

Fluctuation Growth and Saturation in Nonlinear Oscillator on the Threshold of Bifurcation of Spontaneous Symmetry Breaking

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We study pre-bifurcation noise amplification in nonlinear oscillators subject to bifurcations of spontaneous symmetry breaking which are manifested in the doubling of stable equilibrium states. Our theoretical estimates of both the linear growth and the nonlinear saturation of the bifurcations are in good agreement with our results from numerical simulations. We show that in the saturation mode, the fluctuation variance σ_x^2 is proportional to the standard deviation of the external noise σ_η , whereas in the linear mode, σ_x^2 is proportional to the noise variance σ_η^2 . It is demonstrated that the phenomenon of pre-bifurcation noise amplification is more pronounced in the case of a slow transition through the bifurcation point. The amplification of fluctuations in this case makes it easier to form a symmetric probability of the final equilibrium states. In contrast, for a fast transition through the bifurcation point, the effect of amplification is much less pronounced. Under backward and forward passages through the bifurcation point, a loop of noise-dependent hysteresis emerges here. We find that for a fast transition of the nonlinear oscillator through the bifurcation point, the probability symmetry of the final equilibrium states is destroyed.

1. INTRODUCTION

In nonlinear systems, pre-generation noise amplification is commonly observed at the threshold of self-sustained oscillations. This was shown for both radiophysical and optical self-sustained oscillating systems [1]. Noise amplification near the generation threshold is conditioned by the decrease of losses in the oscillator casting the real part of one of Lyapunov indexes of the system from negative to positive values. The initial state of the system loses its stability and the amplified pre-generation noise turns into an effective push for self-sustained oscillations.

Pre-generation of noise amplification is a particular case of a more general phenomenon, the pre-bifurcation amplification of noise and of weak signals due to the decrease (down to zero at the critical point) of the damping strength [2]-[5]. The linear theory developed in [2-5] predicts unlimited growth of fluctuations when approaching the bifurcation point.

Nonlinear analysis of pre-bifurcation noise amplification was performed in [6] for the case of period doubling bifurcations in the logistics map. In this case, similarly to the general one, the Lyapunov exponent changes from negative to positive. However in general, the change of the sign of the real part $Re\lambda$ of Lyapunov exponent λ now leads not to emerging generation of oscillations but to the transition of the system from the unstable state to one of the two possible stable states.

The aim of this work is to study this phenomenon of pre-bifurcation noise amplification in a nonlinear oscillator subject to a bifurcation of spontaneous symmetry breaking. Such a bifurcation is known to lead to two new stable states instead of one stable equilibrium state which loses its stability.

In Section 2, we describe the model of a nonlinear oscillator where doubling of stable equilibrium states may take place. An example of such a system are one-dimensional cross-oscillations of a rod squeezed along its axis (see Sec.2). In Sections 3-5, we present analytical and numerical estimations of the fluctuation level under fast and slow changes of a control parameter. Section 6 presents results of numerical modeling. Section 7 describes the phenomenon of noise-dependent hysteresis in the system under consideration. Finally in Section 8, it is shown that under fast bifurcation transitions in the nonlinear oscillator, the probability symmetry of the stable final states is destroyed.

2. DYNAMIC MODEL

We start by considering oscillations in a nonlinear oscillator described by the second-order equation:

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \frac{\partial U(x)}{\partial x} = \eta(t). \quad (1)$$

Here γ is damping strength, $\eta(t)$ is the noise forcing the system, and $U(x)$ is the potential energy.

As is well-known, the number of minima of the potential energy profile $U(x)$ in the nonlinear oscillator determines the number of equilibrium states [7]. Under the bifurcation of spontaneous symmetry breaking, a transition from a one-minimum potential to a two-minimum one takes place. This is accompanied by a doubling of the number of the stable states. The potential

$$U(x) = Ax^4 + Bx^2, \quad (2)$$

corresponds to bifurcation of spontaneous symmetry breaking. For $A > 0$, it has one minimum $U_{\min}=0$ (at $x=0$), and for $B < 0$, there are two equal negative minima $U_{\min}(x^\pm) = -|B^2|/4A$ located at $x^\pm = \pm\sqrt{|B|/2A}$ with the maximum $U_{\max} = 0$ in between at $x=0$.

Assume parameter B to be dependent on time, $B=B(t)$, taking positive values $B > 0$ for $t < t^*$ and negative ones for $t > t^*$. Such behavior is demonstrated, e. g. by the function

$$B(t) = -\omega_0^2 \arctg \beta(t - t^*), \quad (3)$$

as shown in Fig. 1. The coefficient β characterizes here the speed of the change of the control parameter.

For a physical prototype of the system, described by potential (2), we can take one-dimensional cross-oscillations (along axis x) of a flat rod (ruler) under the influence of a growing in time squeezing force along its axis. At a critical squeeze, when B turns zero, the rod undergoes a bifurcation of spontaneous symmetry breaking and takes a curved shape corresponding to one of the two stable states, x^+ or x^- [7].

The goal of this paper is to determine the variance $\langle [x(t) - \bar{x}]^2 \rangle \equiv \sigma_x^2$ of the response of $x(t)$ to the fluctuation force $\eta(t)$ (the \bar{x} is mean value equal to the stationary value) and in that way we describe both the growth and nonlinear saturation of the pre-bifurcation noise amplification in shortly before the bifurcation.

We will assume that the fluctuation force $\eta(t)$ is a stationary random process with the auto-correlation function

$$\langle \eta(t')\eta(t'') \rangle = \sigma_\eta^2 \Psi_\eta(t' - t''), \quad (4)$$

where σ_η^2 is variance and $\Psi_\eta(\tau)$ is auto-correlation function. We define the correlation time τ_η as the time when the autocorrelation function goes down to the value 0.5.

Below we will restrict shortly-correlated processes, i. g. τ_η is small compared to the period of the free oscillations $2\pi/\omega_0$ of the unperturbed oscillations, so that $\omega_0\tau_\eta \ll 2\pi$. Hence, the noise is similar white noise.

3. ESTIMATION OF FLUCTUATIONS UNDER SLOW CHANGE OF SYSTEM PARAMETERS: WKB APPROXIMATION

Under a low amplitude of the oscillations and for $B(t) > 0$, eq. (1) can be linearized

$$\ddot{x} + 2\gamma\dot{x} + \omega^2(t)x = \eta(t), \quad (5)$$

where $\omega(t) = \sqrt{B(t)}$. For a slow enough decrease of $B(t)$, i. e. when $\beta \ll \omega_0$, $x(t)$ is fairly well described by the Wentzel-Kramers-Brillouin approximation (WKB):

$$x(t) = \frac{A_x}{\sqrt{\omega(t)}} e^{-\gamma t} \cos(\Phi(t) + \varphi_0), \quad (6)$$

where

$$\Phi(t) = \int_0^t \omega(t') dt'.$$

It is easy to apply the same approximation to the Green function of eq. (5):

$$g(t, t') = \begin{cases} 0 & \text{at } t < t', \\ \frac{e^{-\gamma(t-t')}}{\sqrt{\omega(t)\omega(t')}} \sin \Phi_g(t, t') & \text{at } t > t', \end{cases} \quad (7)$$

where

$$\Phi_g(t, t') = \int_{t'}^t \omega(t'') dt''.$$

Using the Green function (7), the solution of the heterogeneous linear equation (5) can be written as

$$x(t) = \int_{-\infty}^t g(t, t') \eta(t') dt', \quad (8)$$

and the variance $\sigma_x^2 = \langle x^2(t) \rangle$ as

$$\sigma_x^2 = \int_{-\infty}^t dt' \int_{-\infty}^t dt'' g(t, t') g(t, t'') \langle \eta(t') \eta(t'') \rangle. \quad (9)$$

For shortly-correlated noise (4) and the Green function (7), this expression takes the form

$$\sigma_x^2 = \tau_\eta \sigma_\eta^2 \int_{-\infty}^t g^2(t, t') dt' = \tau_\eta \sigma_\eta^2 \frac{1}{\omega(t)} \int_{-\infty}^t \frac{e^{-\gamma(t-t')}}{\omega(t')} \sin^2 \Phi_g(t, t') dt'. \quad (10)$$

Substituting for $\sin^2 \Phi_g(t, t')$ its mean value 1/2, we get the estimate

$$\sigma_x^2(t) = \sigma_\eta^2 \frac{\tau_\eta}{2\gamma\omega(t)\Omega(t)}, \quad (11)$$

where $1/\Omega(t)$ denotes the integral

$$\frac{1}{\Omega(t)} = \gamma \int_{-\infty}^t \frac{e^{-\gamma(t-t')}}{\omega(t')} dt', \quad (12)$$

representing the value $1/\omega(t')$ time-averaged with the weight $\gamma \exp[-\gamma(t-t')]$.

It is worth mentioning that for a constant frequency $\omega = \omega_0$, the expression (11) turns into the well known expression for steady fluctuations of the linear oscillator under shortly-correlated fluctuation forcing $\eta(t)$ [1]

$$\sigma_0^2 = \sigma_\eta^2 \frac{\tau_\eta}{2\gamma\omega_0^2}. \quad (13)$$

When approaching the bifurcation point $t=t^*$, where $B(t)$ turns to zero, linear estimate (11) describes a growing fluctuation variance reflecting a pre-bifurcation amplification of the noise. The solid line in Fig. 3 represents this fluctuation growth. The values $1/\omega(t)$ and $1/\Omega(t)$ from (11) go to infinity for $t \rightarrow t^*$. The linear estimate (11) follows the same path, as is shown in Fig. 3 by dash-and-dots. Obviously, the linear estimate (11) based on the VKB approximation is not applicable in the vicinity of the bifurcation point. However, we will demonstrate in the next section, that for $t \rightarrow t^*$, it is possible to obtain a nonlinear estimate for the variance σ_x^2 and will show that it takes the finite values even in the bifurcation point.

A peculiarity of the bifurcation scenario consists in the fact, that while approaching the point of bifurcation, the real part of the Lyapunov exponents $\gamma = \text{Re}\lambda$ do not go down to zero as for the Landau-Hopf bifurcation and the period doubling bifurcation. That is why the fluctuation increase here is not related to loss decrease. Instead, we have a transformation of a pair of conjugate complex exponents $\lambda_{1,2} = -\gamma \pm i\omega$ into a pair of real exponents, one of which is positive. As a result, the pre-bifurcation noise amplification is now connected with the decrease of the frequency $\omega = \text{Im}\lambda$, because for $\omega \rightarrow 0$, the amplitude of the free oscillations grows as $1/\sqrt{\omega(t)}$.

4. NONLINEAR ESTIMATIONS OF FLUCTUATION INTENSITY IN THE VICINITY OF BIFURCATION POINT

As discussed above, expression (11) based on linear theory loses its validity in the vicinity of the bifurcation point. This occurs due to nonlinear effects arising under a infinite growth of the fluctuations. Nonlinear effects can be neglected as far as the fourth-power term in (2) is small compared to the second-order term:

$$A \langle x^4 \rangle \ll |B(t)| \langle x^2 \rangle. \quad (14)$$

Supposing the fluctuations η and x are Gaussian, for estimations we assume

$$\langle x^4 \rangle = 3 \langle x^2 \rangle^2 = 3\sigma_x^4$$

and then re-write inequality (14) as

$$|B(t)| \gg 3A\sigma_x^2 = \frac{3A\sigma_\eta^2\tau_\eta}{2\gamma\omega(t)\Omega(t)}. \quad (15)$$

If we also assume, that the characteristic oscillation attenuation time $1/\gamma$ is less than the characteristic change time $1/\beta$ of the frequency $\omega(t)$, e. g. $\gamma > \beta$, then from eq. (12), we get

$$\frac{1}{\Omega(t)} \approx \frac{1}{\omega(t)} \quad (16)$$

Then eq. (15) takes the form

$$B^2(t) = \omega^4(t) \gg \frac{3A\sigma_\eta^2\tau_\eta}{2\gamma}. \quad (17)$$

From (17), we infer the estimate

$$B_{\min} \sim \left(\frac{3A\sigma_{\eta}^2\tau_{\eta}}{2\gamma} \right)^{1/2} \quad (18)$$

for a permitted distance to the point of bifurcation, and from (11) together with (16), estimate

$$\sigma_{x \max}^2 \sim \sigma_{\eta} \sqrt{\frac{\tau_{\eta}}{6\gamma A}} \quad (19)$$

for the maximum fluctuation intensity. Therefore, for $B < B_{\min}$, the linear effects of fluctuation growth are replaced by nonlinear saturation to be reached at fluctuation intensity $\sigma_{x \max}^2$. A horizontal line in Fig. 2 denotes value $\sigma_{x \max}^2$. A similar saturation of the fluctuation intensity is found for period doubling bifurcations as well [6]. A general approach for estimating the saturation level is discussed in [8].

It is useful to introduce the pre-bifurcation noise amplification factor K as the ratio of $\sigma_{x \max}^2$ to the fluctuation intensity (13) at $\omega = \omega_0$:

$$K = \frac{\sigma_{\max}^2}{\sigma_0^2} = \frac{1}{\sigma_{\eta}} \left(\frac{2\gamma}{3A\tau_{\eta}} \right)^{1/2} \omega_0^2. \quad (20)$$

This value indicates how many times the fluctuation intensity in the saturation zone is greater than the stationary fluctuation intensity of the oscillator.

The phenomenon of pre-bifurcation increase of fluctuation intensity is accompanied by another phenomenon - pre-bifurcation correlation time rise. The latter also experiences saturation in the vicinity of the bifurcation point. In fact, one can speak about pre-bifurcation rise and subsequent saturation of correlation time. This phenomenon is found for period doubling bifurcations [9]. It is noteworthy, that the phenomenon of pre-bifurcation correlation time rise has a general nature. It is as well observed in the nonlinear oscillator subject to bifurcations of spontaneous symmetry breaking and described in the present paper.

This approach looks to be prospective for analysis of different nonlinear geophysical systems in which a period-doubling bifurcations and a bifurcation of spontaneous symmetry breaking take place. Such a system is, for example, the Kuroshio Current System [10].

5. FLUCTUATIONS UNDER A VERY FAST CHANGE OF THE OSCILLATOR PARAMETERS

Above we considered fluctuations for rather slow changes ($\beta < \gamma$) of the oscillator parameters. At a fast passage through bifurcation point, that is at $\beta > \gamma$, one may expect some decrease of σ_{\max}^2 in comparison to the case $\beta < \gamma$.

The trend to a smaller σ_{\max}^2 with growing speed β of the transition through the bifurcation point can be illustrated by the extreme case $\beta \rightarrow \infty$, when for $t < t^*$, $B(t)$ is constant and equals ω_0^2 , while for $t > t^*$, it is constant as well but equals $-\omega_0^2$. In this case fluctuation intensity at $t < t^*$ is constant and equals σ_0^2 , so that the factor of the fluctuation amplification yields : $K=1$.

Naturally, for $t > t^*$, we will observe an exponential growth of the fluctuations due to a stability loss of the equilibrium state $x=0$, but this will have no effect on the fluctuations for $t < t^*$.

6. NUMERICAL SIMULATIONS

The nonlinear oscillator described above was numerically tested for $A=0.5$, and

values of B ranging from $B_0 = 100$ to $B_f = 10^{-8}$, that is almost reaching zero. Changes of B were slow enough (quasi-stationary mode). The damping index γ was taken equal to 0.1. So, to satisfy the quasi-stationary requirement, we needed characteristic change time of B to be small compared to 0.1.

The specified range of parameter B allows to determine the fluctuation variance both in the immediate vicinity of the bifurcation point $B_c=0$ and far away from it (let us remind, that far from the bifurcation point, each value of B corresponds to the oscillation frequency $\omega_0 = \sqrt{B}$). For $B>0$, when the system has only one stable solution (see bifurcation diagram in Fig. 3), this stable point $x(0) = 0$ was taken as the initial value, and the initial value of the derivative $x'(0)$ was taken equal to 0.

A random number generator produced normally distributed values of $\eta(t)$ with zero mean, $\langle \eta(t) \rangle = 0$, and a mean square deviation σ_η ranging from 10^{-6} to 10^{-1} . The auto-correlation function of the random process has characteristic time τ_η in the interval from 10^{-2} to 10^{-3} , being small compared to $\frac{2\pi}{\omega_0}$. In this case, the results do not depend on the shape of the auto-correlation function Ψ_η ; as it must be for processes close to white noise. The use of other random number generators producing, for example, uniformly distributed values of $\eta(t)$, gave results qualitatively similar to those obtained for normally distributed noise. Numerical solution for (5) was obtained using a fourth-order Runge-Kutta method with a fixed step.

Results of the numerical simulations are presented in Fig. 4, where crosses show the dependence of the fluctuation variance σ_x^2 on B . We used a slow change of the parameter B (quasi-stationary mode) and fluctuation variance $\sigma_\eta^2=10^{-8}$. The dash-and-dot line corresponds to the fluctuation variance σ_0^2 estimated in the linear mode (13) describing a fluctuation growth in the direction to the bifurcation threshold. The figure shows that numerical results correspond well the linear estimate for $B > B_{\min}$, which is in line with (18). With a closer approach to the bifurcation point $B=0$, the fluctuation variance reaches saturation at σ_{\max}^2 marked by a horizontal dashed line obtained from the estimate (19).

For the considered quasi-stationary mode, at noise variance $\sigma_\eta^2 = 10^{-8}$, the fluctuation amplification factor K_{\max} (20) was $1.55 \cdot 10^6$. This value agrees by magnitude with the theoretical estimate $K_{\max \text{ teor}} = 6.17 \cdot 10^6$.

The described results are in qualitative agreement with data obtained earlier for period doubling bifurcation [6]: in both cases, the fluctuation mean square σ_x^2 is proportional to the mean square noise deviation σ_η , whereas the amplification factor K_{\max} is inversely proportional to σ_η .

7. NOISE-DEPENDENT HYSTERESIS IN THE VICINITY OF THE BIFURCATION POINT

Next we study the phenomenon of noise-dependent hysteresis in our system. Such a hysteresis occurs when after passing through the bifurcation point, the system for a considerable time remains on the unstable branch and only after some time makes a rather

quick transition to another stable state [11]- [12]. The higher is the control parameter change speed, the more distinct is the hysteretic phenomenon.

In a quasi-stationary mode, when the parameter B changes slowly, the bifurcation in the system occurs at the critical value $B = B^* = 0$. Under a fast change of B , bifurcation of stable state doubling happens only some time after the critical value $B^* = 0$ is passed, the delay time being dependent on the speed of change of the parameter β .

Fig. 5, a presents results of numerical simulations of bifurcation transition in the nonlinear oscillator when B varies via eq. (3). For illustrative purposes, the Fig. 5 combines the bifurcation diagram of the model, that is stable for constant B , but shows a hysteresis of $x(B)$ for varying B . With the change of B , the system having passed the value $B = B^*$, still resides for some time in the vicinity of the unstable branch (this time depends considerably on speed β), and only after that switches to one of the two stable states of the equilibrium. At a high transition speed $\beta = 300$ (plot 3), delay time Δt_3 is considerably greater than delay time Δt_2 at $\beta = 0.3$ (curve 2) and delay time Δt_1 under a very slow transition with the speed $\beta = 0.03$ (plot 1).

Under both forward and backward transitions through the bifurcation point, the system is known to slow down in the vicinity of the former stable points. This delay phenomenon causes the emergence of a hysteretic loop. Note, that under a forward transition through the bifurcation point, the system is more sensitive to noise than under backward transition. Fig. 5, b illustrates the phenomenon of hysteresis for the nonlinear oscillator at $\beta = 3$ (forward sweep) and $\beta = -3$ (backward sweep). As is clearly seen from this figure, hysteretic loop diminishes with noise. This effect can be used to measure weak noise in nonlinear systems as suggested earlier for systems with period doubling bifurcations [12].

8. PROBABILITY SYMMETRY BREAKING IN NONLINEAR OSCILLATOR UNDER BIFURCATION OF STABLE STATE DOUBLING

As is shown in [13], [14] for period doubling bifurcations, dynamic bifurcations under low noise are characterized by a probability symmetry breaking. Under considerable noise, the probabilities of the transition into two equitable final states are the same, each equaling 50%, while in the absence of noise, the final state of the system is fully predictable and depends only on the initial conditions and the speed of change of the control parameter. The phenomenon of probability symmetry breaking is also observed in bifurcations of spontaneous symmetry breaking. In the absence of noise, the system transits with a 100% probability into one of the two possible final states determined by the speed of the transition and the initial conditions. Under the impact of noise, the probabilities of transition of a nonlinear oscillator with varying parameters into either of the two final states tend to become equal.

Fig. 6 illustrates the phenomenon of probability symmetry breaking in the nonlinear oscillator. In the absence of noise, for the initial value $x_0 = 0$ and the initial derivative value $x'(0) = 1$, for the speed $\beta = 3$, the dependence of x on B is shown in plot 1 (Fig. 6a). Under the impact of noise, the system may transit as into the “upper” state (plot 2), as into the “lower” state (plot 3), and it will happen considerably earlier than in the absence of noise (for numerical tests, we took noise variance $\sigma_\eta^2 = 1.87 \cdot 10^{-6}$).

Fig. 6 b shows the dependence of final state probabilities on the initial values x_0 . As seen in this figure, at noise variance $\sigma_\eta^2 = 1.87 \cdot 10^{-7}$, the limits of the final state attraction zones are smeared by noise. In contrast to discrete maps [14], the pattern of the final state attraction zones for a nonlinear oscillator depends not only on initial values and the bifurcation transition speed, but on the initial value of the derivative as well. In the case of

low speeds, the attraction zones are split up resulting in even greater system sensitivity to noise.

When the transition through a bifurcation point slows down, the pre-bifurcation noise amplification pushes the final state probabilities to become equal. Meanwhile, under fast bifurcation transitions, the fluctuations in the vicinity of the bifurcation point decrease and the predictability of the final state rises.

9. CONCLUSION

This work investigates fluctuations in a nonlinear oscillator subject to bifurcations of spontaneous symmetry breaking. Analytical estimates of pre-bifurcation noise amplification are obtained both for linear and nonlinear approximation. It is shown, that the variance of the forced fluctuations σ_x^2 in the saturation mode (near the bifurcation point) is proportional to the mean square of the noise forcing σ_η : $\sigma_x^2 \propto \sigma_\eta$, whereas in the linear mode (far from the bifurcation threshold), it is proportional to the noise variance σ_η^2 : $\sigma_x^2 \propto \sigma_\eta^2$. Analytical estimates are in good agreement with numerical results.

The pre-bifurcation fluctuation amplification is shown to facilitate the establishment of probability symmetry of final equilibrium states. Under a slow change of the oscillator parameter, the impact of weak noise results in equalizing the probabilities of the two possible final states. Under a fast bifurcation transition, the effect of pre-bifurcation noise amplification weakens, the system becomes less sensitive to noise and the final states become more predictable (probability symmetry breaking).

Finally, it is demonstrated, that the bifurcation of spontaneous symmetry breaking in a nonlinear oscillator with varying control parameter is accompanied by a delay phenomenon, whose parameters strongly depend on the level of noise, a noise-dependent hysteresis.

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Fluctuation growth and saturation in nonlinear oscillator on the threshold of bifurcation of spontaneous symmetry breaking

FIG. 1. Model law for parameter B in accordance with (3).

FIG. 2. Nonlinear estimate of the fluctuation variance.

FIG. 3. Bifurcation diagram of the nonlinear oscillator (Eq. 5)

FIG. 4. Pre-bifurcation noise amplification for a bifurcation of spontaneous symmetry breaking. Fluctuation variance σ_x^2 versus parameter B in a quasi-stationary mode ($\sigma_\eta^2 = 10^{-8}$, $\tau_\eta = 3.5 \cdot 10^{-3}$, $\gamma = 0.1$, $A = 0.5$).

FIG. 5. Bifurcation diagram and solutions for the differential equation (1): a) – slow transition through the bifurcation point, $\beta = 0.03$ (plot 1); fast transition, $\beta = 0.3$ (plot 2); very fast transition, $\beta = 300$ (plot 3);
b) – forward and backward transition through the bifurcation point at speeds $\beta = 3$ (plot 1) and $\beta = -3$ (plot 2); the impact of noise $\sigma_\eta^2 = 1.87 \cdot 10^{-7}$ at $\beta = 3$ (plot 3).

FIG 6. Probability symmetry breaking in the nonlinear oscillator.

- a) – bifurcation diagram and solution for equation (1) at $\beta = 3$ (plot 1), solutions in the presence of noise with variance $\sigma_\eta^2 = 1.87 \cdot 10^{-7}$ (plots 2 and 3);
б) – final state attraction zones – solid lines; smearing of attraction zone limits by noise with variance $\sigma_\eta^2 = 1.87 \cdot 10^{-7}$ at $\beta = 3$ – dots.

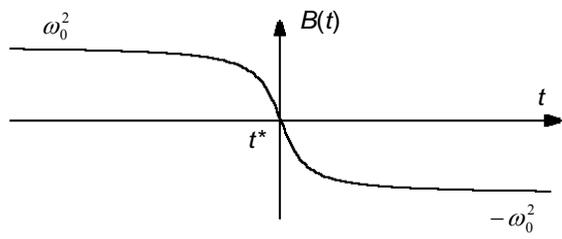


FIG. 1.

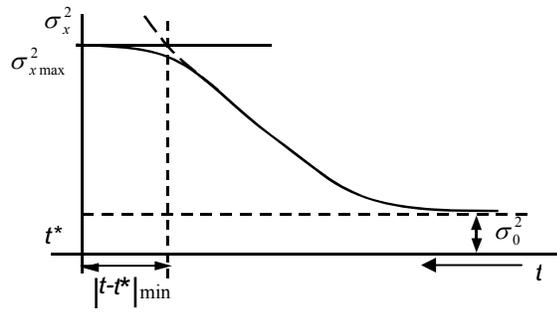


FIG. 2.

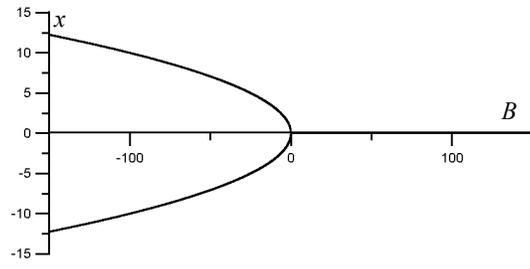


FIG. 3.

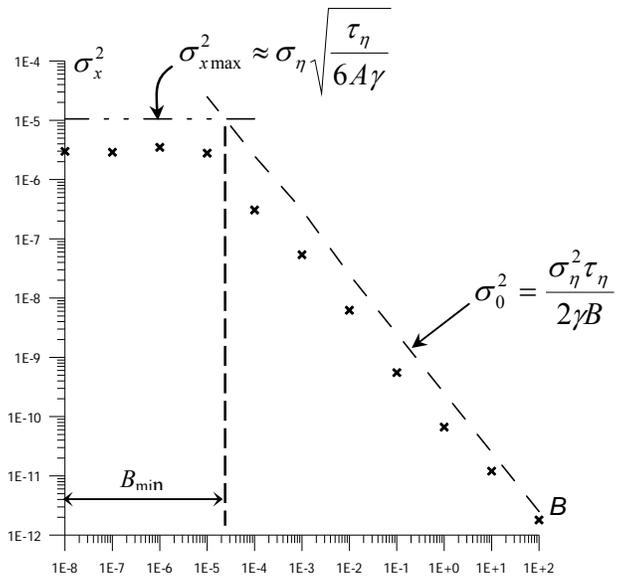


FIG. 4.

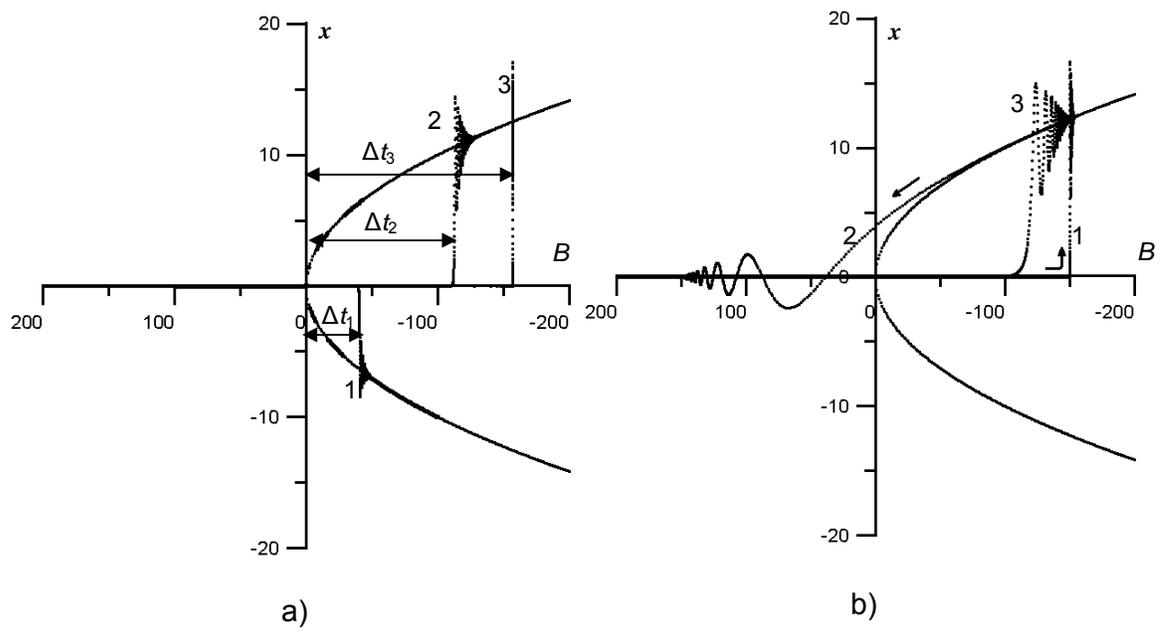


FIG. 5.

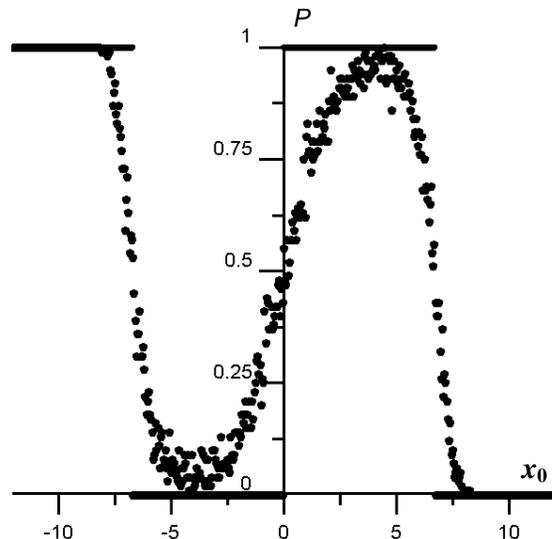
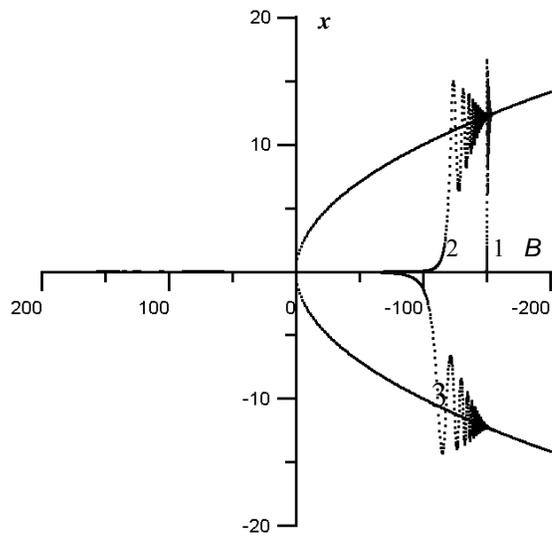


FIG. 6.

a)

b)