Dependently-Typed Programming in Economic Modelling

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PIK: Potsdam Institute for Climate Impact Research
"At PIK researchers in the natural and social sciences work together to study global change and its impacts on ecological, economic and social systems. They examine the Earth system's capacity for withstanding human interventions and devise strategies for a sustainable development of humankind and nature. PIK research projects are interdisciplinary and undertaken by scientists from the following Research Domains: Earth System Analysis, Climate Impacts and Vulnerabilities, Sustainable Solutions and Transdisciplinary Concepts and Methods."

"Who sent you hither? Wherefore do you come?" R3 I.iv.174

Richard Hamming (as echoed by Paul Graham):

1. What are the most important problems in your field?

2. Are you working on one of them?

3. Why not?

"ben trovato , *may I say*" *TS I.ii.24*

Important problem: "end software crisis by ensuring software correctness".

"Have you no modesty?" MND III.ii.285

Somewhat more modest goal: improve correctness of models used for integrated assessments.

Integrated assessment models are used to answer questions such as "how will phasing out nuclear power plants affect the unemployment in Germany?".

"too ambitious" TNK prologue.23

More modest still: improve correctness of economic models used for integrated assessments.

Why economic models?

1. Most integrated assessment models contain an economic component.
2. Economic models have more structure than some of the other components.(More opportunity to reuse software components).
3. There is more need for improving economic theory than physical theory.

"not without ambition" Mac I.v.17

Starting point: improve correctness of models that compute an equilibrium.

Mainstream economic models are based on the idea that agents will interact in a way that leads to a kind of optimal state, an "equilibrium". Non-mainstream models attempt rather to simulate plausible behavior of agents, in order to see whether an equilibrium is reached or not.

"there it begins" Cym V.v.179

Characteristics of economic models:

1. "Theoretical" side: we can formalize it
2. "Computational" side: we can implement it

Hopefully, dependently-typed programming allows us to join both sides.

"how soon confusion / May enter 'twixt the gap of both" Cor III.i.111

The quintessential economic situation: exchange of goods.

1. Two agents, two goods, X units of the first good, Y units of the second.
2. Agent i has x_i unit of the first good, and y_i units of the second.
3. A distribution of goods to agents, such as $((x_1, y_1), (x_2, y_2))$ is called an *allocation*. Agents have preferences over allocations.
4. Agents are allowed to exchange their goods in order to find a better allocation: no throwing goods away, and no creation of goods: $x_1 + x_2 = X$, $y_1 + y_2 = Y$.

What is a good allocation?

"Why, then, we'll make exchange; here, take you this." TG II.ii.6

Definitions of Pareto efficiency. A feasible allocation $\mathbf{x}$ is a **weakly Pareto efficient** allocation if there is no feasible allocation $\mathbf{x}'$ such that all agents strictly prefer $\mathbf{x}'$ to $\mathbf{x}$. A feasible allocation $\mathbf{x}$ is a **strongly Pareto efficient** allocation if there is no feasible allocation $\mathbf{x}'$ such that all agents weakly prefer $\mathbf{x}'$ to $\mathbf{x}$, and some agent strictly prefers $\mathbf{x}'$ to $\mathbf{x}$.

Varian, p. 323

An allocation x is weakly Pareto efficient, if there exists no feasible allocation that dominates it *strictly everywhere*.

An allocation x is strongly Pareto efficient, if there exists no feasible allocation that dominates it weakly everywhere and *strictly somewhere*.

"None better" AW III.vi.17

Obviously (?), strong Pareto efficiency implies weak Pareto efficiency.

Idris formalization of this property...

"the adage must be verified" 3H6 I.iv.126

A typical example is the Cobb-Douglas economy, in which the agents preferences induced by the utility functions

$$u_1(x, y) = x^a y^{(1-a)}$$

$$u_2(x, y) = x^b y^{(1-b)}$$

where $0 < a, b < 1$.

"Then mark th' inducement." H8 II.iv.169

How can we find Pareto-efficient allocations?

An idea: start with any *feasible* allocation $((x_1, y_1), (x_2, y_2))$. Solve:

maximize $u_1(x, y)$ such that

$$u_2(X - x, Y - y) = u_2(x_2, y_2)$$

where $X = x_1 + x_2$ and $Y = y_1 + y_2$

The solution will be a Pareto-efficient allocation.

"so find we profit" AC II.i.7

In the example we had, the solution of the maximization problem is given by

$$\frac{a}{1-a}\frac{y}{x} = \frac{b}{1-b}\frac{Y-y}{X-x}$$

In more complex exchange economies, finding Pareto-efficient points is hard.

"A cunning man did calculate" 2H6 IV.i.34

If goods have prices p_x, p_y then an initial allocation gives each agent a *budget*:

$$B_i = p_x x_i + p_y y_i.$$

An agent has to solve:

maximize $u(x, y)$ such that

$$p_x x + p_2 y = B_i$$

Whether the resulting allocation is feasible depends on the prices.

"We can afford no more at such a price." LLL V.ii.223

Definition of Walrasian equilibrium.

An allocation-price pair $(\mathbf{x}, \mathbf{p})$ is a **Walrasian equilibrium** if (1) the allocation is feasible, and (2) each agent is making an optimal choice from its budget set. In equations:

1. $\sum_{i=1}^{n} \mathbf{x}_i = \sum_{i=1}^{n} \boldsymbol{\omega}_i$
2. If $\mathbf{x'}_i$ is preferred by agent i to $\mathbf{x}_i$, then $\mathbf{p}\mathbf{x'}_i > \mathbf{p}\boldsymbol{\omega}_i$.

Varian, p. 325

“in equal balance justly weighed” 2H4 IV.i.67

Idris formalization of Walrasian equilibrium.

"formally, according to our law" R2 I.iii.29

Walrasian equilibria are Pareto-efficient.

Informal proof:

If $\mathbf{x'}$ is feasible, then $\sum_{i=1}^{n} \mathbf{x'}_i = \sum_{i=1}^{n} \boldsymbol{\omega}_i$.

Multiplying both sums by $\mathbf{p}$, we have that allocation $\mathbf{x'}$ costs just as much as $\boldsymbol{\omega}$.

However, since $\mathbf{x}$ is optimal, and since we are given that every agent strictly prefers $\mathbf{x'}$ to $\mathbf{x}$, we have that each individual bundle in $\mathbf{x'}$ costs more that the respective bundle in $\boldsymbol{\omega}$. Therefore, the allocation $\mathbf{x'}$ costs strictly more that the initial endowment.

Contradiction.

"This must be so." Ham I.ii.106

We can check the informal argument by implementing it in Idris . . .

"With untired spirits and formal constancy." JC II.i.227

How can we compute Walrasian equilibria?

For the special case of the Cobb-Douglas economy, the solution can be computed analytically:

$$\frac{p_y}{p_x} = \frac{(1-a)x_1+(1-b)x_2}{ay_1+by_2}$$

$$x_1^* = \frac{B_1 a}{p_x}$$

$$y_1^* = \frac{B_1(1-a)}{p_y}$$

In general, however, computing Walrasian equilibria involves a lot of numerical methods (optimization, solving linear systems, etc.).

"hard, hard!" KL III.vii.32

We continue to develop the formalization of economic theory: Nash, correlated equilibria, etc.

Work has begun on a DSL for numerical methods.

"Further to boast were neither true nor modest" Cym V.v.18

Ideally, one would like to have numerical methods implemented in terms of constructive reals, used in a constructive economic theory.

"Go thy ways. I begin to be aweary of thee" AW IV.v.54