

Network mutual information and synchronization under time transformations

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Abstract. We investigate the effect of general time transformations on the phase synchronization (PS) phenomenon and the mutual information rate (MIR) between pairs of nodes in dynamical networks. We demonstrate two important results concerning the invariance of both PS and the MIR. Under time transformations PS can neither be introduced nor destroyed and the MIR cannot be raised from zero. On the other hand, for proper time transformations the timing between the cycles of the coupled oscillators can be largely improved. Finally, we discuss the relevance of our findings for communication in dynamical networks.

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1. Introduction

Time and complex dynamics play a major role in biological, social, economical and physical systems. Cycles of different periods often govern their dynamical behavior and determine their intrinsic activity. A variety of processes require a precise timing between the oscillator cycles for a proper functioning, as for example, the respiratory and cardiac systems [1], spike discharges and information transmission [2, 3] in neuron networks, ecology [4], fireflies blinking after dark and pacemaker cells of the human heart [5]. Synchronization is an efficient mechanism to generate such a timing [1]–[11]. Among several types of synchronization recently found in complex systems [8], chaotic phase synchronization (PS) displays special importance because of its weak constraints on the dynamics and coupling strength. It has been reported that PS mediates the process of information transmission and collective behavior in neural and active networks [2, 11, 12], as well as communication processes in the human brain [2, 13, 14].

In real systems, PS is the most common type of synchronization [1, 2, 4, 5], [8]–[11], [13, 14]. The main reason for PS to be so common relies on the fact that real oscillators are not identical, but have some parameter mismatch. When real coupled oscillators undergo a transition to PS the timing is not precise. In many situations one wishes to improve the timing condition, but in fact, one cannot systematically control the oscillator parameters to drive them to a higher level of PS. The question is then how to improve the timing without changing the oscillator parameters. The natural candidate is a time transformation. Could one enhance a better timing by changing the time? Or even better, could one introduce PS by time transformations?

Coupled dynamical systems under time transformations are important in physics without an absolute time as well as in situations where the time cannot be directly obtained, as in the study of sedimental cores in the field of geophysics. In the latter case, the time at which the sedimentation took place is usually unknown. Only a proxy for the time can be derived from the measurements, which does not yield the ‘real’ time but only a monotonic transformation of it [15]. In the study of synchronization phenomenon in such a system the natural question is whether not having access to the *real* time could effect synchronization.

Such time transformations (typically nonlinear) have attracted a great deal of attention (see [16, 17] and references therein). They cause no change in the topology of the dynamics, but the duration of the cycles can be drastically modified. An important problem is to analyze whether the dynamical properties are invariant under time transformations [16]. Recent results have shown that dynamical systems under time transformation can present nontrivial and counterintuitive properties. For example, a nonmixing dynamics can be converted to a mixing one [17].

In this work, we show that time transformations, satisfying simple conditions of integrability, can neither introduce nor destroy the phenomenon of PS. We also explore the natural connection between synchronization and information exchange in coupled oscillators. We uncover the transformation law for the mutual information rate (MIR), the rate with which the information about a node can be retrieved in another node. If the MIR is zero in one time frame it will remain zero for any other. On the other hand, if the MIR is nonzero it can be drastically modified by a time transformation. *Surprisingly, if there is no synchronization (to any extent) between the nodes forming a network, time transformations containing information about a particular oscillator (node) of the network cannot be used to carry this information to another oscillator.*

2. Two oscillators case: enhancing a precise timing

We first illustrate our approach for the paradigmatic example of two coupled Rössler oscillators:

$$\dot{x}_{1,2} = -\alpha_{1,2}y_{1,2} - z_{1,2} + \epsilon(x_{2,1} - x_{1,2}), \quad (1)$$

$$\dot{y}_{1,2} = \alpha_{1,2}x_{1,2} + 0.15y_{1,2}, \quad (2)$$

$$\dot{z}_{1,2} = 0.2 + z_{1,2}(x_{1,2} - 10) \quad (3)$$

with $\alpha_1 = 1$ and $\alpha_2 = \alpha_1 + \Delta\alpha_2$. We shall denote $\mathbf{x}_j = (x_j, y_j, z_j)$, where $j = 1, 2$, and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2)$. Since for these oscillators the trajectory revolves around one specific point (figure 1(a)), we can simply define a phase by $\tan \phi_j = y_j/x_j$, which yields [18]

$$\phi_j(\mathbf{x}, t) = \int_0^t (\dot{y}_j x_j - \dot{x}_j y_j) / (x_j^2 + y_j^2) dt. \quad (4)$$

Furthermore, let us denote the time at which the oscillator \mathbf{x}_j completes its i th cycle by t_j^i . That is, the times at which the phase is increased by 2π (see appendix A for more details). We can show that there is PS if, and only if, we have

$$|t_1^i - t_2^i| \leq \kappa, \quad (5)$$

where κ is the minimum finite number that bounds the inequality. For more details concerning this equivalence see appendix A. The value of κ shows how well paced both oscillators are. The smaller the value of κ the better the timing between \mathbf{x}_1 and \mathbf{x}_2 .

For $\epsilon = 0.0015$ and $\Delta\alpha_2 = 0.001$, the two oscillators are in PS, which means that the phase difference $\Delta\phi = \phi_1(t) - \phi_2(t)$ is bounded for all times. Consequently, equation (5) holds (figures 1(b) and (c)). In the PS regime, the oscillators have the same mean frequency, namely

$$\langle \dot{\phi}_1 \rangle_t = \langle \dot{\phi}_2 \rangle_t \approx 1.035,$$

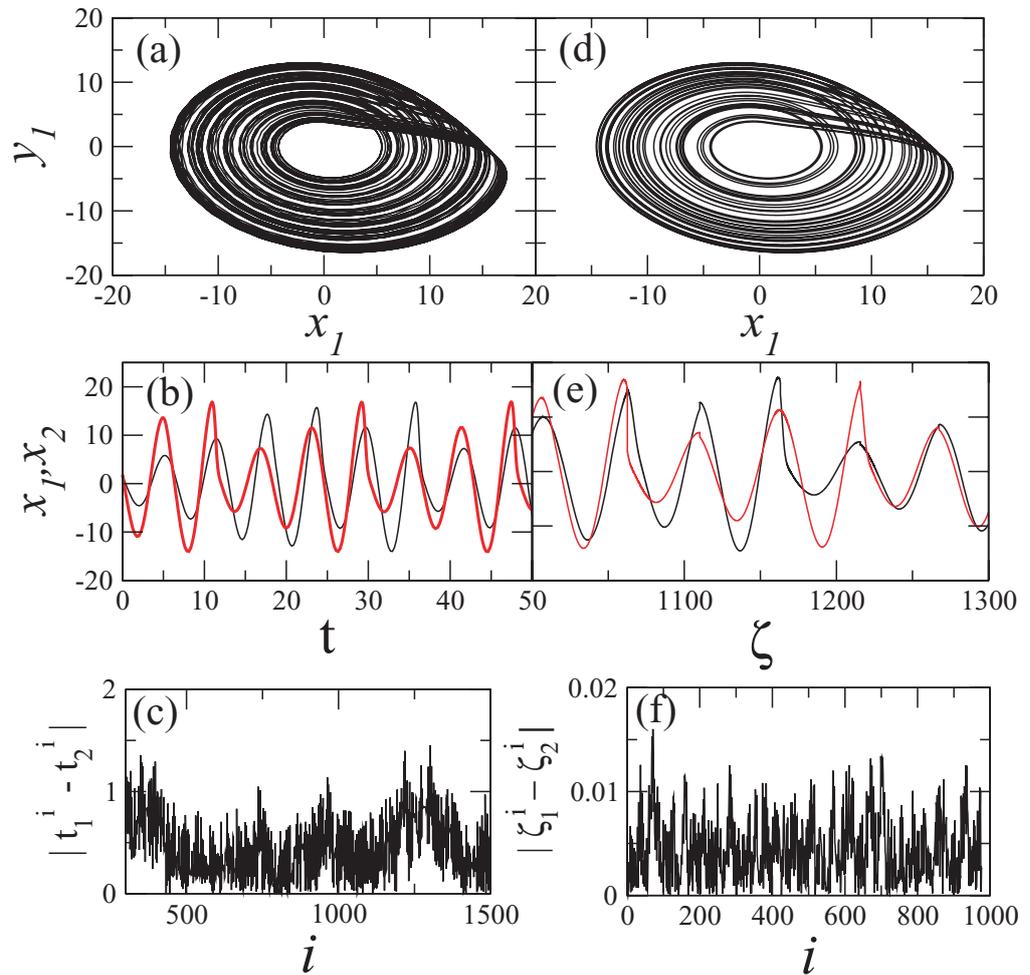


Figure 1. The effect of time transformation in two synchronous Rössler oscillators. In (a), we show the attractor projection onto the (x, y) -plane, and in (b) the time series x_1 and x_2 versus t . In (c), we depict the time difference $|t_1^i - t_2^i|$ for the i th period between both oscillators. One can see that even though the quantity $|t_1^i - t_2^i|$ is bounded, it has large fluctuations. In (d)–(f), we proceed a time transformation $t \rightarrow \zeta$ given by equation (7). In (d), we show that the attractor projection in the subspace (x, y) , and the time series x_1 and x_2 versus ζ in (e), whereas the amount $|\zeta_1^i - \zeta_2^i|$ is shown in (f). One can see that after the time transformation the timing condition is drastically improved.

where $\langle \cdot \rangle_t$ is the time average with respect to t . The average period is given by

$$\langle T_j \rangle_t = 2\pi / \langle \dot{\phi}_j \rangle_t \approx 6.067.$$

We have that $\max |t_1^i - t_2^i|$ corresponds approximately to $\langle T_j \rangle_t / 4$ (figure 1(c)), which can be rather problematic for a reliable communication system based on chaos synchronization, since the two oscillators do not reach the Poincaré section with a precise timing. See [11] for a detailed discussion.

The main question is then what could happen when we transform the time. Could we produce an effective improvement of the timing? If we linearly scale the time introducing $\zeta = \alpha t$, then the average period transforms as

$$\langle T_j \rangle_\zeta = \frac{\langle T_j \rangle_t}{\alpha},$$

whereas the timing

$$|\zeta_1^i - \zeta_2^i| = \frac{|t_1^i - t_2^i|}{\alpha}.$$

Thus, there is no effective improvement of the timing with respect to the average period, since

$$\frac{|\zeta_1^i - \zeta_2^i|}{\langle T_j \rangle_\zeta} = \frac{|t_1^i - t_2^i|}{\langle T_j \rangle_t}.$$

The situation can be altered to improve the timing condition in equation (5) by using a nonlinear time transformation, namely $t \rightarrow \zeta$ of the form:

$$d\zeta = \lambda(\mathbf{x}, t) dt. \quad (6)$$

Such a transformation may distort directly the synchronization phenomenon acting on the times t_j^i . To improve the timing between the oscillators given $\gamma \gg 1$ and $\sigma < 1$, we perform the time transformation:

$$\lambda(\mathbf{x}, t) = \begin{cases} \gamma, & \text{if } x_{1,2} > 0, x_{2,1} < 0 \text{ and } \dot{y}_1 > 0, \\ \sigma, & \text{otherwise,} \end{cases} \quad (7)$$

which shrinks the time between t_1^i and t_2^i enhancing a more accurate pacing between the oscillators. γ may be chosen according to the pacing condition desired. For our purpose, we fix $\gamma = 100$. We can use the parameter σ to control the average period. In the following, we fix $\sigma = 0.11$. The new time is given by $\zeta_j = \int_0^{t_j} \lambda(\mathbf{x}, t) dt$. The equation of motion now takes the form

$$\dot{x}_{1,2} = \lambda^{-1}(\mathbf{x}, t)[- \alpha_{1,2} y_{1,2} - z_{1,2} + \epsilon(x_{2,1} - x_{1,2})], \quad (8)$$

$$\dot{y}_{1,2} = \lambda^{-1}(\mathbf{x}, t)[\alpha_{1,2} x_{1,2} + 0.15 y_{1,2}], \quad (9)$$

$$\dot{z}_{1,2} = \lambda^{-1}(\mathbf{x}, t)[0.2 + z_{1,2}(x_{1,2} - 10)]. \quad (10)$$

The time transformation causes no changes in the state space, compare figures 1(a) and (b). However, the time series of $x_1 \times t$ and $x_1 \times \zeta$ are drastically modified (figures 1(b) and (e)). Although, the time transformation is not able to interfere with the PS phenomenon (figure 1(f)), it changes the frequency of the oscillators

$$\langle \dot{\phi}_1 \rangle_\zeta = \langle \dot{\phi}_2 \rangle_\zeta \approx 0.998,$$

which implies that $\langle T_{1,2} \rangle_\zeta \approx 6.296$. On the other hand, now $\max|\zeta_1^i - \zeta_2^i| \approx \langle T_j \rangle_\zeta / 420$. Remembering that $\max|t_1^i - t_2^i| \approx 2\langle T_j \rangle_t / 3$, we conclude that this time transformation yields an improvement of a factor of 280 for the timing. Of course, equation (7) can be altered to have an even better timing. These ideas can also be applied to a network. Whenever there is a cluster of oscillators in PS within the network, one can transform the time by equation (6) suitably choosing $\lambda(\mathbf{x}, t)$ to have a precise timing among all oscillators of the PS cluster.

3. Phase diffusion and coherence

Time transformation cannot destroy the synchronization. On the other hand, it does alter important characteristics of the dynamics, for example the coherence of the oscillators and the phase diffusion. By a time transformation we can transform an oscillator that originally is endowed with phase diffusion into an oscillator with an arbitrarily small phase diffusion. An interesting point is that in data analysis the phase diffusion plays a role in order to detect PS [19]. The idea is that one can detect PS by variations in the phase diffusion.

In general the phase depends on the amplitude of the oscillator and the frequency can be written as: $\dot{\phi}(t) = \omega + \xi(\mathbf{x}, t)$, where ω is the average frequency of the oscillator and $\xi(\mathbf{x}, t)$, in many cases, acts as an effective noise due to the chaotic nature of the oscillator [8]. Therefore, the phase dynamics is generally diffusive, which means that for large time intervals one expects $\langle [\phi(t) - \omega t]^2 \rangle_\mu \approx \Gamma t$, where $\langle \cdot \rangle_\mu$ denotes the ensemble average, and Γ the diffusion constant. Having the time of the i th cycle of the oscillator \mathbf{x}_j , we can write $t_j^i = i \langle T_j \rangle + v_j^i$, where $\langle T_j \rangle$ is the average period. By calculating the phase diffusion, we have

$$\langle [\phi_j(t_j^i) - \omega_j(i \langle T_j \rangle + v_j^i)]^2 \rangle_\mu = \omega_j^2 \langle [v_j^i]^2 \rangle_\mu \approx \Gamma_j t.$$

Hence, $t_\ell^i - i \langle T_\ell \rangle$ gives the phase diffusion properties.

Let us analyze the distortions in the phase diffusion by a time transformation and its effect on PS. Supposing that the oscillators \mathbf{x}_1 and \mathbf{x}_2 are not in PS, we write $t_1^i - t_2^i = \alpha \times i + \xi^i$, where $\alpha, \xi^i \in \mathbb{R}$ are chosen to hold the equality. By performing a time coordinate change we endow the oscillator \mathbf{x}_1 with zero phase diffusion. This means that we have a new time ζ with

$$\Delta \zeta_1^i = \zeta_1^i - \zeta_1^{i-1} = 1, \quad (11)$$

i.e. $\Delta \zeta_1^i = \Delta t_1^i / \Delta t_1^i$, where $\Delta t_1^i = t_1^i - t_1^{i-1}$. The new time coordinate is given by

$$\zeta_1^i = \sum_{n=0}^i \Delta t_1^n / \Delta t_1^n,$$

$$\zeta_2^i = \sum_{n=0}^i \Delta t_2^n / \Delta t_1^n.$$

We have

$$|\zeta_1^i - \zeta_2^i| = \left| \sum_i (\Delta t_1^i - \Delta t_2^i) / \Delta t_1^i \right|. \quad (12)$$

Next, consider the maximum Δt_1^i , namely

$$\max_i \Delta t_1^i = \gamma^{-1}.$$

Thus, we have

$$\left| \sum_i (\Delta t_2^i - \Delta t_1^i) / \Delta t_1^i \right| \geq \gamma |t_2^i - t_1^i|, \quad (13)$$

which can be written as:

$$|\zeta_1^i - \zeta_2^i| \geq \gamma (\alpha \times i + \xi^i). \quad (14)$$

Therefore, as the number of periods tends to infinity, the time event difference $|\zeta_1^i - \zeta_2^i|$ diverges. Thus, enhancing coherence in the oscillator does not introduce PS.

4. Breaking down the hypotheses on λ

By violating the conditions (ii) and (iii), which guarantees the boundedness of $\lambda(\mathbf{x}, t)$, PS can be introduced. Considering our former case where

$$t_1^i - t_2^i = \alpha \times i + \xi^i, \quad (15)$$

we could transform the time by

$$\lambda(\mathbf{x}, t) = \frac{1}{i}, \quad \text{if } t_1^i < t \leq t_1^{i+1}. \quad (16)$$

The timing condition is given by

$$|\zeta_1^i - \zeta_2^i| = \left| \int_{t_1^i}^{t_1^{i+1} - \alpha i - \xi^i} \lambda(\mathbf{x}, t) dt \right| \leq |\alpha i / i| + |\xi^i / i|. \quad (17)$$

Thus,

$$\lim_{i \rightarrow \infty} |\zeta_1^i - \zeta_2^i| \leq \alpha, \quad (18)$$

the time difference is bounded by α . Hence, this time transformation allows us to introduce PS between \mathbf{x}_1 and \mathbf{x}_2 . This does not contradict our results, because this time transformation is not bounded, violating the assumptions (ii) and (iii). When one uses a non-bounded transformation $\lambda(\mathbf{x}, t)$, like the latter one, the time is shrunk and becomes meaningless; there is no long term behavior with respect to ζ , since $\lim_{i \rightarrow \infty} \zeta_\ell^i$ is still bounded. The function $\lambda(\mathbf{x}, t)$ can be made smooth without changing equation (18). Here, we have considered $\lambda(\mathbf{x}, t)$ to be a step function without loss of generality.

5. Feigning PS

PS is invariant whenever $\lambda(\mathbf{x}, t)$ fulfills the conditions (i)–(iii). However, it is important to mention that the phase definition must be defined consistently. This means that one must have the same phase definition before and after the time transformation, otherwise one could predict that PS is not invariant, due to the changing of the phase definition. Let us consider two spiking neurons \mathcal{N}_1 and \mathcal{N}_2 . We assume that the spike times t_1^i and t_2^i are independent, with neuron \mathcal{N}_1 having a higher frequency, figure 2(a), in such a way that there is no $n : m$ PS between \mathcal{N}_1 and \mathcal{N}_2 . Let $t_2^{n_i}$ be the spike time of \mathcal{N}_2 that precedes the $i + 1$ th spike of \mathcal{N}_1 , figure 2(a). Then, given $\sigma_i \ll \gamma_i$, we perform the following transformation:

$$\lambda(t) = \begin{cases} \sigma_i, & \text{if } t_1^i < t \leq t_2^{n_i}, \\ \gamma_i, & \text{if } t_2^{n_i} < t \leq t_1^{i+1}. \end{cases} \quad (19)$$

This shrinks the time between t_1^i and $t_2^{n_i}$ and stretches between $t_2^{n_i}$ and t_1^{i+1} creating bursts in \mathcal{N}_2 . In the rescaled time ζ , the bursts of \mathcal{N}_2 is synchronized with the spikes of \mathcal{N}_1 . However, there is no synchronization between the spikes, since $\lim_{i \rightarrow \infty} |\zeta_1^i - \zeta_2^i| \rightarrow \infty$. On the other hand, after the time coordinate change, it is very tempting to introduce a phase for the bursts that increases 2π between two successive bursts of \mathcal{N}_2 . Changing the phase definition, the phase difference between \mathcal{N}_1 and \mathcal{N}_2 becomes bounded. Therefore, it seems possible to introduce PS between two asynchronous neurons. However, this is a fake PS once the phase definition is changed.

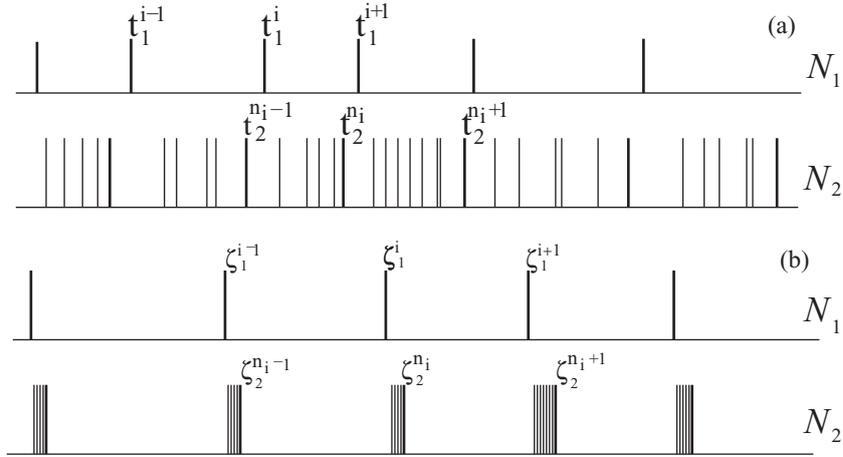


Figure 2. The effect of time transformation in two asynchronous neurons. In (a), the spikes in both neurons are governed by stochastic processes. They present no synchronization. From (a) to (b) we perform the transformation given by equation (19), which shrinks the time between t_1^i and $t_2^{n_i}$ and stretches the time between $t_2^{n_i}$ and t_1^{i+1} . As a consequence, it seems that PS between the spikes in \mathcal{N}_1 and the bursts of \mathcal{N}_2 is enhanced.

6. Network information transmission

Let us analyze the effect of time transformations in the information transmission in networks. For every pair of oscillators \mathbf{x}_j and \mathbf{x}_k , we can define a coordinate transformation

$$\mathbf{x}_{jk}^{\parallel} = \mathbf{x}_j + \mathbf{x}_k, \quad (20)$$

$$\mathbf{x}_{jk}^{\perp} = \mathbf{x}_j - \mathbf{x}_k, \quad (21)$$

that produces two positive conditional Lyapunov exponents (in units of bits/unit time) $\sigma^{\parallel}(t)$ and $\sigma^{\perp}(t)$. The MIR is bounded from above [12]

$$I_C(t) \leq \sigma^{\parallel}(t) - \sigma^{\perp}(t). \quad (22)$$

The main goal is to know how the MIR behaves as we implement a time transformation. By choosing a proper nonlinear $\lambda(\mathbf{x}, t)$ in equation (6), we can introduce different timescales in the oscillators time series as well as endow the time transformation with as much information about the dynamics as we want. The main question is whether, under such nonlinear $\lambda(\mathbf{x}, t)$, the information contained in $\lambda(\mathbf{x}, t)$ could be transmitted to the oscillators.

To answer this question we need to uncover the general transformation law for I_C . After some manipulations we can uncover the transformation law of $I_C(t)$:

$$I_C(\zeta) \leq \frac{I_C(t)}{\langle \lambda \rangle_t}, \quad (23)$$

where, again $\langle \cdot \rangle_t$ stands for the time average. For details see appendix B.

Equation (23) shows an invariant character of I_C . If $I_C(t) = 0$, what happens in the absence of synchronization (correlation) between oscillators, no time transformation that respects conditions (i)–(iii) can raise $I_C(t)$ from zero. Hence, no matter how much information is

contained in $\lambda(\mathbf{x}, t)$, if there is no synchronization this information cannot be used. If, on the other hand, $I_C(t)$ is positive, then $I_C(\zeta)$ can be made arbitrarily large.

To illustrate our findings, we consider a network of four identical Hindmarsh–Rose chaotic neurons electrically coupled in an all-to-all topology,

$$\dot{x}_j = y_j + 3x_j^2 - x_j^3 - z_j + I_j + \sum_k C_{jk}(x_k - x_j), \quad (24)$$

$$\dot{y}_j = 1 - 5x_j^2 - y_j, \quad (25)$$

$$\dot{z}_j = -rz_j + 4r(x_j + 1.6), \quad (26)$$

where C_{jk} stands for the coupling matrix. We use $r = 0.005$, $I_i = 3.2$ and random initial conditions.

We define the following time transformation

$$\lambda(\mathbf{x}_1, t) = \begin{cases} \alpha, & \text{if } x_1 = 0 \quad \text{and} \quad y_1 > -4.6, \\ \beta, & \text{if } x_1 = 0 \quad \text{and} \quad y_1 \leq -4.6. \end{cases} \quad (27)$$

This shrinks the time between spikes when the $y_1 > -4.6$ and stretches when $y_1 < -4.6$ creating a frequency modulation (FM) between the spikes, which depends on the trajectory position. Hence, the transformation carries information about \mathbf{x}_1 . In our analysis, we keep fix $\alpha = 0.5$ and $\beta = 2$. t_j^i denotes the time of the i th crossing of the trajectory of \mathbf{x}_j with the section $x_j = 0$ (an spike event). The time interval between two crossings is $\Delta t_j^i = t_j^{i+1} - t_j^i$.

We introduce a symbolic dynamics which exhibits rather easily the results for the distinct synchronization regimes. We can encode the binary information about the transformation $\lambda(\mathbf{x}_1, t)$ by setting α to the symbol ‘0’ and β to ‘1’. Hence, we have for two consecutive $\lambda = \alpha$: ‘00’; one $\lambda = \alpha$ followed by $\lambda = \beta$: ‘01’; one $\lambda = \beta$ followed by $\lambda = \alpha$: ‘10’; and finally two consecutive $\lambda = \beta$: ‘11’. Whenever the time transformation is able to transmit the information about the symbols, we can access the information about \mathbf{x}_1 in the spike time intervals of the other neurons.

Figures 3(a) and (b) show return maps Δt_2^i versus Δt_2^{i+1} of the neuron \mathbf{x}_2 . We split this map into four return maps, depending on the value of the transformation $\lambda(\mathbf{x}_1, t)$. That is, distinguished by the different symbols ‘00’, ‘01’, ‘10’ and ‘11’. The information about the values of $\lambda(\mathbf{x}_1, t)$ should be considered to be unknown, but here we make use of it to illustrate our ideas.

By measuring Δt_2^i we should be able to infer the time interval Δt_1^i , if the time transformation can transmit information. Figure 3(a) shows that return maps Δt_2^i versus Δt_2^{i+1} for the different values of $\lambda(\mathbf{x}_1, t)$ superimpose, and as a consequence it is impossible to discern whether the region that encodes for 00 is mapped to either 01 or 00 and so on. That leads to a complete uncertainty about $\lambda(\mathbf{x}_1, t)$ by measuring Δt_2^i . Therefore, there is no exchange of information between \mathbf{x}_1 and \mathbf{x}_2 .

The timescale of \mathbf{x}_2 is being rescaled according to a function that contains information about the position of \mathbf{x}_2 . From the way the function λ is constructed, whenever $y_1 > -4.6$ and $x_1 = 0$, the oscillation frequency of the oscillator \mathbf{x}_2 in the time- ζ frame is increased. Whenever $y_1 \leq -4.6$ and $x_1 = 0$, the oscillation frequency of the oscillator \mathbf{x}_2 in the time- ζ frame is decreased. So, the oscillation frequency of \mathbf{x}_2 is being modulated. FM is a typical procedure to transmit information, a protocol in which the information signal is carried by the frequency of a wave. It would be natural to imagine that by modulating the oscillator \mathbf{x}_2 using a time transformation based on the position of \mathbf{x}_1 one could realize at least partially information about

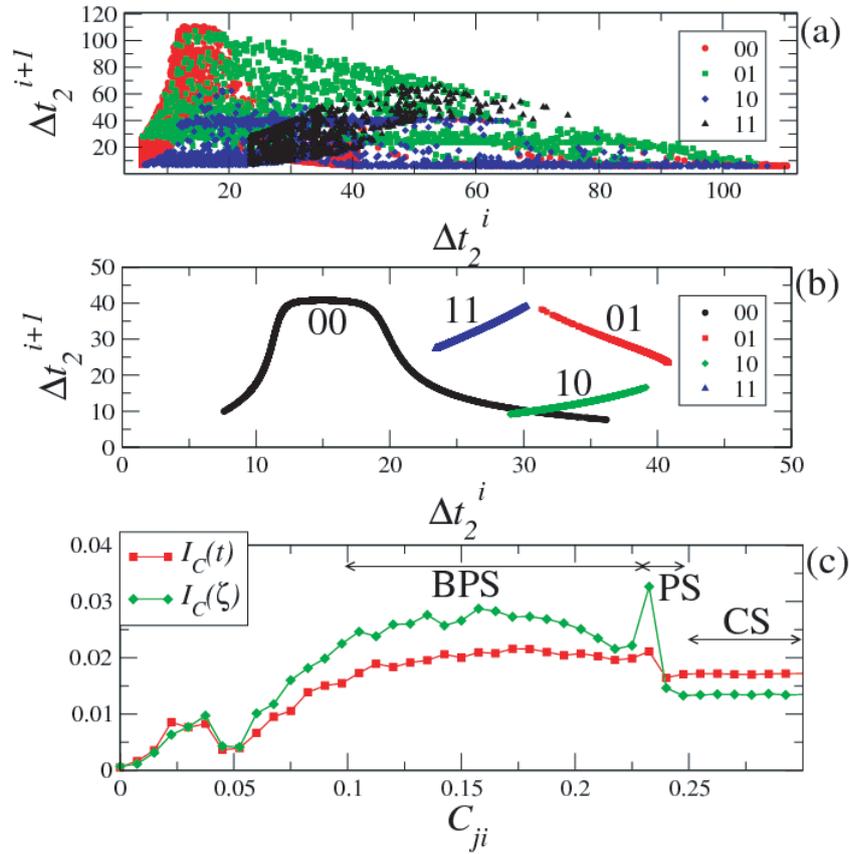


Figure 3. Encoded time intervals between two spikes in neuron \mathbf{x}_2 for $C_{jk} = 0$ (a), and for $C_{jk} = 0.3$ (b). In (c), the MIR between neurons \mathbf{x}_1 and \mathbf{x}_2 , for the time- t (filled squares), and for the time- ζ (filled diamonds). BPS (burst PS) is found for $C_{jk} = [0.1, 0.23]$. In this regime, only the burst are phase synchronized. PS is found for $C_{jk} = [0.23, 0.25]$, and CS (complete synchronization) is found for $C_{jk} = [0.25, 0.3]$. For $C_{jk} \in [0.05, 0.23]$ $\lambda(\mathbf{x}, t)$ is smaller than 1, which provides an increase in I_C up to 60%. For CS $\langle \lambda \rangle_t = 1.29$ provides a decrease in I_C .

\mathbf{x}_1 by making measures in \mathbf{x}_2 . However, surprisingly, that is not the case in dynamical networks. Therefore, if elements in a dynamical network do not exchange information among themselves, there is no time transformation that can change this scenario.

When the neurons are completely synchronized (for $C_{jk} = 0.3$), we see in figure 3(b) that except for one point, the return maps Δt_2^i versus Δt_2^{i+1} for different values of $\lambda(\mathbf{x}_1, t)$ are disjoint, which means that by measuring Δt_2^i we have complete knowledge about the trajectory of the neuron \mathbf{x}_1 .

6.1. Effect of $\lambda(\mathbf{x}_1, t)$ on $I_C(t)$

We keep $\lambda(\mathbf{x}_1, t)$ fixed and vary the coupling strength C_{jk} . Equation (23) states that whenever $\langle \lambda(\mathbf{x}_1, t) \rangle_t < 1$ the time transformation increases the MIR. In figure 3(c), we show the MIR

between Δt_1^i and Δt_2^i using the Shannon mutual information⁶, for the two time frames. $I_C(t)$ denotes the MIR in the time- t frame and $I_C(\zeta)$ the MIR in the time- ζ frame. For $C_{ij} \in [0.5, 0.23]$ $\langle \lambda(\mathbf{x}_1, t) \rangle_t < 1$ which provides an effective increase in the MIR.

In equation (27), λ is defined to contain information about \mathbf{x}_1 . However, λ could be defined to contain information about an arbitrary information signal to be transmitted. In such a case, each disjoint region (such as the ones shown in figure 3(b)) would encode information about this signal, which can be retrieved somewhere else in the network.

$\lambda(\mathbf{x}, t)$ can be constructed using information about some particular node of the network, a group of nodes. Whenever the oscillators are phase synchronized, we can improve the MIR by using $\lambda(\mathbf{x}, t)$ that contains information about the dynamics of the phase synchronized oscillators.

7. Conclusions

In summary, we have shown that for general dynamical oscillators it is neither possible to introduce nor to destroy PS by a time transformation. Furthermore, we have discussed the possible application of these ideas to relevant technological problems such as nonlinear digital communication [11]. Moreover, we have illustrated these results for nonsynchronized oscillators, showing that the enhancement of zero phase diffusion does not enhance PS. We have also discussed that by breaking the boundedness condition imposed on λ , PS can be enhanced. However, such a transformation is physically meaningless. Finally, we have shown that the time transformation can introduce the presence of distinct timescales, which can feign PS. Our findings might be relevant to several areas of natural science for the study of synchronization where the exact time the phenomenon took place is unknown and only a proxy for the time can be derived from the measurements. Examples can be found in geophysics when sediment cores are studied. Such situations may arise in dendrochronology, ice cores and tree rings.

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Appendix A. PS invariance under time transformations

We consider two general oscillators $\dot{\mathbf{x}}_j = \mathbf{F}_j(\mathbf{x}_j)$, where $\mathbf{x}_j \in \mathbb{R}^n$ and $\mathbf{F}_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and analyze a general coupling scheme:

$$\dot{\mathbf{x}}_{1,2} = \mathbf{F}_{1,2}(\mathbf{x}_{1,2}) + \mathbf{C}_{1,2}(\mathbf{x}, t)[\mathbf{H}_{2,1}(\mathbf{x}) - \mathbf{H}_{1,2}(\mathbf{x})], \quad (\text{A.1})$$

where $\mathbf{H}_j(\mathbf{x})$ is the coupling vector function, and $\mathbf{C}_j(\mathbf{x}, t)$ is the coupling matrix. Note that this scheme also takes unidirectional couplings (master–slave configuration) into account. We

⁶ We encode the spikes using the following rule. The i th symbol of the encoding is ‘1’ if a spike is found in the time interval $[i\delta, (i+1)\delta]$, and ‘0’ otherwise. A symbolic sequence is split into words of length $L = 4$. The Shannon entropy is $H = -\sum_m P_m \log_2 P_m$, where P_m is the probability of finding one of the 2^L words. The MIR is estimated by $I_C = [H(\mathbf{x}_1) + H(\mathbf{x}_2) - H(\mathbf{x}_1; \mathbf{x}_2)]/(\delta \times L)$. We choose $\delta \in [\min(\Delta t_j^i), \max(\Delta t_j^i)]$ to maximize I_C .

suppose that each \mathbf{x}_j has a stable attractor and a frequency $\dot{\phi}_j = \Omega_j(\mathbf{x}, t)$, where $\Omega_j(\mathbf{x}, t)$ is a continuous function (or Riemann integrable). Furthermore, we assume that there is a number M such that $\Omega_j(\mathbf{x}, t) \leq M$. From now on, slightly abusing the notation we shall omit the dependence of the functions on the coordinates and on time, whenever there are no problems with the notation. Given a finite real number c , the condition for PS between \mathbf{x}_1 and \mathbf{x}_2 can then be written as:

$$|\phi_1(t) - \phi_2(t)| < c. \quad (\text{A.2})$$

First, we formalize the relation between PS and the timing condition given by equation (5). We have

$$|\phi_1(t) - \phi_2(t)| = \left| \int_0^{t_1^i} \Omega_1 dt - \int_0^{t_2^i} \Omega_2 dt - \int_{t_2^i}^{t_1^i} \Omega_2 dt + \beta^i(t) \right|,$$

where

$$\beta^i(t) = \int_{t_1^i}^t \Omega_1 dt - \int_{t_1^i}^t \Omega_2 dt.$$

Next, since $\phi_j(t_j^i)$ is equal to $i \times 2\pi$,⁷ it yields

$$|\phi_1(t) - \phi_2(t)| \leq M|t_1^i - t_2^i| + \max_i |\beta^i(t)|. \quad (\text{A.3})$$

The term $\max_i |\beta^i|$ is always bounded. By hypothesis we have

$$|t_j^i - t_j^{i-1}| \leq \Lambda,$$

we have

$$|\beta^i| = \left| \int_{t_1^i}^t \Omega_1 dt - \int_{t_1^i}^t \Omega_2 dt \right|, \quad (\text{A.4})$$

$$\leq \left| \int_{t_1^i}^t \Omega_1 dt \right| + \left| \int_{t_1^i}^t \Omega_2 dt \right|. \quad (\text{A.5})$$

But now remembering that $\max_{j,t} \{\Omega_j(t)\} = M$, then

$$|\beta^i| \leq 2M\Lambda. \quad (\text{A.6})$$

Therefore, a bounded time event difference $|t_1^i - t_2^i|$ implies the boundedness of the phase difference. A similar argument shows that the boundedness of the phase difference implies the boundedness of the time event difference. Therefore, equation (5) is equivalent to PS.

We analyze the effect of time transformation PS. We assume $\lambda(\mathbf{x}, t)$ to be (i) *at least Riemann integrable* (ii) *finite*, and (iii) *bounded away from zero*. The two latter conditions are equivalent to the existence of two numbers $\delta^{-1}, \eta \in \mathbb{R}_+$ such that $\delta^{-1} \leq \lambda(\mathbf{x}, t) \leq \eta$. Under the assumptions (i)–(iii) we can demonstrate that PS is invariant under time transformations. First, we show that

$$|t_1^i - t_2^i| \leq \kappa \quad \Rightarrow \quad |\zeta_1^i - \zeta_2^i| \leq \tilde{\kappa}.$$

⁷ Note that this is also true for t_j^1 . This means that $\phi_j(t) = \int_0^{t_j^1} \Omega_j = 2\pi$. Thus, in this construction, there is no contribution of the initial phase $\phi_j(0)$.

Noting that $\zeta_1^i = \int_0^{t_1^i} \lambda(\mathbf{x}, t) dt$ and $\zeta_2^i = \int_0^{t_2^i} \lambda(\mathbf{x}, t) dt$, we find

$$|\zeta_1^i - \zeta_2^i| = \left| \int_0^{t_1^i} \lambda(\mathbf{x}, t) dt - \int_0^{t_2^i} \lambda(\mathbf{x}, t) dt \right|.$$

This may be written as $|\zeta_1^i - \zeta_2^i| = \left| \int_{t_2^i}^{t_1^i} \lambda(\mathbf{x}, t) dt \right|$. However, since $\lambda(\mathbf{x}, t) \leq \eta$ we have

$$|\zeta_1^i - \zeta_2^i| \leq \eta |t_1^i - t_2^i|.$$

Thus, the boundedness of $|t_1^i - t_2^i|$ implies the boundedness of $|\zeta_1^i - \zeta_2^i|$. Now, we show that

$$|\zeta_1^i - \zeta_2^i| \leq \tilde{\kappa} \quad \Rightarrow \quad |t_1^i - t_2^i| \leq \kappa.$$

We have

$$|t_1^i - t_2^i| = \left| \int_0^{\zeta_1^i} \lambda^{-1}(\mathbf{x}, t) dt - \int_0^{\zeta_2^i} \lambda^{-1}(\mathbf{x}, t) dt \right|,$$

which equals $\left| \int_{\zeta_2^i}^{\zeta_1^i} \lambda^{-1}(\mathbf{x}, t) dt \right|$. As $\delta^{-1} < \lambda(\mathbf{x}, t)$, we get

$$|t_1^i - t_2^i| \leq \delta |\zeta_1^i - \zeta_2^i|.$$

Therefore, we conclude that there is PS in the ‘new’ time frame ζ if and only if there is PS in the original time t .

The results stated in this section are general and do not depend on the attractor topology or coherent properties, as long as a phase can be introduced. Note that we do not have to know the phase equation, but only assume that it exists.

The onset of PS, and even the phase equation, depends on the attractor topology and coherence [8]. If the attractor has a simple topology, that is, it has proper rotation, then the onset of PS is given by a transition of the zero Lyapunov exponent to negative values. For such a case, the results of [16], concerning the invariance of the sign of the Lyapunov exponents, can be used to state the invariance of PS under time transformations.

Appendix B. Transformation law for MIR

Representing one node dynamics of the network by

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \quad (\text{B.1})$$

the Lyapunov exponents of an invariant set of the phase space are defined as

$$h_t^i = \lim_{t \rightarrow \infty} \ln \frac{|\mathbf{y}_t^i|}{|\mathbf{y}_{t_0}^i|} \quad (\text{B.2})$$

with

$$\dot{\mathbf{y}}_t^i = \mathbf{D}\mathbf{F}\mathbf{y}_t^i. \quad (\text{B.3})$$

$\mathbf{x}(0)$ is a typical initial condition and $\mathbf{y}_{t_0}^i = \mathbf{y}_t^i(0)\sigma_t^\perp$ are the tangent vector at $\mathbf{x}(0)$. In [16], the transformation law for the Lyapunov spectrum is shown. Under a time transformation λ fulfilling the hypotheses (i)–(iii) we have

$$h_\zeta^i = \frac{h_t^i}{\langle \lambda \rangle_t}. \quad (\text{B.4})$$

Here, we show that the conditional Lyapunov exponents σ_\parallel and σ_\perp follow the same transformation law, since the conditional Lyapunov exponents are the Lyapunov exponents of the network considering that all the initial conditions are equal, and therefore, the same results from [16] apply.

Expanding equation (B.1) linearly around the synchronous state s , and using the parallel coordinate defined in (20), we arrive at

$$\dot{\mathbf{x}}^\parallel = 2[\mathbf{F}(s) - \mathbf{DF}(s)s] + \mathbf{DF}(s)\mathbf{x}^\parallel \quad (\text{B.5})$$

$$= \mathbf{G}(s, \mathbf{x}^\parallel). \quad (\text{B.6})$$

Proceeding in the same way for the perpendicular coordinate in (21), we arrive at

$$\dot{\mathbf{x}}^\perp = \mathbf{DF}(s)\mathbf{x}^\perp \quad (\text{B.7})$$

$$= \mathbf{M}(s, \mathbf{x}^\perp). \quad (\text{B.8})$$

Now by means of \mathbf{G} and \mathbf{M} one can obtain the variational equations of (B.6) and (B.8), which provide the way small perturbations propagate along the parallel and perpendicular directions. From them, we obtain the Lyapunov conditional exponent σ_t^\parallel along the parallel direction and the Lyapunov conditional exponent σ_t^\perp along the transversal direction, respectively.

Then, by applying the results of [16], we conclude that

$$\sigma_\zeta^\parallel = \frac{\sigma_t^\parallel}{\langle \lambda \rangle_t}, \quad (\text{B.9})$$

$$\sigma_\zeta^\perp = \frac{\sigma_t^\perp}{\langle \lambda \rangle_t}. \quad (\text{B.10})$$

While the parallel conditional exponents are the Lyapunov exponents of the synchronization manifold, whose positive exponents measure the rate of information produced by the nodes if they were completely synchronous, the transversal conditional exponents are the Lyapunov exponents along the directions transversal (orthogonal) to the synchronization manifold, whose positive exponents measure the rate of information that can be erroneously transmitted between nodes.

For a matter of simplicity in the notation, we denote the sum of all the positive parallel exponents by σ_t^\parallel and the sum of all positive transversal exponents by σ_t^\perp .

Then, by using the results of [12], we can write

$$I_C(t) \leq \sigma_t^\parallel - \sigma_t^\perp, \quad (\text{B.11})$$

which in other words means that the MIR, i.e. the rate with which information is exchanged between two nodes of the network, is given by the rate of information produced by the synchronous trajectories minus the rate of information produced by the desynchronous trajectories, the error in the transmission of information.

To intuitively understand equation (B.11), one can compare the right-hand side of it with the usual definition of MIR between a source of information, denoted by S , and the receiver of information, denoted by R , given by $H(S) - H(S|R)$. The term $H(S)$, which can be compared with σ_t^{\parallel} , represents the rate with which information is produced in the source and the term $H(S|R)$, which can be compared with σ_t^{\perp} , represents the rate of uncertainty remaining about the transmitted information after observing the received information, i.e. the rate with which information is erroneously transmitted.

Then, taking into account equations (B.9) and (B.10) we have

$$I_C(\xi) \leq \frac{1}{\langle \lambda \rangle_t} (\sigma_t^{\parallel} - \sigma_t^{\perp}), \quad (\text{B.12})$$

concluding

$$I_C(\xi) \leq \frac{I_C(t)}{\langle \lambda \rangle_t}. \quad (\text{B.13})$$

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