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# Phase and average period of chaotic oscillators

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**Abstract**

Recently, there has been a great effort to extract the phase of chaotic attractors and complex oscillators. As a consequence many phases have been introduced, as example the standard phase $\theta$ based on the rotation of the vector position, and the phase $\phi$ based on the rotation of the tangent vector. Despite of the large interest in the phase dynamics of coupled oscillators there is still a lack of approaches that analyze whether these phase are equivalent and on what conditions these phases work. In this work, we show that the phase $\phi$ generalizes the standard phase $\theta$, and it is equal to the length of the Gauss map, the generator of the curvature in differential geometry. Furthermore, we demonstrate, for a broad class of attractors, that the phase synchronization phenomenon between two coherent chaotic oscillators is invariant under the phase definition. Moreover, we discuss to which classes of oscillators the defined phases can be used to calculate quantities as the average frequency and the average period of oscillators. Finally, we generalize the phase $\phi$ which allows its use also to homoclinic attractors.


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*Keywords:* Phase of chaotic attractors; Coupled oscillators; Neural dynamics

Phase of chaotic oscillators has attracted a great deal of attention since it was found that not only the amplitude of the trajectory but also the phase play an important role to understand the onset of synchronization in coupled oscillators [1]. In particular, the phenomenon of phase synchronization (PS) between coupled oscillators, for which the phase difference becomes bounded while the amplitudes may be uncorrelated [2]. In addition, the study of the phase of chaotic flows can provide applications to important technological problems, e.g. communication with chaos [3,4], new insights into the collective behavior in networks of coupled chaotic oscillators [5,6] as well as pattern formation [2,7].

In a common sense phase is related to rotations of a vector of the dynamical system. For chaotic attractors in which the trajectory has a proper center of rotation, i.e. it spirals around of a fixed point [4], it is possible to define a phase $\theta$ as the angular displacement of the vector position. Moreover, whenever the chaotic oscillators present a coherent character,[1] one can transform the original equation of motion $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$, where $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{G} : \mathbb{R}^m \to \mathbb{R}^m$, to a new equation of motion that carries the information of the radius and the phase [8]. So, given a neighborhood $N$ of the attractor of the oscillator $\mathbf{y}$, the new equations take the form $\dot{\mathbf{R}} = \mathbf{G}(\mathbf{R}, \vartheta)$ and $\dot{\vartheta} = 1 + \delta(\mathbf{R}, \vartheta)$, where $\mathbf{R} : N \to \mathbb{R}^{m-1}$, $\vartheta : N \to S^1$, and $\delta(\mathbf{R}, \vartheta) \ll O(1)$. The phase $\vartheta$, after this coordinate changing, is not unique, and it will depend on how one approximates the function $\delta(\mathbf{R}, \vartheta)$. Thus, one can define many phases on the attractor, however, all having a physical meaning, e.g. giving the correct average frequency [8].

When the trajectory does not present a coherent motion, or a proper rotation, in Refs. [9,10], instead of analyzing phase in the phase-space, a phase $\phi$ was introduced as the displacement of the velocity vector. This is useful since in many cases

[1] Following Ref. [8], coherence means that there is a section, in which given a time interval for the $i$ return $T^i$, there are numbers $\epsilon$ and $\delta$ such that $|T^i - \epsilon| \leqslant \delta$, and $\delta \ll O(1)$ and $\epsilon = \langle T \rangle$.

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even though the trajectory may be non-coherent in the phase-space, it may present a coherent motion in the velocity space. Such a phase definition avoids the need of proper rotations since a center of rotation is not necessary. Indeed, such a phase definition can be interpreted as follows: the center of rotation is the trajectory itself. Thus, given the trajectory at a time $t + \Delta t$ the center of rotation is the trajectory at a time $t$. Therefore, one avoids the need of a proper rotation. However, it is still unclear what is the physical meaning of this phase, and whether this phase generalizes the phase for coherent oscillators. It is also unknown to what extent this phase works.

In this Letter, we analyze the phase $\phi$ and its geometrical meaning. More specifically, we show that this phase is the length of the Gauss map, the generator of the curvature in differential geometry. $\phi$ generalizes $\theta$, which implies also that the phenomenon of PS is invariant under the change $\theta \to \phi$. In general, the phase $\phi$ will generate negative frequencies, and therefore, the phase is not an one-to-one transformation with the trajectory. However, this non-positiveness is important in order to obtain the average time $\langle T \rangle$ from this phase. Then, we introduce an approach to recover the one-to-one correspondence between phase and trajectory, and still being able to calculate this quantity from the phase. Finally, we show to what extent one could use this phase.

Let us start by noting that the phase $\phi$ can be derived by analyzing the dynamics in the tangent field. Herein, we consider low dimension dynamical systems, namely in $\mathbb{R}^3$. So, given the trajectory $\mathbf{y}_t$, with $\mathbf{y}_t = (x_t, y_t, z_t,)$, we project it into the subspace $(x_t, y_t)$. From now on, we shall denote the projected flow as $\mathbf{x}_t = (x_t, y_t)$. Let us give a geometrical interpretation for $\phi$. Before, we need two basic definition from differential geometry.

**Definition 1.** Suppose the plane curve $C$ is parameterized by $\mathbf{x}: \mathbb{R} \to \mathbb{R}^2$. The Gauss map $\mathcal{G}: C \to S^1$ assigns to the point $\mathbf{x}_t$ the unity tangent vector $\mathcal{G}_t = \dot{\mathbf{x}}_t / |\dot{\mathbf{x}}_t|$.

**Definition 2.** The curvature of the curve $C$ at the point P is

$$\kappa(\mathbf{x}_t) = \lim_{\alpha \downarrow \mathbf{x}_t} \frac{\text{length of } \mathcal{G}}{\text{length of } \alpha},$$

where

$$\text{length of } \mathcal{G}_t = \int_{t-\epsilon}^{t+\epsilon} |\dot{\mathcal{G}}_t| \, dt, \quad \text{and} \quad \text{length of } \alpha = \int_{t-\epsilon}^{t+\epsilon} |\dot{\mathbf{x}}_t| \, dt.$$

The geometrical idea of the transformation $\mathcal{G}$ in Definition 1 is given in Fig. 1.

Let us note that a small increment in the phase $d\phi$ is given by the angle between $\dot{\mathbf{x}}_t$ and $\dot{\mathbf{x}}_{t+dt}$. Now, since the multiplication by a scalar does not change the angle between vectors, measuring the angle between $\dot{\mathbf{x}}_t$ and $\dot{\mathbf{x}}_{t+dt}$ is the same as measuring the angle between $\mathcal{G}_t$ and $\mathcal{G}_{t+dt}$. So, we can interpret the phase as being the angular displacement of $\mathcal{G}_t$. In general, even though $\mathbf{x}_t$ may not present singularities (zero velocities) $\mathcal{G}_t$ will do, due to the inflections in the trajectory, and therefore, we expect to have negative frequencies for the angular displacement of $\mathcal{G}_t$. We can write an equation for the phase $\phi_t$, by noting that $|\mathcal{G}_t \wedge \mathcal{G}_{t+dt}| = \sin d\phi_t$, where $\wedge$ represents the vectorial product, by expanding the sinus in Taylor series $|\mathcal{G}_t \wedge \dot{\mathcal{G}}_t| \, dt \approx d\phi_t + O(d\phi_t^2)$, we have:

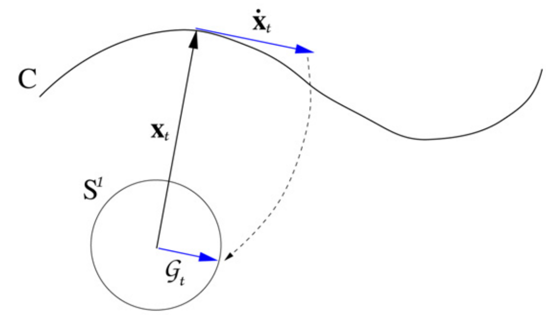


Fig. 1. Illustration of the action of the Gauss map $\mathcal{G}: C \to S^1$.

$$\dot{\phi}_t = |\mathcal{G}_t \wedge \dot{\mathcal{G}}_t|. \tag{1}$$

However, since $\mathcal{G}_t \in (\dot{x}, \dot{y})$, the vectorial product points to the direction $(0, 0, 1)$, presenting only one component whose signal changes when the tangent field changes the direction. To take this signal change into account, we neglect the modulus. Therefore, phase in terms of the vector field takes the form:

$$\dot{\phi}_t = \left. \frac{\dot{\mathbf{x}}_t \wedge \ddot{\mathbf{x}}_t}{\dot{\mathbf{x}}_t^2} \right|_k \tag{2}$$

where $k$ denotes the component $(0, 0, 1)$, the only non-zero component. Next, we explore the relation of the phase $\phi$ and the curvature. Using $ds = r\, d\phi$, and noting that $ds = |\mathcal{G}_{t+dt} - \mathcal{G}_t|$, and $r = |\mathcal{G}_t| = 1$, we have that $\dot{\phi} = |\dot{\mathcal{G}}_t|$. So, the curvature may be written as $\kappa(\mathbf{x}_t) = \int_{t-\epsilon}^{t+\epsilon} \dot{\phi}_t \, dt / \int_{t-\epsilon}^{t+\epsilon} |\dot{\mathbf{x}}_t| \, dt$, which for $\epsilon \ll O(1)$ can be written as $\kappa(\mathbf{x}_t) = \dot{\phi}_t \times 2\epsilon / |\dot{\mathbf{x}}_t| \times 2\epsilon$. Thus, the curvature takes the form:

$$\kappa(\mathbf{x}_t) = \frac{\dot{\phi}_t}{|\dot{\mathbf{x}}_t|}. \tag{3}$$

We conclude three important points here, (i) the phase $\phi$ is equal to the length of $\mathcal{G}_t$, taking into account the rotation direction. In general, $\mathcal{G}_t$ will have many singularities even though the trajectory $\mathbf{x}_t$ has none. This means that if $\mathcal{G}_t$ rotates in a clockwise (respectively counterclockwise) direction it may eventually rotates in a counterclockwise direction (respectively clockwise), producing negative frequencies, what makes the phase not to be a one-to-one transformation with the trajectory. (ii) when the dynamical system presents cusps, e.g. to due the projection, the curvature will be a sharp function on the cusps, while the phase may not. Note that the curvature can be written as

$$\kappa(\mathbf{x}_t) = \left. \frac{\dot{\mathbf{x}}_t \wedge \ddot{\mathbf{x}}_t}{|\dot{\mathbf{x}}_t|^3} \right|_k. \tag{4}$$

At the cusps $|\dot{\mathbf{x}}_t| \approx 0$, so, the divergency in the phase is given by $\ddot{\mathbf{x}}_t / |\dot{\mathbf{x}}_t|$, while in the curvature $\ddot{\mathbf{x}}_t / |\dot{\mathbf{x}}_t|^2$. (iii) the phase measures how much a curve changes its direction, and it is equal to the curvature when normalized by a scalar $|\dot{\mathbf{x}}_t|$. Therefore, the phase is not the curvature. Next, one last analysis may be suitable.

Let us note that since $\langle \mathcal{G}_t, \mathcal{G}_t \rangle = 1$, with $\langle \cdot, \cdot \rangle$ denoting the scalar product, thus, we have $\langle \mathcal{G}_t, \dot{\mathcal{G}}_t \rangle = 0$. Hence, $\mathcal{G}_t$ and $\dot{\mathcal{G}}_t$ are orthogonal vectors, but the phase is the length of $\mathcal{G}_t$, taking into account the rotation direction, which means $\dot{\phi} = |\dot{\mathcal{G}}_t| \times \mathrm{sgn}(\dot{\mathbf{x}}_t)$, where $\mathrm{sgn}(\dot{\mathbf{x}}_t)$ is 1 if $\dot{\mathbf{x}}_t$ rotates in a clockwise direction and $-1$ if $\dot{\mathbf{x}}_t$ rotates in a counterclockwise direction. An explicitly expression for $\dot{\mathcal{G}}_t$ can be easily obtained by derivating $\mathcal{G}_t$, which yields:

$$\dot{\mathcal{G}}_t = \frac{1}{|\dot{\mathbf{x}}_t|}\big(\ddot{\mathbf{x}}_t - \mathcal{G}_t \langle \mathcal{G}_t, \ddot{\mathbf{x}}_t \rangle\big). \tag{5}$$

Thus, from the perspective of an observer being led by the velocity vector (using the interpretation of a moving reference point, as we have discussed in the introduction) the phase $\phi$ is the integral with respect to time of the orthogonal part of the acceleration normalized by $|\dot{\mathbf{x}}_t|$. The next step is to analyze whether the phase $\phi$ generalizes the phase $\theta$.

*Periodic oscillators*: A necessary condition to state that a certain phase can be used to chaotic oscillators is that its application to period oscillators has to give the correct frequency and period. The phase $\phi$ fulfills these requirements. In fact, it is easy to see that $\langle \dot{\phi} \rangle = \langle \dot{\theta} \rangle = 2\pi/T_0$, if $\phi$ and $\theta$ increase $2\pi$ per oscillation, where $T_0$ is the period of the oscillator. To show this relation, let us say that $\dot{\phi} = \tilde{\nu}(\phi)$ and $\dot{\theta} = \nu(\theta)$, where $\nu$ and $\tilde{\nu}$ are continuous functions with period $2\pi$. Thus, taking the averages, we have $\langle \dot{\theta} \rangle = \frac{1}{T_0}\int_0^{T_0} \nu(\theta)\,dt = \frac{1}{T_0}\int_0^{2\pi} d\theta = 2\pi/T_0$ and $\langle \dot{\phi} \rangle = \frac{1}{T_0}\int_0^{T_0} \tilde{\nu}(\phi)\,dt = \frac{1}{T_0}\int_0^{2\pi} d\phi = 2\pi/T_0$. A trivial example where both phases can be analytically computed is for the oscillator $\mathcal{Y} = [V\cos\omega t, V\sin\omega t]$. In this case we have $\dot{\phi}\mathcal{Y} = \dot{\theta}\mathcal{Y} = w$.

*Coherent chaotic oscillators*: We demonstrate that for two coherent chaotic attractors, the phase synchronization phenomena can be observed independently on the phase definition, which means that one has boundness in the phase difference of the two oscillators using either $\theta$ or $\phi$. Under some hypotheses one can show that there are numbers $\sigma$ and $\gamma$, bounded away from zero, for which holds:

$$\sigma\theta \geqslant \phi \geqslant \gamma\theta. \tag{6}$$

This implies, in particular, that PS between two chaotic systems exist, with the phase being measured by $\theta$, if and only if, it exists PS with the phase being measured by $\phi$. So, the phenomenon of PS is, as it should be, invariant under the change $\theta \to \phi$.

In order to prove Eq. (6), we suppose that there are numbers $\alpha_1, \alpha_2$ and $\delta_1, \delta_2$ bounded away from zero, such that for all times it holds: (i) $\alpha_2^{-1}\mathbf{x} \geqslant \dot{\mathbf{x}} \geqslant \alpha_1^{-1}\mathbf{x}$ and (ii) $\delta_2\dot{\mathbf{x}} \geqslant \ddot{\mathbf{x}} \geqslant \delta_1\dot{\mathbf{x}}$. Conditions (i) and (ii) guarantees that the trajectory does not visit a singularity, i.e. a fixed point. At the singularity $\dot{\mathbf{x}} = 0$, which breaks our hypotheses.[2] Next, note that $\tan\theta = y/x$. Taking the derivative we have $\frac{\partial \tan\theta}{\partial\theta} \times \dot{\theta} = \frac{d}{dt} y/x$, which yields $\sec^2\theta \times \dot{\theta} = (\dot{y}x - y\dot{x})/x^2$. Then, we get:

$$\theta = \frac{\mathbf{x}_t \wedge \dot{\mathbf{x}}_t}{\mathbf{x}_t^2}. \tag{7}$$

So, by using Eq. (2) and the conditions (i) and (ii) we have:

$$\begin{aligned}\phi &= \int_{t_0}^{t} \frac{\dot{\mathbf{x}}_t \wedge \ddot{\mathbf{x}}_t}{\dot{\mathbf{x}}_t^2}\,dt \\ &\geqslant \int_{t_0}^{t} \alpha_1^{-1}\alpha_2^2\delta_1 \frac{\mathbf{x}_t \wedge \dot{\mathbf{x}}_t}{\mathbf{x}_t^2}\,dt = \delta_1\alpha_2^2\alpha_1^{-1}\theta.\end{aligned} \tag{8}$$

Now, identifying $\delta_1\alpha_2^2\alpha_1^{-1} = \gamma$, we conclude the first side of Eq. (6). To show the other inequality, one can use the same ideas to conclude that $\phi \leqslant \delta_2\alpha_1^2\alpha_2^{-1}\theta = \sigma\theta$. Next, we show that under conditions (i) and (ii) the phases $\theta$ and $\phi$ are equivalent for PS detection. Having Eq. (6), we suppose that there is PS, and therefore, the phase difference $\Delta\theta$ is bounded, i.e. there is a number $\varrho$ such that $|\theta_1 - \theta_2| < \varrho$. First, we show that the boundness in the phase difference $\Delta\theta$ implies the boundness in the phase difference $\Delta\phi$. Indeed, we have $\sigma\theta_1 \geqslant \phi_1$, we have $\sigma\theta_1 - \sigma\theta_2 \geqslant \phi_1 - \sigma\theta_2$, however, since $\sigma\theta_2 \geqslant \phi_2 \geqslant 0$, we get $\sigma(\theta_1 - \theta_2) \geqslant \phi_1 - \phi_2$. Next, once that $|\theta_1 - \theta_2| \leqslant \varrho$, it yields $\max(\phi_1 - \phi_2) \leqslant \sigma\varrho$, concluding the boundness in the phase difference $\Delta\phi$. Using the same arguments, we show that the boundness of the phase difference $|\phi_1 - \phi_2|$ implies that $|\theta_1 - \theta_2|$ is bounded. This result holds for a phase that depends smoothly on the coordinates.

The exact relation between $\phi$ and $\theta$ is less clear. $\phi$ depends on the first and the second derivative, while $\theta$ only on the first. So, in general, they differ by an amount $\Lambda_\phi$ which may be written as: $\Lambda_\phi = \dot{\phi} - \dot{\theta}$, where $\Lambda_\phi$ is a function of $\mathbf{x}$, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$, which in terms of the vector field takes the form:

$$\Lambda_\phi = \frac{\dot{\mathbf{x}}_t \wedge \ddot{\mathbf{x}}_t}{\dot{\mathbf{x}}_t^2} - \frac{\mathbf{x}_t \wedge \dot{\mathbf{x}}_t}{\mathbf{x}_t^2}. \tag{9}$$

An analytical calculation of the deviation for a general case might be impossible. However, if we bring about the fact that coherent attractors have a well defined average period $\langle T \rangle$,[1] in such a way that both phases grow governed by this average time, then both phases are equivalent, the averages frequencies are equal, due to the fact that both phases increase $2\pi$ for cycle, which implies that $\langle \Lambda_\phi \rangle = 0$.

*Positively defined frequency*: $\dot{\phi}$ assumes negative values due to the coordinate system. In principle one could proceed a coordinate change in order to avoid the negativeness of $\dot{\phi}$. However, such a procedure is rather difficult, since even though the transformation exists, it is unknown. Here we explore the simplest way to avoid the fact that $\dot{\phi}$ is not positively defined. We map the negative values to positive ones by a moduli operation, in other words, we have a new frequency $\dot{\psi} = |\dot{\phi}|$. Of course, now we have that the average increasing of the phase $\psi$ per cycle, namely $\langle r_\psi \rangle$, is no longer equal to $2\pi$, which is reflected in a positive average deviation $\langle \Lambda_\psi \rangle$ of $\dot{\psi}$ from $\dot{\theta}$. The phase in terms of the vector field takes the form

[2] These hypotheses are violated by attractors which contain an equilibrium point within the attractor, as an example the Lorentz attractor. Such a phenomenon does not interfere in the chaotic behavior [11], however, it does interfere in the PS, since in a small neighborhood of the equilibrium the velocity approaches arbitrarily zero.

$\dot{\psi} = \sqrt{\ddot{\mathbf{x}}_t^2 - \langle \dot{\mathbf{x}}_t, \ddot{\mathbf{x}}_t \rangle^2} / |\dot{\mathbf{x}}_t|$. The frequency $\dot{\theta}$ can be written as

$$\dot{\theta} = \frac{1}{|\mathbf{x}_t|} \sqrt{\dot{\mathbf{x}}_t^2 - \frac{\langle \mathbf{x}_t, \dot{\mathbf{x}}_t \rangle^2}{\mathbf{x}_t^2}}. \tag{10}$$

Having an equation for the phase $\theta$[3] the deviation $\Lambda_\psi$ can be written as:

$$\Lambda_\psi = \frac{1}{|\dot{\mathbf{x}}_t|} \sqrt{\ddot{\mathbf{x}}_t^2 - \frac{\langle \dot{\mathbf{x}}_t, \ddot{\mathbf{x}}_t \rangle^2}{\dot{\mathbf{x}}_t^2}} - \frac{1}{|\mathbf{x}_t|} \sqrt{\dot{\mathbf{x}}_t^2 - \frac{\langle \mathbf{x}_t, \dot{\mathbf{x}}_t \rangle^2}{\mathbf{x}_t^2}}. \tag{11}$$

In general, $\langle \Lambda_\psi \rangle > 0$ implies that $\langle \dot{\psi} \rangle$ is no longer equal to $2\pi / \langle T \rangle$, what does not constitute a problem because we can analytically calculate the average increasing of the phase.

Let us call $T_k$, the time spent between two successive crossing, namely $k$ and $k-1$, of the trajectory and a defined Poincaré section. The average period is given by $\langle T \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} T_i = \frac{t}{N}$. The average frequency, $\langle \dot{\psi} \rangle = \lim_{t\to\infty} \frac{1}{t} \int_0^t \dot{\psi}\, dt = \lim_{N\to\infty} \frac{1}{\langle T \rangle} (\frac{1}{N} \int_0^t \dot{\psi}\, dt)$. Thus, introducing $\langle r_\psi \rangle = \lim_{\tau_N\to\infty} \frac{1}{N} \int_0^t \dot{\psi}\, dt$, which is the average increasing of the phase in one typical cycle, we get:

$$\langle T \rangle = \frac{\langle r_\psi \rangle}{\langle \dot{\psi} \rangle}. \tag{12}$$

Thus, we can also calculate the average period by $\dot{\psi}$, however, first, we have to calculate $\langle r_\psi \rangle$ as a function of the vector field. Noting that for coherent attractors $\langle T \rangle = \frac{2\pi}{\langle \dot{\theta} \rangle}$, which is also equal to $\frac{\langle r_\psi \rangle}{\langle \dot{\psi} \rangle}$, we have that $\langle r \rangle = 2\pi \frac{\langle \dot{\theta} \rangle + \langle \Lambda_\psi \rangle}{\langle \dot{\theta} \rangle}$, and finally:

$$\langle r_\psi \rangle = 2\pi \left( 1 + \frac{\langle \Lambda_\psi \rangle}{\langle \dot{\theta} \rangle} \right). \tag{13}$$

The average frequency and the average period are no longer straightforwardly related, as in periodic oscillators. However, we can still obtain the average period $\langle T \rangle$. In order to analyze PS between oscillators using the phase $\psi$, we must subtract from $\psi$, for each single oscillator, a drift that appears due to the fact that $\langle r_\psi \rangle \geqslant 2\pi$. The drift is given by $\delta\psi = (\langle r_\psi \rangle - 2\pi) \times t$.

We illustrate these ideas using two non-identical coupled Rössler oscillators $\Sigma_1$ and $\Sigma_2$, given by $\dot{x}_{1,2} = -\alpha_{1,2} y_{1,2} - z_{1,2} + \epsilon[x_{2,1} - x_{1,2}]$, $\dot{y}_{1,2} = \alpha_{1,2} x_{1,2} + a y_{1,2}$, $\dot{z}_{1,2} = b + z_{1,2}(x_{1,2} - d)$, with $\alpha_1 = 1$ and $\alpha_2 = \alpha_1 + \delta\alpha$. The constants $\alpha = 1.00$, $a = 0.15$, $b = 0.4$, and $d = 8.5$ are chosen such that we have a chaotic attractor in a phase coherent regime.

Let us consider the two coupled Rössler. For $\delta\alpha = 0.001$ and $\epsilon = 0.01$. First, we calculate the average frequency for the oscillator $\Sigma_1$, Fig. 2(a), (b) using $\dot{\theta}_1$ and $\dot{\phi}_1$; the two phases give the same average frequency $\langle \dot{\theta}_1 \rangle = \langle \dot{\phi}_1 \rangle \approx 1.0343$, but, $\phi$ is not a monotonic function, since $\dot{\phi}$ can be negative, Fig. 2(c).

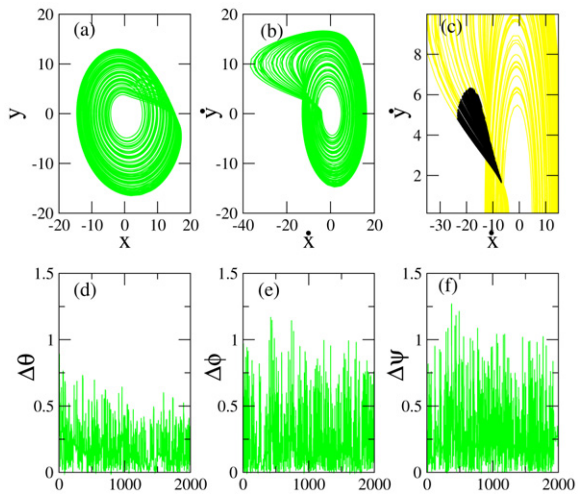


Fig. 2. In (a) the projection of the Rössler oscillator on the plane $(x, y)$. In (b) the projection on the plane $(\dot{x}, \dot{y})$. The trajectory presents a coherent character in both projections, however the instantaneous frequency $\dot{\phi}$ assumes negative values. In (c) it is depicted, in black, the region of the attractor where $\dot{\phi} < 0$, and in gray, the attractor. We use three types of phases $\theta$, $\phi$, and $\psi$ to measure the phase difference $\Delta\theta$, $\Delta\phi$, and $\Delta\psi$ respectively, and to detect PS. In (d) the phase is measured by $\theta$, while in (e) by $\phi$. For such oscillator we can also use the phase $\psi$ after removing the drift, i.e. one must analyze the quantity $|\Delta\psi| = |(\psi_1 - \delta\psi_1) - (\psi_2 - \delta\psi_2)|$ which in PS is bounded $(f)$.

Therefore, the one-to-one correspondence of the phase with the trajectory is lost. We can use the phase $\psi$ which is one-to-one, however, $\langle \dot{\psi}_1 \rangle = 1.0373$ which is not equal to $\langle \dot{\theta}_1 \rangle$. For these values of $\delta\alpha$ and $\epsilon$, the two oscillators are phase synchronized, which means that the phase difference remains bounded for all the time, and the average quantities, e.g. average period and average frequency, are equal in both oscillators. As we showed, both phases $\theta$ and $\phi$ can be used to detect PS, Fig. 2(d), (e). $\psi$ can also be used once that one removes the drifting term $(\langle r \rangle - 2\pi) \times t$, Fig. 2(f).

Next, we analyze the average period. First, we calculate $\langle T \rangle$ by defining a Poincaré section, and then we compare the average period obtained from the Poincaré section technique with the average period given by Eq. (12). The Poincaré section is defined at $y = 0$. We compute the return time to this section, namely $T_k$, and then we take the average, which provides us $\langle T \rangle \approx 6.0710$. Next, from Eq. (13), we have that the average growing of the phase $\phi$ is $2\pi$, which means that $\langle \Lambda_\phi \rangle = 0$. Calculating the average period by Eq. (12), we have $\langle T \rangle = \langle r_\phi \rangle / \langle \dot{\phi} \rangle \approx 6.0710$. Similarly, we calculate $\langle r_\psi \rangle$, from Eq. (13), which give us $\langle r_\psi \rangle \approx 6.2984 > 2\pi$, and therefore $\langle \Lambda_\psi \rangle > 0$, (actually from Eq. (11), we have that $\langle \Lambda_\psi \rangle \approx 0.005$). So, using Eq. (12), we obtain the average period $\langle T \rangle = \langle r_\psi \rangle / \langle \dot{\psi} \rangle \approx 6.0710$, which shows that the approach is able to give the correct average time.

*Non-coherent chaotic oscillators*: By non-coherent oscillators, we mean those oscillators in which the trajectory does not possess a proper rotation or has a broad distribution in the power spectra.[1] So, the phase $\theta$ cannot be defined. Here we classify these attractors into two classes:

[3] Using the same ideas as before, $ds = r\, d\theta$, where $d\theta$ is the angle between $\mathbf{x}_t / |\mathbf{x}_t|$ and $\mathbf{x}_{t+dt} / |\mathbf{x}_{t+dt}|$, we can write the equation stated. An interesting case to use this equation for $\dot{\theta}$ is the central forces, where the frequency is $\frac{M}{r^2}$. Noting that $\vec{r} = r\mathbf{e}_r \Rightarrow \dot{\vec{r}} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi$. We have $\dot{\theta} = \frac{1}{r} \sqrt{\dot{\mathbf{r}}^2 + \frac{\langle \mathbf{r}, \dot{\mathbf{r}} \rangle^2}{\mathbf{r}^2}} = \frac{1}{r} \sqrt{\dot{r}^2 + \frac{M}{r^2} - \frac{r^2 \dot{r}^2}{r^2}} = \frac{M}{r^2}$.

**Definition 3.** Class I is composed by attractors in which the trajectory presents a coherent motion in the velocity space, and $\dot{\phi}$ is always positive.

**Definition 4.** Class II is composed by attractors in which the trajectory presents weak non-coherent motion in the velocity space, i.e. the trajectory is coherent, but $\dot{\phi}$ presents negative values for short time intervals.

As example, the standard Rössler attractor is a coherent attractor in the phase space, and also in the velocity space, but for some other parameters it belongs to the *class I*. The attractors from the forced Chua's circuit, studied here, belong to the *class II*. Our ideas hold for these two classes of attractors.

For oscillators of *class I*, $\dot{\phi} > 0$, which implies $\phi = \psi$. We have also that $\langle r_{\psi,\phi} \rangle = 2\pi$. This is so, because a rotation of the velocity vector increases $2\pi$ for a typical return time. Actually, the increasing of $2\pi$ in the rotation of the velocity vector defines a Poincaré section, since the motion is coherent in the velocity space. So, we can compute the average period by $\langle T \rangle = 2\pi/\langle\dot{\psi}\rangle$.

For attractors of *class II*, the phases $\psi$ and $\phi$ give different results. Indeed, for this class $\langle r_\psi \rangle > 2\pi$, while $\langle r_\phi \rangle = 2\pi$. This occurs because the attractors of *class II* present a weak non-coherence, characterized by non-coherences in the velocity space, where the frequency $\dot{\phi}$ is negative, which causes the average growing of the phase $\psi$ to be different than $2\pi$. Since frequency $\dot{\psi}$ is positively defined, these small regions are incorporated in the phase $\psi$. On the other hand, the phase $\phi$ filters this weak non-coherence, since it can be negative. Roughly speaking, the phase $\phi$ increases in the first part of these regions and decreases in the second part, and as a result these non-coherent regions do not contribute to the phase growing, as if the weak non-coherent attractors were actually a coherent attractor in the velocity space.

An appropriate case to illustrate our ideas is the phase dynamics of a non-coherent chaotic attractor driven by a periodic forcing. Let us denote the chaotic dynamics by $\mathcal{X}$ and the periodic forcing by $\mathcal{Y}$. Thus, phase synchronization between the forcing and the chaotic attractor implies $\langle\dot{\phi}_{[\mathcal{X}]}\rangle = \langle\dot{\theta}_{[\mathcal{Y}]}\rangle$. Keeping this in mind, we analyze the case of the Chua's circuit driven by a periodic forcing, whose state equations are given by: $a\dot{x} = g(y-x) - i_{NR}(x)$, $b\dot{y} = g(x-y)+z$, $c\dot{z} = -y - V_p \sin\omega$ and $\dot{\omega} = 2\pi f_p$. The term $i_{NR}$ is the non-linear electrical current injected in the circuit, mathematically represented by: $i_{NL}(x) = m_0 x + 0.5(m_1 - m_0)\{|x + B_p| - |x - B_p|\}$. The parameters used in the numerical analyzes are $a = 0.1$, $b = 1.0$, $c = 1/6$, $g = 0.574$, $m_0 = -0.5$, $m_1 = -0.8$, and $B_p = 1.0$.

The non-coherent attractors are found in the driven Chua's oscillator when the coupling strength increases. In particular, for $f_p = 0.275$ and $V_p = 0.097$. An usual stroboscopic technique shows that the two oscillators are phase synchronized, Fig. 3(a), i.e. the stroboscopic map does not fulfill the attractor [12,13]. Using the phase $\psi$, we have that $\langle\dot{\psi}\rangle = 0.275229$. The phase $\psi$ was averaged in the time range of $10^{10}$. The phase $\phi$ gives $\langle\dot{\phi}\rangle = 0.275000$, Fig. 3(c). The phase $\psi$, in this situation, misleads the results, see Fig. 3(d), because it considers all non-coherence of the attractor in the velocity space, see Fig. 3(b). So, for this class, the average period can be calculated by $\langle T \rangle = 2\pi/\langle\dot{\phi}\rangle$.

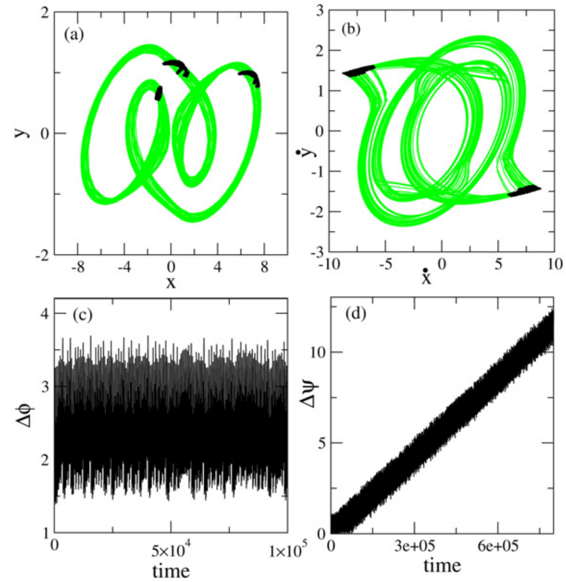


Fig. 3. The non-coherent phase attractor of the periodic driven Chua's circuit. In (a) we depict, in gray, the attractor in the projection $(x, y)$ and, in black, a stroboscopic map. The stroboscopic map is confined which shows that the attractor is phase synchronized. In (b) we show, in gray, the attractor in the projection $(\dot{x}, \dot{y})$ and, in black, the region where $\dot{\phi}$ has negative values. In (c) we show the phase difference $|\Delta\phi| = |\phi_{[\mathcal{X}]} - \phi_{[\mathcal{Y}]}|$, which is bounded due to PS. In (d) the phase difference $|\Delta\psi| = |\psi_{[\mathcal{X}]} - \psi_{[\mathcal{Y}]}|$ diverges. In this case, we cannot correct the drift in the phase $\psi$ since we do not have an equation for $\langle r_\psi \rangle$ in terms of the vector field. The perturbation parameters are $f = 0.2750$ and $V = 0.097$.

*Homo(hetero)clinic chaotic attractor*: For such attractors [14] the phases $\phi$ and $\psi$ cannot be used to calculate the average period. That is so, due to the fact that these dynamics get arbitrarily close to the "rest" state, i.e. near the unstable homoclinic point. The phase depends on the derivatives of the trajectory which vanishes in the homoclinic points causing the phase to diverge.

We illustrate these ideas in the perturbed Hindmarsh–Rose neuronal (HR) model [15] given by: $\dot{x} = y + 3x^2 - x^3 - z + I + A\sin(\omega t)$, $\dot{y} = 1 - 5x^2 - y$, $\dot{z} = -rz + 4r(x + 1.6)$, where, $A$ is the amplitude and $\omega$ the frequency of the perturbation. $(x, y)$ have a fast dynamics and $z$ a slow one. $r$ is the ratio of fast/slow time scales. First, we set $r = 0.005$ and $I = 3.2499$, in order to produce a chaotic attractor with two distinct time scales, the spikes and the bursts. In our numerical analysis we use the following initial conditions: $x = -1.31$, $y = -7.32$, and $z = 3.35$. Next, we set $A = 0.015$ and $\omega = 0.00607$, such that the HR neuron phase synchronizes with the perturbation.

In Fig. 4(a) we depict the time series from $x$ (in black) and the sinusoidal perturbation (in gray). So, one can clearly see that the HR neuron phase synchronizes with the perturbation. Even though the HR neuron presents two time-scales, the phase $\theta$ can be applied [16], since the trajectory has a proper rotation, see Fig. 4(b).

The HR neuron synchronizes with the perturbation in the slow time-scale, the bursts, which means that the average period

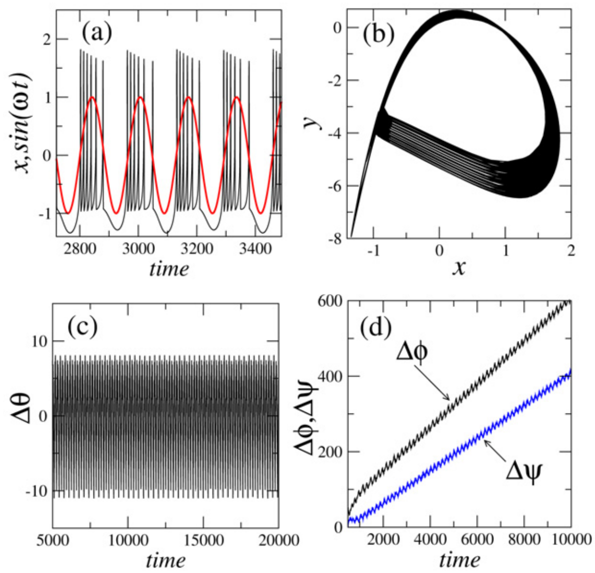

Fig. 4. In (a) we plot the time series $x$ of the neuron in black line and the sinusoidal perturbation in gray. In (b) we depict the attractor of the HR neuron projected in the subspace $(x, y)$. As one can see the attractor has a proper rotation. In (c) we plot the phase difference $\Delta\theta = \theta - 6 \times \omega t$, which is bounded, showing the PS between the HR neuron and the periodic perturbation. In (d) we show the phase differences $\Delta\phi = \phi - 6 \times \omega t$ and $\Delta\psi = \psi - 6 \times \omega t$, that diverge. So these phases fail to detect PS between the HR neuron and the perturbation.

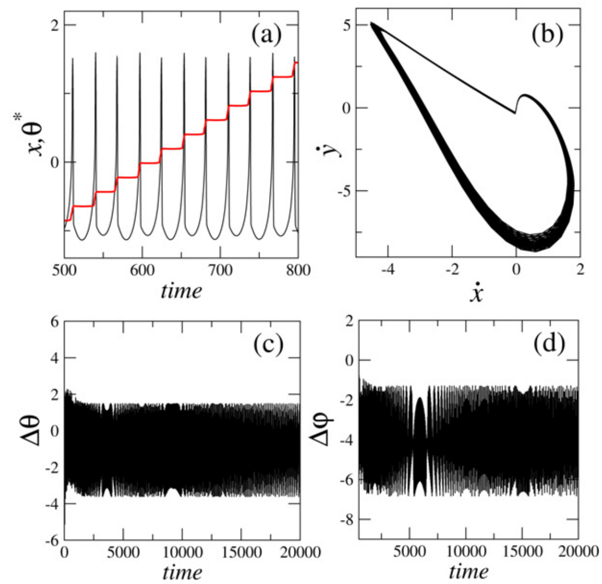

Fig. 5. In (a) the time series $x$ of the neuron in black line and the phase $\theta^*$ in bold gray, where $\theta^* = \theta/30$, for a better visualization. In (b) the projection of the attractor of the HR neuron in the subspace $(\dot{x}, \dot{y})$. In (c) the phase differences $\Delta\theta = \theta - \omega t$. In (d) we plot $\Delta\varphi = \varphi - \omega t$, which is a bounded quantity.

between two bursts $\langle T_{\text{burst}}\rangle$ is equal to the average period of the perturbation $2\pi/\omega$. The phases $\theta$, $\phi$ and $\psi$ capture only the dynamics of the fast time scale.[4] Therefore, we must have a $m : n$ PS, that is, the difference $|m\theta - n\omega t|$ is bounded. Since within each burst there is on average 6 spikes, we have $m = 1$ and $n = 6$. In this case, the phase $\theta$ can be applied and it gives the correct results, indeed by applying Eq. (10), it yields $\langle\dot{\theta}\rangle/6 = 0.00607 = \omega$, and there is PS, see Fig. 4(c). However, the phases $\phi$ and $\psi$ mislead the results. Our numerical analysis shows that $\langle\dot{\phi}\rangle/6 \approx 0.0045$ and $\langle\dot{\psi}\rangle/6 \approx 0.00715$, and these phases are not able to detect PS between the HR neuron and the perturbation [Fig. 4(d)]. Of course, in this case the boundness in the phase difference $\Delta\theta$ does not imply the boundness in the phase difference $\Delta\phi$, since $\dot{\mathbf{x}}_t = 0$, which violates our results for the equivalence of $\theta$ and $\phi$.

This problem is not restricted to multi-time-scale oscillators. For instance, as the ratio $r$ between the fast and slow time scale is increased, the average interval between bursts is of the same order of the average interval between spikes. In particular, for $r = 0.04$ the two time-scale merge into one [Fig. 5(a)], e.g. only spikes are present. In addition, for $A = 0.015$ and $\omega = 0.0353$ the HR neuron phase synchronizes with the perturbation. Again, even though the oscillator now presents only one time scale and presents a coherent motion and proper rotation [Fig. 5(b)] in the phase space, the dynamics is mainly constituted by spikes, and therefore very often $\dot{\mathbf{x}}_t \approx 0$, see Fig. 5(b), which misleads the results obtained from the phase $\phi$. The trajectory in the phase space is coherent and has a proper rotation, therefore, Eq. (10) can be used, and indeed, it gives the correct results. Actually, we have $\langle\dot{\theta}\rangle = 0.0353$, and PS can be detected by means of $\theta$ [Fig. 5(c)].

The trajectory in the velocity space $(\dot{x}, \dot{y})$ has a coherent motion, however, it does not have a proper rotation, since the trajectory in the velocity space visits the origin $(0, 0)$. Hence the phase $\phi$ cannot be applied. This problem can be overcome by a translation of the attractor on the velocity space $(\dot{x}, \dot{y})$. Therefore, the introduction of a phase $\varphi = \tan^{-1}[(\dot{y}_t - \dot{y}_0)/(\dot{x}_t - \dot{x}_0)]$ is able to correct the problems of $\phi$ in this case. $\varphi$ yields the following equation for the frequency:

$$\dot{\varphi}_t = \frac{\dot{\mathbf{x}}_t \wedge \ddot{\mathbf{x}}_t + \ddot{\mathbf{x}}_t \wedge \dot{\mathbf{x}}_0}{(\dot{\mathbf{x}}_t - \dot{\mathbf{x}}_0)^2}, \tag{14}$$

where, we denote $\dot{\mathbf{x}}_0 = (\dot{x}_0, \dot{y}_0)$. This phase generalizes the phase $\phi$, with the condition that (i) $(\dot{\mathbf{x}}_t - \dot{\mathbf{x}}_0)^2$ is bounded away from zero, (ii) the trajectory spirals around $\mathbf{x}_0$ with a proper rotation.

In the previous case of the HR neuron [Figs. 5(a) and (b)], if we set $\dot{\mathbf{x}}_0 = (0, -4)$, $\varphi$ succeeds to extract the correct average frequency, namely $\langle\dot{\varphi}\rangle = 0.0353$. Next, if we analyze the phase difference $\Delta\varphi = \varphi - \omega t$, this quantity is bounded [Fig. 5(d)]. Therefore, the phase $\varphi$ is able to detect PS. $\varphi$ generalizes $\phi$ and can be applied whenever the attractor on the velocity space has a proper rotation. $\varphi$ can extract the correct average frequency and average period whenever the trajectory in the velocity space possesses a coherent character.

Moreover, the ideas herein can be also applied to a case where the oscillator is not coherent but present a proper rotation, e.g. the Sprott E chaotic flow [18]. Both phases $\theta$ and $\phi$

[4] For a numerical analysis of the system behavior on different time scales, the continuous wavelet analysis is able to characterize each time scale by means of its own phase. An approach to chaotic synchronization detection is developed in Ref. [17] in terms of wavelet analysis.

are able to extract the average period of the attractor. Since the attractor has a proper rotation we can define a Poincaré section, and then, calculate the average period of the attractor. In particular, for $\alpha = 4.0$ (see [18] for more details) we have that $\langle T \rangle = 6.484$, whereas $\langle \theta \rangle = \langle \phi \rangle = 0.969$, which is equal to $2\pi/\langle T \rangle$.

In conclusion, we have analyzed a phase for chaotic attractors based on the rotation of the tangent vector of the trajectory. We discussed the natural link between this phase, the curvature, and the average period of the chaotic attractors. Moreover, we demonstrated that for a broad class of chaotic attractors phase synchronization is invariant under the phase definitions. We adopt two new ideas: (i) the frequency might reach negative values but still possessing a physical meaning, however, losing its one-to-one character with the trajectory, (ii) we can use a phase which average growing per cycle can be bigger than $2\pi$. Our results holds for a broad class of chaotic attractors, basically all the attractors whose dynamics possesses a coherent behavior either in the phase space or in the velocity space. For homoclinic type of attractors, we showed that in general the phase $\phi$ fails to extract the correct average frequency of the attractor, due to the fact that the trajectory goes arbitrarily close to the homoclinic point. We overcame this problem by generalizing the phase $\phi$. Therefore, we have introduced the phase $\varphi$ which is basically the phase $\phi$ with a translation in the velocity space. This phase solves the problems of the phase $\phi$ and works whenever the trajectory presents a coherent behavior and a proper rotation after the translation.

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## References

[1] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence, Springer, Berlin, 1984.
[2] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge Univ. Press, Cambridge, 2001; S. Boccaletti, J. Kurths, G. Osipov, D. Valladares, C. Zhou, Phys. Rep. 366 (2002) 1.
[3] S. Hayes, C. Grebogi, E. Ott, A. Mark, Phys. Rev. Lett. 73 (1994) 1781.
[4] T. Yalçinkaya, Y.C. Lai, Phys. Rev. Lett. 79 (1997) 3885.
[5] G.V. Osipov, M.V. Ivanchenko, J. Kurths, B. Hu, Phys. Rev. E 71 (2005) 056209.
[6] M. Dhamala, V.K. Jirsa, M. Ding, Phys. Rev. Lett. 92 (2004) 028101.
[7] R. Ellison, V. Gardner, J. Lepak, et al., Int. J. Bifur. Chaos 15 (2005) 2283; W.J. Freeman, J.M. Barrie, J. Neurophys. 84 (2000) 1266.
[8] K. Josić, M. Beck, Chaos 13 (2003) 247; K. Josić, D.J. Mar, Phys. Rev. E 64 (2001) 056234.
[9] J.Y. Chen, K.W. Wong, J.W. Shuai, Phys. Lett. A 285 (2001) 312.
[10] G.V. Osipov, B. Hu, C. Zhou, M.V. Ivanchenko, J. Kurths, Phys. Rev. Lett. 91 (2003) 024101.
[11] C. Bonatti, L.J. Diaz, M. Viana, Dynamics Beyond Uniform Hyperbolicity, Springer, Berlin, 2004.
[12] M.S. Baptista, T. Pereira, J.C. Sartorelli, I.L. Caldas, J. Kurths, Physica D 212 (2005) 216.
[13] C.M. Ticos, E. Rosa Jr., W.B. Pardo, J.A. Walkenstein, M. Monti, Phys. Rev. Lett. 85 (2000) 2929.
[14] F.T. Arecchi, R. Meucci, A. Di Garbo, E. Allaria, Opt. Lasers Eng. 39 (2003) 393; F.T. Arecchi, A. Lapucci, R. Meucci, et al., Europhys. Lett. 6 (1988) 677.
[15] J.L. Hindmarsh, R.M. Rose, Proc. R. Soc. London B 221 (1984) 87; J.L. Hindmarsh, R.M. Rose, Nature 296 (1982) 162.
[16] J.W. Shuai, D. Durand, Phys. Lett. A 264 (1999) 289.
[17] A.E. Hramov, A.A. Koronovskii, Chaos 14 (2004) 603; M. Chavez, C. Adam, V. Navarro, S. Boccaletti, J. Martinerie, Chaos 15 (2005) 023904.
[18] J.M. González-Miranda, Phys. Lett. A 352 (2006) 83.