for a variety of coupling topologies.

We believe that the origin of the anomalous phase synchronization lies in the fact that in coupled nonidentical nonlinear systems, regions of parameter space where synchronization can and can-

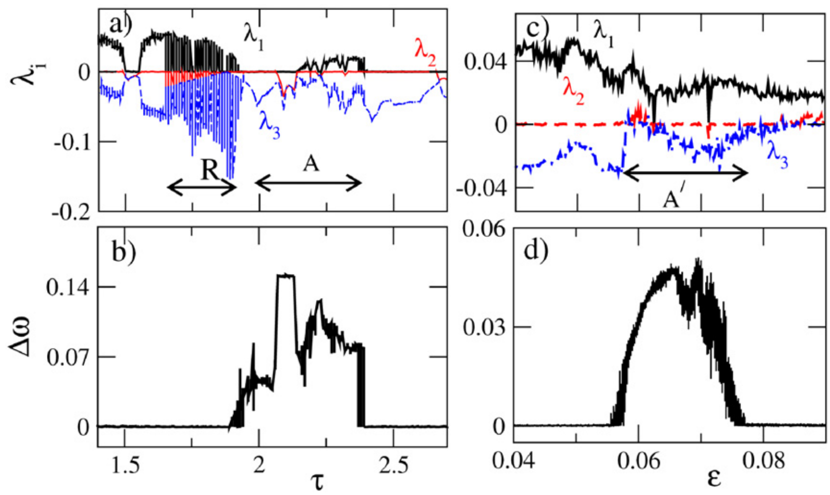

Fig. 6. For the coupled food–web system, (a), (c) the three largest Lyapunov Exponents (λ_1 (black), λ_2 (red) and λ_3 (blue)), and (b), (d) difference in individual frequencies, $\Delta\omega$, between the oscillators. The left panel shows the variation with τ at fixed $\epsilon = 0.12$ (vertical arrow in Fig. 5) while right panel has variable ϵ for fixed time-delay $\tau = 3$ (horizontal arrow in Fig. 5). In (b) and (d) the frequency difference curve shown is a composite of 50 different simulations. (For interpretation of the references to color in this figure legend, the reader in referred to the web version of this Letter.)

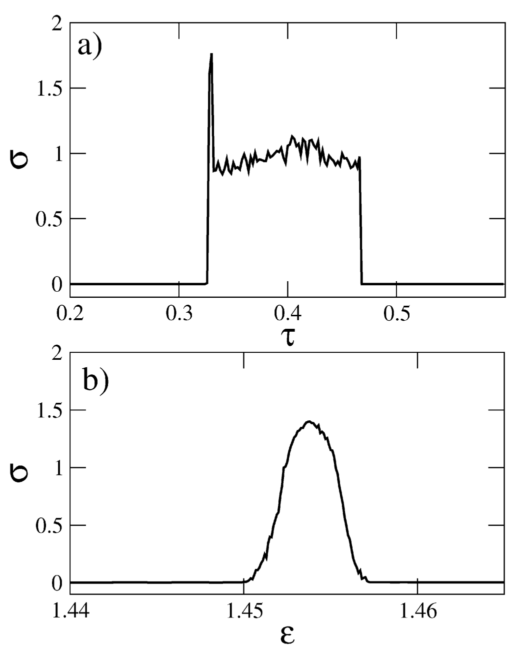

Fig. 7. Variance of individual frequencies σ for $N = 100$ globally coupled Landau–Stuart oscillators, Eq. (1) as a function of (a) τ for fixed $\epsilon = 1.4$ and (b) with ϵ for fixed $\tau = 1.6$.

not occur are interwoven in a complex manner. The two examples studied here (Figs. 2 and 5 are representative; for larger numbers of oscillators, the corresponding diagrams will be much more complex). Thus, when any single parameter is varied, there can be transitions from phase synchrony to phase disorder and back to phase synchrony.

4. Summary

Anomalous effects in phase synchronization in a group of nonidentical coupled systems can occur in two ways. Starting from the situation of no coupling, the approach to a synchronized regime can be nonmonotonic: the variance in the frequencies of the individual systems can increase before eventually vanishing. Alternately, given an ensemble of time-delayed coupled nonidentical oscillators in phase synchrony, variation of the parameters can take the system out of synchrony before eventually restoring it. In this latter case, the variance in the frequencies goes from zero to a finite value before again vanishing. In both situations, the origin of phase disorder lies in the complex manner in which regions of synchronization/desynchronization are arranged in parameter space.

In the present work we have considered time-delay in the interaction between nonidentical oscillators and seen that the phenomenon of anomalous phase synchronization can be enhanced in comparison with the case of instantaneous coupling ($\tau = 0$). Further, a system which does not show APS for the case of instantaneous coupling may show APS when time-delay is introduced. It should be possible to verify this effect experimentally; anomalous synchronization effects have a parallel in the phase-flip bifurcation which occurs for identical time-delay coupled systems [20]. The fact that APS can be observed through variation of additional parameters suggests that it can be controlled. In particular, it may be of interest to use such strategies when phase disorder is required [21].

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