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Universal occurrence of the phase-flip bifurcation in time-delay coupled systems

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Recently, the phase-flip bifurcation has been described as a fundamental transition in time-delay coupled, phase-synchronized nonlinear dynamical systems. The bifurcation is characterized by a change of the synchronized dynamics from being in-phase to antiphase, or vice versa; the phase-difference between the oscillators undergoes a jump of π as a function of the coupling strength or the time delay. This phase-flip is accompanied by discontinuous changes in the frequency of the synchronized oscillators, and in the largest negative Lyapunov exponent or its derivative. Here we illustrate the phenomenology of the bifurcation for several classes of nonlinear oscillators, in the regimes of both periodic and chaotic dynamics. We present extensive numerical simulations and compute the oscillation frequencies and the Lyapunov spectra as a function of the coupling strength. In particular, our simulations provide clear evidence of the phase-flip bifurcation in excitable laser and Fitzhugh–Nagumo neuronal models, and in diffusively coupled predator-prey models with either limit cycle or chaotic dynamics. Our analysis demonstrates marked jumps of the time-delayed and instantaneous fluxes between the two interacting oscillators across the bifurcation; this has strong implications for the performance of the system as well as for practical applications. We further construct an electronic circuit consisting of two coupled Chua oscillators and provide the first formal experimental demonstration of the bifurcation. In totality, our study demonstrates that the phase-flip phenomenon is of broad relevance and importance for a wide range of physical and natural systems. [DOI: 10.1063/1.2905146]

The spontaneous onset of synchronization in coupled nonlinear oscillators is a remarkable and ubiquitous phenomenon. In recent years there has been much effort to explore various types of synchronization, including phase synchronization, when the dynamics is correlated in phase but uncorrelated in amplitude. Phase synchronization has been increasingly studied in the context of potential applications in disciplines ranging from physics and chemistry to biology and medical sciences. It was found that the degree of synchronization can be an important part of the function or malfunction of a given system, and in many cases synchronization in phase turned out to be undesirable. In mechanical systems synchronization may result in dangerous jams or overloads. Similarly, several neurological disorders such as epileptic seizures or Parkinson's disease are associated with synchronized firings of neurons, while in ecological systems, the synchronization of populations is often seen as detrimental because it enhances the chances of global species extinctions. These findings highlight the need to explore reliable methods for preventing the formation of phase synchronization in coupled oscillatory systems. One such possibility was recently identified if the coupling between the subsystems allows for time delay. In this case the system can undergo a phase-flip bifurcation,[1,2] where the coupled oscillators alternate from a state of in-phase to antiphase, with the emergence of large phase differences between the interacting systems. Here we demonstrate that the phase-flip bifurcation occurs in a wide and important class of systems, including excitable dynamics, that apply in laser and neuronal systems, and regular and chaotic cycling ecological models. Further we provide the first explicit experimental verification of this bifurcation in coupled electronic circuits. Taken together, our results suggest the phase-flip bifurcation to be a general and important property of time-delay coupled nonlinear systems.

I. INTRODUCTION

Consider coupled nonlinear oscillators which are identical or nearly identical. It is well known that under certain circumstances weak coupling suffices to synchronize the oscillators, namely causing them to oscillate in unison.[3] A variety of coupling scenarios have been extensively investigated in recent years and now fairly general prescriptions are available that will ensure synchronous dynamics regardless of the nature of the motion itself. The possibility of having synchronous chaotic motion,[4] in particular, has been a discovery of considerable potential significance. Depending on the nature and the strength of the coupling the synchronous oscillation can be similar to the isolated dynamics of one of the oscillators or it can be quite distinct.

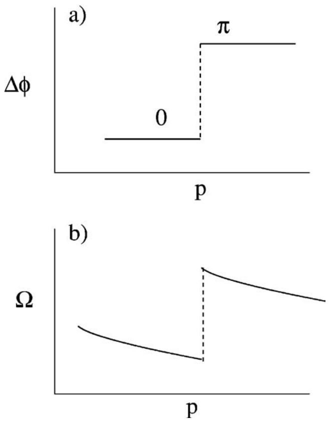

FIG. 1. Schematic figure for phase-flip bifurcation showing the variation of (a) phase difference and (b) frequency as a function of system parameter p.

When physically coupling two systems, it is often necessary to consider time delays. These arise, for example, when the interaction between the systems is effected by transmitted signals of finite velocity, as is common in a variety of practical situations. In biological, ecological or social systems time-delayed coupling typically arises from diffusive or migratory processes which are not instantaneous. Since the interaction frequently acts over time scales that are comparable to any intrinsic time scales in the system, time delay is an essential property of the coupling which cannot be neglected. The synchronization of time delay coupled oscillators has been investigated in a number of studies.[2,5–7] It was found that the systems in the presence of time delay continue to show synchronization, although new dynamical phenomena such as amplitude death[1,2,5] also become possible.

The phase-flip bifurcation[1,2] is a phenomenon that has been recently observed in such systems, in the *synchronized* dynamics of time delay coupled oscillators. In this bifurcation the relative phase between the oscillators jumps from zero to π or vice versa as the coupling parameters or time delay is varied. A schematic of this bifurcation is shown in Fig. 1. The phase-flip is also accompanied by a discontinuous change in the frequency of the synchronized oscillators [see Fig. 1(b)], as well in a spectrum of Lyapunov exponents which measure the stability of the system.[1,2] A detailed study of this bifurcation in coupled Landau–Stuart oscillators has been presented in Ref. 1. It was shown analytically that there was a discontinuity in the frequency, which could also be estimated.

In the present paper, we extend these previous works and investigate the phenomenology of the phase-flip bifurcation by exhaustive numerical simulations in a general model for time-delayed coupled nonlinear oscillators,

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{F}(x) + \epsilon\mathbf{g}[\mathbf{y}(t-\tau), x(t)],\\ \dot{\mathbf{y}} &= \mathbf{F}'(y) + \epsilon\mathbf{g}[\mathbf{x}(t-\tau), y(t)].\end{aligned} \tag{1}$$

Here $\mathbf{x}$ and $\mathbf{y}$ denote the variables of the two n-dimensional subsystems. The dynamical equations are specified by $\mathbf{F}$ and $\mathbf{F}'$ (the prime denoting that the parameters of the two systems could be different), and $\mathbf{g}$ specifies the manner in which the subsystems are coupled. The parameters ϵ and τ are the coupling strength and the time delay, respectively, and we have considered the case when the coupling is symmetric. Earlier studies have examined the cases of coupled limit-cycle oscillators[2] and Rössler chaotic dynamical systems.[1]

We believe the bifurcation should be easy to detect and that this effect should be observed in any dynamical, oscillatory system. Our main motivation in the present work is thus to show this bifurcation in some important paradigmatic classes of systems, and we begin in Sec. II by exploring the phase-flip bifurcation in excitable systems. Two models we consider are an excitable laser system and the Fitzhugh–Nagumo model of a neuron. Similarly, in Sec. III we investigate two different ecological predator-prey models where the bifurcation manifests itself. Our analysis is complemented by a calculation of the fluxes between the two populations. This measure is useful in estimating the potential of a rescue effect, namely the chance that immigration from neighboring populations may help prevent species extinction. As mentioned in the abstract, being able to change the relative phase between two oscillating populations may have potentially important practical applications. An experimental verification of the bifurcation in two coupled Chua circuits is then presented in Sec. IV, and the paper concludes with a discussion and summary in Sec. V.

II. EXCITABLE SYSTEMS

Many physical and physiological systems are excitable[8–10] namely, when an external perturbation is below a threshold, the dynamics remains in a quiescent state, with drastically different dynamics results when the perturbation is above the threshold. The nature of the dynamics in excitable systems is intrinsically different from that in nonlinear oscillators owing to the two time scales that are present in the motion.[11] A variety of accessible experimental systems (such as semiconductor lasers[9,11] or neuronal systems[10,11]) are properly described through excitable models.

Consider for instance the equations which mimic the dynamics of low-frequency fluctuations in semiconductor laser with optical feedback,[9]

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_1 - x_2 - x_1^3 + x_1x_2 + \beta_1 + \beta_2x_1^2.\end{aligned}$$

Now we couple two such laser systems with delay,

$$\begin{aligned}\dot{x}_1(t) &= x_2,\\ \dot{x}_2(t) &= x_1 - x_2 - x_1^3 + x_1x_2 + \beta_1 + \beta_2x_1^2\\ &\quad + \epsilon[y_2(t-\tau) - x_2(t)],\\ \dot{y}_1(t) &= y_2,\\ \dot{y}_2(t) &= y_1 - y_2 - y_1^3 + y_1y_2 + \beta_1 + \beta_2x_1^2\\ &\quad + \epsilon[x_2(t-\tau) - y_2(t)].\end{aligned} \tag{2}$$

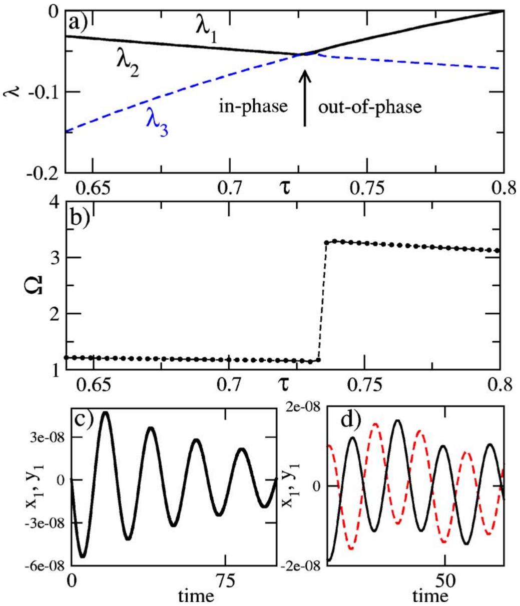

FIG. 2. (Color online) (a) Spectrum of Lyapunov exponents for the excitable laser model, Eq. (2): λ_1 (black), λ_2 (black-overlapped), and λ_3 (red) with time delay, τ, at fixed coupling strength $\epsilon=3$ (for $\beta_1=0.08$ and $\beta_2=1$). In the amplitude death region all Lyapunov exponents are negative and $\lambda_1=\lambda_2$. (b) Frequency Ω as a function of time-delay, τ. (c) In- and (d) out-of-phase motion before and after the bifurcation at $\tau=0.7$ and $\tau=0.75$, respectively.

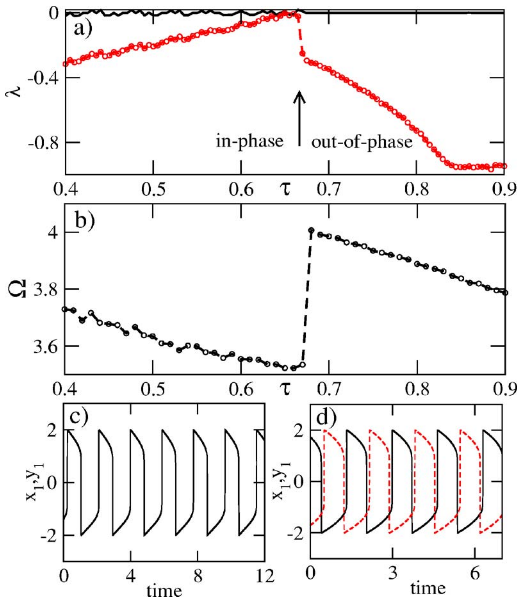

FIG. 3. (Color online) (a) Spectrum of Lyapunov exponents for coupled FHN oscillators, Eq. (3): λ_1 (black) and λ_2 (red, dashed) as a function of the time delay, τ, at fixed $\alpha=0.2$, $\mu=0.001$ and $\epsilon=0.3$. (b) Average frequency Ω of oscillator 1 as a function of τ. (c) In- and (d) out-of-phase dynamics before and after the bifurcation at $\tau=0.6$ and $\tau=0.7$, respectively.

Each of the (uncoupled) oscillators has three fixed points: a saddle, an unstable focus, and a stable node (for details, see Refs. 9 and 11). The unstable manifold of the saddle terminates in a stable node, while the stable manifold of the saddle connects to the coexisting unstable focus. Since the only attractor is the stable node, any trajectory that is made to cross the stable manifold of the saddle by a perturbation necessarily makes a long excursion to reach the attractor. This dynamical behavior, which is independent of the strength of the perturbation so long as it is above the threshold, makes this system excitable. In the coupled system, Eq. (2), the perturbation itself is introduced via the time-delayed coupling. When the coupling is instantaneous, namely, $\tau=0$, the dynamical system has dimension 4, while with time-delay in the coupling the system is infinite-dimensional. Numerical simulations are performed for a (sufficiently) high-order discretization of these equations, and can be carried out by adapting standard numerical techniques.[12]

From the spectrum of the Lyapunov exponents[13] the phase-flip bifurcation point is indicated (see the arrow) by a discontinuity in the slope of the Lyapunov exponents [Fig. 2(a)] and in the frequency of oscillation [Fig. 2(b)]. Trajectories before and after the bifurcation are shown in Figs. 2(c) and 2(d), respectively. These findings clearly confirm the existence of a phase-flip bifurcation in this system.

Neurons, wherein the dynamics exhibits spikes are another important example of excitable systems, and a widely used mathematical description is provided by the simplified van der Pol–Fitzhugh–Nagumo[10] excitable neuronal model. Consider two such neurons which are delay-coupled,

$$\begin{aligned}
\dot{x}_1(t) &= \frac{x_1 - x_1^3/3 - x_2}{\mu},\\
\dot{x}_2(t) &= x_1 + \alpha + \epsilon[y_2(t-\tau) - x_2(t)],\\
\dot{y}_1(t) &= \frac{y_1 - y_1^3/3 - y_2}{\mu},\\
\dot{y}_2(t) &= y_1 + \alpha + \epsilon[x_2(t-\tau) - y_2(t)].
\end{aligned} \tag{3}$$

The fast variable x_1 corresponds to the membrane potential and the variable x_2 is related to recovery or refractoriness. Here the parameters μ and α represent the square root of the quotient inductance/capacitance and the potential supplied to the membrane, respectively. This system has a fixed point which is stable for $\alpha>1$ and an unstable focus for $\alpha<1$, which gives rise to limit cycle oscillations. A finite perturbation in α near $\alpha=1$ shows the excitability. Shown in Fig. 3(a) are the largest two Lyapunov exponents for $\alpha=0.2$. The discontinuity in λ_2, indicated by the arrow in Fig. 3(a), locates the bifurcation, which here lies in the region of periodic motion.[1,2,5] Typical trajectories, before and after the bifurcation are shown in Figs. 3(c) and 3(d): in- and out-of-phase motions. The clear characteristic jump in the oscillation frequency $\Omega(\tau)$ is shown in Fig. 3(b) which confirms this bifurcation.

Phase synchronized oscillations in neuronal ensembles are a topic of considerable current interest in neurobiology. For instance, in a recent study of induced focal epilepsy Li *et al.*[14] have observed both in- and out-of-phase oscillations in spatially separate regions of a rat brain. Since the time-delay in the information transmission between different brain regions will depend on their spatial separation, the phase-flip bifurcation may prove useful in understanding the detailed mechanism of dynamical diseases such as epilepsy, or other neurological disorders that are associated with synchronized regions in the brain such as Parkinson's disease.

III. ECOLOGICAL SYSTEMS

An important issue in modeling ecological systems is understanding the behavior of interacting populations. A standard approach in studying extended systems is to investigate metapopulations, namely, an ensemble of local populations which are coupled by diffusive migration of individuals. Since the individuals need a time span τ to disperse to the other population, such systems naturally involve time delays depending on the spatial separation between the two populations and the migration velocity.

We first consider the predator-prey model introduced by Rosenzweig and MacArthur[15] which is known to exhibit limit-cycle oscillations (and has played an important role in studying phenomena such as the so-called paradox of enrichment). In order to investigate the time-delayed synchronization in such systems we study two predator-prey populations which are coupled in the following manner:

$$\begin{aligned}
\dot{x}_1(t) &= x_1(1-x_1/b_1) - b_2x_1x_2/(b_3+x_1),\\
\dot{x}_2(t) &= b_4x_1x_2/(b_3+x_1) - b_5x_2 + \epsilon[y_2(t-\tau)-x_2(t)],\\
\dot{y}_1(t) &= y_1(1-y_1/b_1) - b_2y_1y_2/(b_3+y_1),\\
\dot{y}_2(t) &= b_4y_1y_2/(b_3+y_1) - b_5y_2 + \epsilon[x_2(t-\tau)-y_2(t)].
\end{aligned} \tag{4}$$

Here x_1 and x_2 represent the population densities of the prey and predator species, b_1 is the carrying capacity for the prey, b_2 is the maximum rate at which an individual predator can consume prey, b_3 is the predator's half-saturation constant, b_4 is the conversion ratio of prey eaten to predator reproduction, and b_5 is the predator mortality. The populations are diffusively coupled by migration of the predator species over a time span τ.

The parameters are fixed in the limit-cycle regime of the model at $b_1=1$, $b_2=0.05, b_3=0.5$, $b_4=0.5$, and $b_5=0.1$. Shown in Fig. 4(a) is the spectrum of Lyapunov exponents, where the discontinuous change in the second Lyapunov exponent clearly predicts the phase-flip bifurcation. This is confirmed by the sudden increase of the oscillation frequency $\Omega(\tau)$ which jumps by a factor of about 3, from $\Omega \approx 0.1$ to $\Omega \approx 0.3$ at the bifurcation [see Fig. 4(b)]. The change in the dynamics across this transition is depicted in Figs. 4(c) and 4(d). Prior to the bifurcation for time delays $\tau < \tau_c$, both predator-prey systems oscillate in perfect synchrony, while for $\tau > \tau_c$ the oscillations are changed to antiphase synchronization. The population maxima of the first species arise during the troughs of the second species and vice versa. These results provide an univocal confirmation for the presence of the phase-flip bifurcation in this system. Note that the period length $T=2\pi/\Omega$ is much larger than the critical time delay at the bifurcation, $T \gg \tau_c$.

The difference in the populations is an estimate of the migration flux. Shown in Figs. 5(c) and 5(d) are the time delayed population fluxes $|x_1(t)-y_1(t-\tau)|$ for the prey abundances between the two populations, corresponding to the in- and out-of-phase motions of Figs. 5(a) and 5(b). It can be seen that the flux reaches a minimum near each extremum of the prey. Similar behavior [Figs. 5(g) and 5(h)] is also observed for the predators with abundances as shown in Figs. 5(e) and 5(f). Note that the flux in Figs. 5(d), namely the prey flux when the motion is in antiphase, takes particularly small values. This is quite remarkable since the two population are coupled by migration of the predator species.

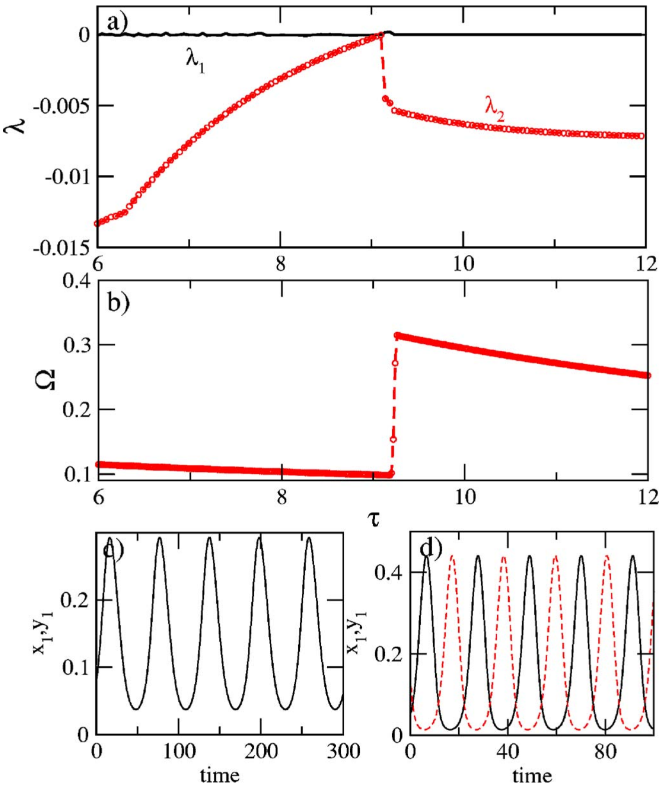

FIG. 4. (Color online) (a) Spectrum of Lyapunov exponents for the ecological model, Eq. (4): λ_1 (solid black line) and λ_2 (dashed red line) as a function of the time-delay τ at fixed coupling strength $\epsilon=0.6$. (b) Frequency of oscillation Ω as a function of the time delay τ. Trajectories of the oscillators 1 (solid line) and 2 (dashed line) with periodic motion at (c) $\tau=8$ (in phase) and (d) $\tau=10$ (out-of-phase).

To obtain more insight into the migration process in the time-delay coupled populations, we take time-averages. Shown in Fig. 6 is the quantity $\langle |x_i(t)-y_i(t-\tau)| \rangle$, i.e., the τ-delayed average of prey or predator fluxes as a function of the delay time τ. As can be seen in Fig. 6(a), there is a noticeable jump in this quantity at the bifurcation $\tau=\tau_c$. For the predator species ($i=2$) the flux increases somewhat across the bifurcation (and is decreasing only for larger values of τ), while the averaged flux for the prey ($i=1$) decreases at the bifurcation. In contrast the averaged instantaneous flux increases drastically at the bifurcation; see Fig. 6(b).

Numerous predator-prey model have been studied over the past few decades, and one that has been recently introduced as a "minimal" model that describes realistic

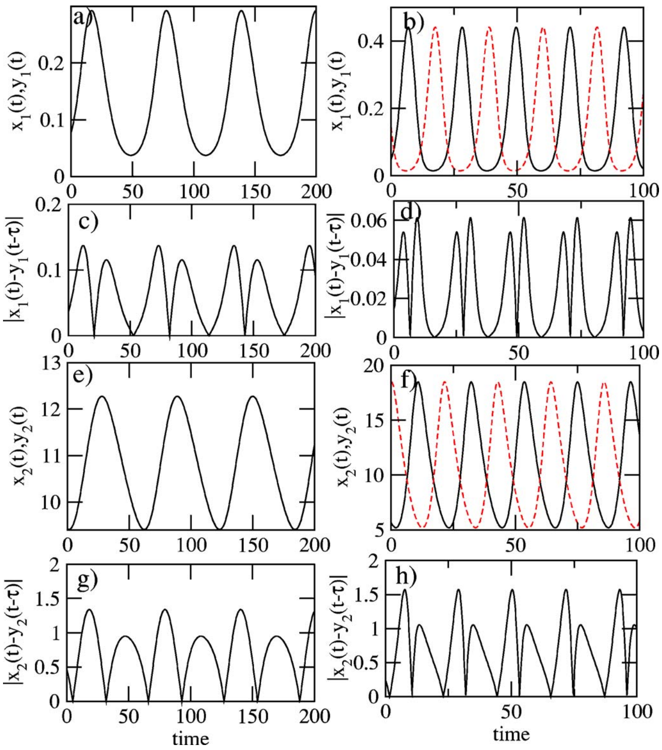

FIG. 5. (Color online) Change of the time-delayed population-fluxes at the phase-flip bifurcation. The left panel corresponds to in-phase synchronized dynamics ($\tau=8$) and the right panel to antiphase synchronized dynamics ($\tau=10$) in the ecological model, Eq. (4). Other details remain the same as in Fig. 4(a). Shown in (a) and (b) is the time course of the prey-species, for population $x_1(t)$ (solid line) and population $y_1(t)$ (dashed line). Panels (c) and (d) show the absolute values of the corresponding time-delayed fluxes $\langle|x_1(t)-y_1(t-\tau)|\rangle$. Similarly, panels (e) and (f) depict the time course of the predators [$x_2(t)$ and $y_2(t)$], and (g) and (h) show the time-delayed predator fluxes $\langle|x_2(t)-y_2(t-\tau)|\rangle$.

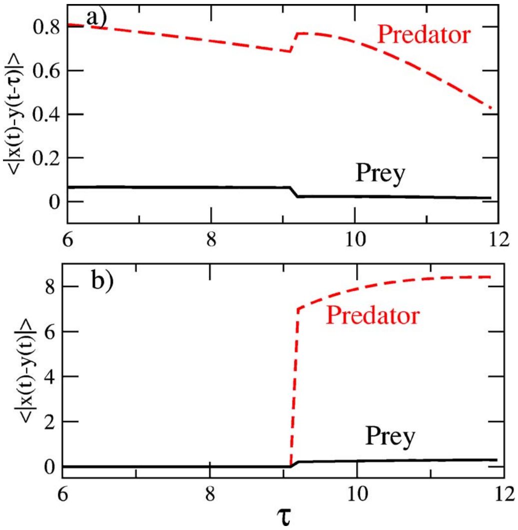

FIG. 6. (Color online) Change of the averaged τ-delayed fluxes $\langle|x_i(t)-y_i(t-\tau)|\rangle$ as a function of the time-delay τ. Shown are (a) the average delayed flux and (b) the average instantaneous flux for both prey $i=1$ (solid line) and predators $i=2$ (dashed line).

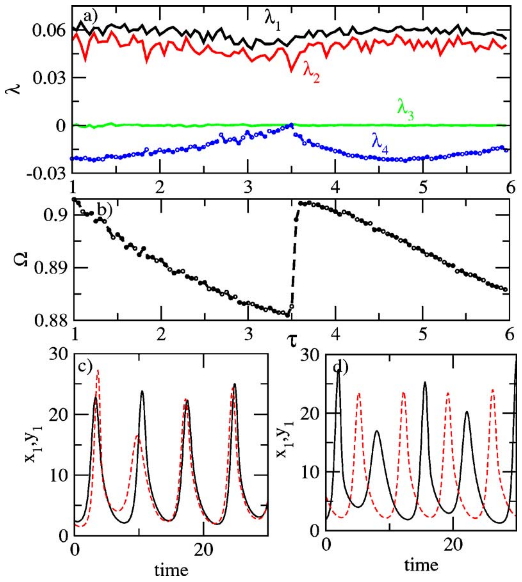

FIG. 7. (Color online) (a) Spectrum of Lyapunov exponents for the ecological model, Eq. (5): λ_1 (solid black line) and λ_2 (solid red line), λ_3 (solid green line) and λ_4 (dashed blue line) as a function of the time-delay τ at fixed coupling strength $\epsilon=0.01$, and (b) variation of the frequency with time delay. Trajectories of oscillators 1 (solid line) and 2 (dashed line) when the dynamics is chaotic at (c) $\tau=3$ (nearly in-phase) and at (d) $\tau=4$ (nearly out-of-phase).

population oscillations with chaotic amplitudes but uniform phase evolution[16] is a three-level vertical food chain: vegetation is consumed by a herbivore which in turn is preyed upon by the top predator. Consider two such coupled systems,

$$
\begin{aligned}
\dot{x}_1(t) &= x_1 - 1.5 - 0.1x_1x_2,\\
\dot{x}_2(t) &= -0.945x_2 + 0.1x_1x_2 - 0.6x_2x_3\\
&\quad + \epsilon[y_2(t-\tau) - x_2(t)],\\
\dot{x}_3(t) &= -10x_3 + 0.1 + 0.6x_2x_3,\\
\dot{y}_1(t) &= y_1 - 1.5 - 0.1y_1y_2,\\
\dot{y}_2(t) &= -0.945y_2 + 0.1y_1y_2 - 0.6y_2y_3\\
&\quad + \epsilon[x_2(t-\tau) - y_2(t)],\\
\dot{y}_3(t) &= -10y_3 + 0.1 + 0.6y_2y_3.
\end{aligned}
\tag{5}
$$

x_1, y_1 are vegetation variables, x_2, y_2 are herbivores, and x_3, y_3 are the top predators. Details of the parameters and the specific choices for the values above can be found in Ref. 16. The two systems are time-delay coupled via the migration of the two herbivore species.

This model is able to produce several dynamical regimes, such as limit-cycle oscillations and a period-2 cascade towards phase coherent chaos.[16] In all these dynamical regimes of the model we have observed the phase-flip bifurcation to arise naturally for sufficient time delay τ. An example is shown in Fig. 7, where both before and after the transition the dynamics is chaotic. All the characteristic hallmarks of the phase-flip bifurcation, namely the change in the spectrum of Lyapunov exponents [Fig. 7(a)] and of the oscillation frequency $\Omega(\tau)$ [Fig. 7(b)] are evident here. The fourth largest Lyapunov exponent is discontinuous at the transition, in contrast to the examples studied above.

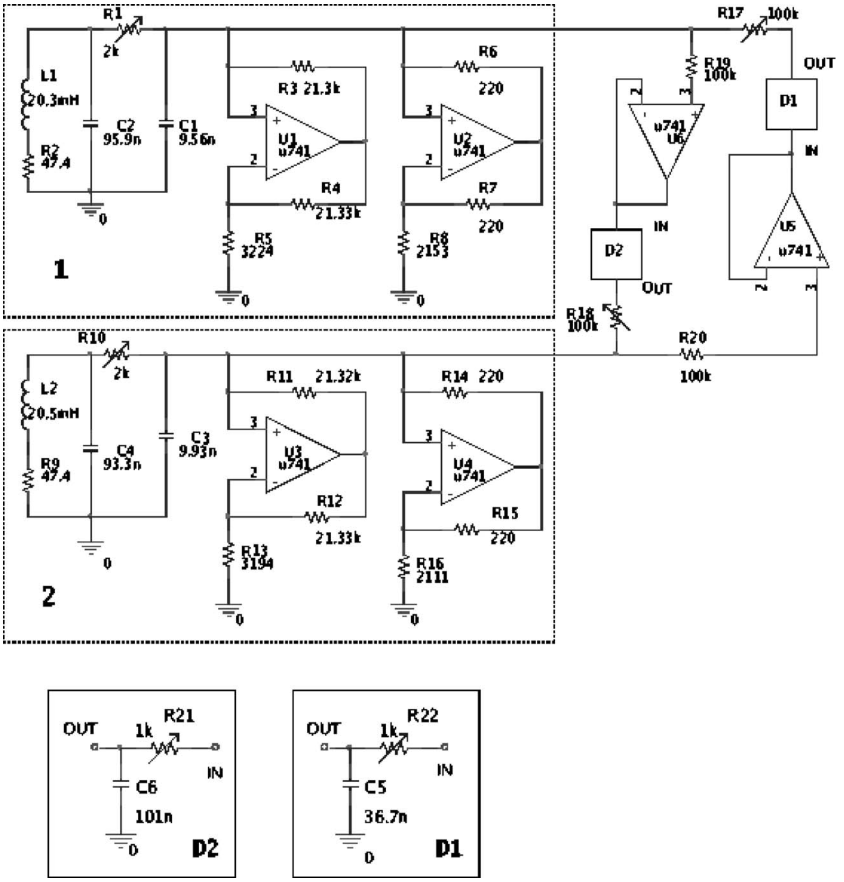

FIG. 8. Circuit of two delay coupled Chua oscillators: the oscillators are drawn inside dotted boxes labeled 1 and 2. Delay networks D_1 and D_2 are separately drawn with capacitors C_5 $=36.7$ nf and $C_6=10$ nf, respectively.

While further studies certainly are required in order to properly model actual field data, it should be pointed out that existing ecological data frequently show antiphase dynamics, similar to that exhibited in Figs. 4(d) and 7(d). One prominent example arises in the dynamics of epidemic outbreaks of measles, which appear to show two forms of synchronization: the number of cases of measles in neighboring cities in the UK are either in-phase (e.g., Birmingham and Newcastle) or out-of-phase (e.g., Cambridge and Norwich); see Fig. 1 of Ref. 17. Such different forms of synchronization are not only of academic interest, but are of utmost practical importance and there has been some indication that differential phase behavior can be of use in population control. For instance, it is believed that for interacting populations, the out-of-phase behavior is beneficial since it promotes species conservation. The reason is that asynchrony enhances the global persistence of a population through the rescue effect, even when there are local extinctions. In contrast, in-phase dynamics usually are thought to promote species extinction because the rescue effect is then not effective. It is likely that a detailed analysis of the phase-flip bifurcation in oscillating predator-prey metacommunities will be of much interest in this context.

IV. EXPERIMENT: COUPLED CHUA OSCILLATORS

In this section we present experimental evidence of the phase-flip bifurcation in an electronic circuit implementation of two delay-coupled Chua oscillators. The two oscillators are only approximately identical since in reality it is not possible to ensure that the parameters are exactly equal. Further, unlike the model systems discussed above, the coupling is asymmetric, and the time-delays are not equal. However, this does not appear to affect the phase-flip phenomenon. The coupled circuit is shown in Fig. 8.

Each Chua oscillator[18,19] consists of a passive resistor $R_{1,10}$, two capacitors $C_{1,3}$ and $C_{2,4}$ and inductors $L_{1,2}$ with leakage resistance $R_{2,9}$. The piecewise linear function f is simulated by using two op-amp U_1-U_2 (or U_2-U_4) and associated resistances. All component values are noted in the circuit diagram.

The op-amp U_5 is used for unidirectional current flow from the node of the C_3 capacitor of one oscillator (say II) to the node of the C_1 capacitor of another oscillator (I). The *RC* network D_1 (a combination of resistor R_{22} and capacitor C_5) with a series resistance R_{17} is used to introduce delay coupling from oscillator II to oscillator I. The resistor R_{22} is tuned to control the time delay τ_1 while R_{17} controls the coupling strength ϵ_1. Similarly, the op-amp U_6 allows unidirectional current to flow from the node of the C_1 capacitor of oscillator I to the node of the C_3 capacitor of oscillator II via another *RC* network D_2 (a combination of resistor R_{21} and capacitor C_6) and the series resistance R_{18}. Accordingly, the resistor R_{21} controls the delay time τ_2 and R_{18} controls the coupling strength ϵ_2. A bidirectional delay coupling is thus established between the two Chua oscillators using delay networks (D_1, D_2) and resistances (R_{17}, R_{18}).

In the experiment, we adjust the dynamical state of the two uncoupled Chua oscillators by fixing the resistances R_1

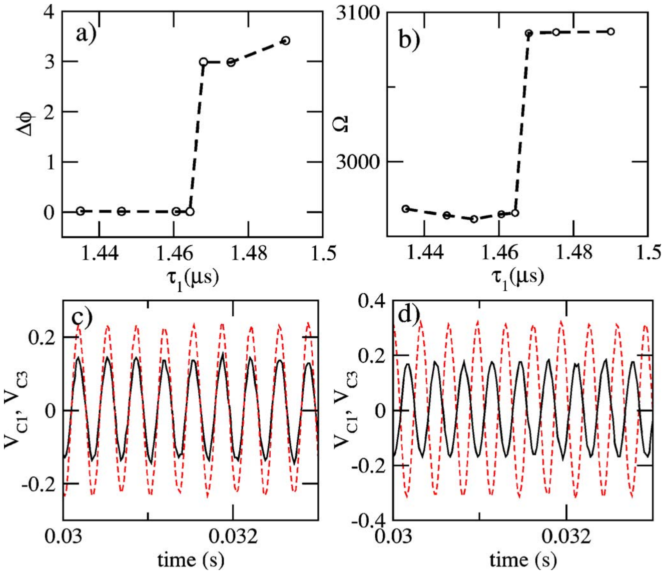

FIG. 9. (Color online) (a) Phase difference and (b) average oscillation frequency of an individual circuit as a function of the time delay τ_1. The experimental time series of the voltages V_{C1} (solid line) and V_{C3} (dashed line) for (c) in-phase dynamics at $R_{22}=394\ \Omega$ and (d) out-of-phase dynamics at $R_{22}=402\ \Omega$.

$=1525\ \Omega$, so that oscillator I is in the period-2 regime, and $R_{10}=1507\ \Omega$, so that oscillator II is in period 4. After coupling, we fix the coupling strength $\epsilon_1=1/R_{17}$ and $\epsilon_2=1/R_{18}$ by appropriate selection of the resistances $R_{17}=21.36\ \mathrm{k}\Omega$ and $R_{18}=65.2\ \mathrm{k}\Omega$ and also fix one of the time delays $\tau_2=R_{21}C_6$ by the choice of $R_{21}=523\ \Omega$ and $C_6=1008$ nf. The delay $\tau_1=R_{22}C_5$ is only varied by varying R_{22} while keeping the capacitor $C_5=36.7$ nf fixed. The voltage VC_1 and VC_3 at capacitor nodes C_1 and C_3, respectively, are monitored using a 2-channel digital oscilloscope (Tektronix, TDS 220, 100 MHz) with a maximum sampling rate of 1.0 GS/s and record length of 2500 data points in each snapshot. The instantaneous phases $\phi_i(t)$ of the measured oscillatory voltages, VC_1 and VC_3, are estimated using the Hilbert transform[20] separately. The resulting phase difference $\Delta\phi(t)$ of the coupled oscillators is then plotted with delay time τ_1 as shown in Fig. 9(a) which reveals a sharp transition from 0 to π as indicated by the discontinuity of the line plot with solid black circles. This transition is accompanied by a sharp increase in the frequency of the coupled system as shown in Fig. 9(b). The frequency is estimated from the average rate of change in instantaneous phase $\langle \Delta\phi(t)/dt \rangle$.

Our experimental results, shown in Fig. 9, confirm the existence of the phase flip bifurcation as defined in the previous sections. In fact, the Chua oscillators become chaotic under bidirectional coupling with small time delay, however, the coupled oscillators move to a periodic state when the delay time τ_1 is increased. The coupled Chua oscillators remain periodic with increase in delay τ_1 and maintain in-phase synchrony until the switch to antiphase synchrony above a critical time delay. The measured voltage time series of VC_1 and VC_3 are plotted in Fig. 9(c) in solid and dotted lines, respectively, for $\tau_1=14.46\ \mu$s ($R_{22}=394\ \Omega$). The two circuits show in-phase synchrony, but they switch over to antiphase (or out-of-phase) behavior for $\tau_1=14.75\ \mu$s ($R_{22}=402\ \Omega$) as shown in Fig. 9(d). The phase flip bifurcation, the sharp transition from in-phase to antiphase, occurs above a critical time delay of $\tau_1=14.64\ \mu$s ($R_{22}=399\ \Omega$) as shown in Fig. 9.

V. SUMMARY AND DISCUSSION

When systems that are spatially separated are coupled, it is natural to introduce time delays in the interactions to account for the finite velocity with which signals are transmitted between them. In the present paper, we have shown that such systems, when synchronized, can undergo a bifurcation that affects their relative phases while not otherwise influencing the dynamics in any significant manner.

Time delay appears to be necessary for this bifurcation, although it is clearly not sufficient. The nature of the coupling and of the nonlinearity is crucial for effecting synchrony.[1,2] Indeed, when systems are coupled via conjugate or dissimilar variables some effects similar to those observed in time-delay coupled systems can occur.[21] In earlier work we have discussed the phenomenology of the phase-flip bifurcation in coupled Landau–Stuart limit cycle oscillators and demonstrated this bifurcation within different types of dynamics in chaotic systems such as coupled Rössler oscillators. This transition can be from in-phase motion to out-of-phase motion or vice versa, and typically, the pattern repeats with increasing delay. At each phase flip though, the oscillator frequency *always* appears to increase. Further, the bifurcation is robust with respect to additive random noise (results not shown here).

It is important to note that this bifurcation occurs in any dynamical regime. Since the only change is one of relative phase, the actual dynamics can be periodic (see Figs. 3, 4, and 9), or even chaotic (Fig. 7), in addition to the case of amplitude death (Fig. 2). What distinguishes these cases is the spectrum of the Lyapunov exponents. When the dynamics is attracted to fixed points, all Lyapunov exponents are negative. When the motion is periodic the largest Lyapunov exponent is zero, when quasiperiodic Refs. 1 and 2 (not shown here) the largest two exponents are zero, and when the motion is chaotic the largest two Lyapunov exponents are positive.

In the present paper we have extended these previous findings to different classes of oscillators, including excitable dynamics such as laser or neuronal models, and periodic and chaotic ecological models. The occurrence of the phase-flip bifurcation in a wide class of systems suggests that these effects can be of consequence; earlier experimental results on time-delay coupled lasers appear to have probed the dynamics on either side of the bifurcation point.[7] Here, we have implemented time-delay coupling in a model electronic circuit and have been able to give a clear experimental evidence for the phase-flip bifurcation. Note that transitions to antiphase synchronization in two coupled Chua circuits have been reported before.[22] Even though these studies did not involve time delays, it was found that the transition to antiphase always goes together with an increase in the oscillation frequency.

Exploration of the dynamics of spatially extended delay-coupled systems wherein we expect that new effects will arise from the interaction of phase and relay synchronization[23] are currently underway.[24]

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