Economic Equilibria in Type Theory

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Motivation

Integrated assessment models are used to answer questions such as "how will phasing out nuclear power plants affect Germany's unemployment?".

We focus on the economic models because

1. Most integrated assessment models contain an economic component.
2. Economic models have more structure than some of the other components. (More opportunity to reuse software components).
3. There is more need for improving economic theory than physical theory.
4. My boss is an economist.

Game theory 101

Example: A two-person finite game.

Strategy sets: first player can choose from $\{T, B\}$, second player from $\{L, R\}$.

A *play* consists of each agent making a choice from their strategy set.

Agents have *preferences* on plays (often induced by a *payoff function*).

Typical representation:

	L	R
T	(TL, LT)	(TR, RT)
B	(BL, LB)	(BR, RB)

Essential question: what are "good" plays?

Game context

$$
\begin{array}{l}
\textbf{module}\ \mathit{GameContext}''\ (nA' : \mathit{Nat})\ \textbf{where} \\
\quad \begin{array}{lcl} nA & : & \mathit{Nat} \\ nA & = & S\ nA' \\ \mathit{Agent} & : & \mathit{Set} \\ \mathit{Agent} & = & \mathit{Fin}\ nA \end{array}
\end{array}
$$

$$
\begin{array}{l}
\textbf{module}\ \mathit{GameContext}'\ (\mathit{Strategy} : \mathit{Agent} \to \mathit{Set})\ \textbf{where} \\
\quad \begin{array}{lcl} \mathit{Play} & : & \mathit{Set} \\ \mathit{Play} & = & (a : \mathit{Agent}) \to \mathit{Strategy}\ a \end{array}
\end{array}
$$

$$
\begin{array}{l}
\textbf{module}\ \mathit{GameContext}\ (\\
\quad \mathit{prefs} : (a : \mathit{Agent}) \to \mathit{TotalPreorder}\ \mathit{Play})\ \textbf{where}
\end{array}
$$

Some examples

Sometimes, it's clear what a good play is . . .

	L	R
T	(1, 1)	(0, 0)
B	(0, 0)	(0, 0)

Some examples

Sometimes, it's clear what a good play is . . .

	L	R
T	(1, 1)	(0, 0)
B	(0, 0)	(0, 0)

Some examples

Other times, it's clear what a bad play is . . .

	L	R
T	(1, 0)	(0, 4)
B	(3, 0)	(2, 3)

Some examples

Other times, it's clear what a bad play is . . .

	L	R
T	(1, 0)	(0, 4)
B	(3, 0)	(2, 3)

Pareto optimality (efficiency)

$$
\begin{aligned}
&\mathit{Pareto}' \quad : \; \mathit{Play} \to \mathit{Set}\\
&\mathit{Pareto}'\; p* \; = \\
&\quad (\forall (p : \mathit{Play}) \to \\
&\qquad \neg\, ((\, \forall\, (a : \mathit{Agent}) \quad \to (\mathit{prefs}\; a) \vdash p \geqslant p*) \bigwedge \\
&\qquad\quad \exists\, (\lambda(a : \mathit{Agent}) \to (\mathit{prefs}\; a) \vdash p > p*)))
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{Pareto} \quad : \; \mathit{Play} \to \mathit{Set}\\
&\mathit{Pareto}\; p* \; = \\
&\quad \forall\, (p : \mathit{Play}) \to \\
&\qquad (\forall (a : \mathit{Agent}) \to \quad (\mathit{prefs}\; a) \vdash p* \geqslant p) \bigvee \\
&\qquad (\exists (\lambda (a : \mathit{Agent}) \to (\mathit{prefs}\; a) \vdash p* > p))
\end{aligned}
$$

Computing Pareto points

For finite games, we can use brute-force enumeration or somewhat cleverer methods such as sieving.

Pareto points

As expected . . .

	L	R
T	(1, 1)	(0, 0)
B	(0, 0)	(0, 0)

Pareto points

As expected . . .

	L	R
T	(1, 0)	(0, 4)
B	(3, 0)	(2, 3)

Pareto points

There's something unsatisfactory here ...

	L	R
T	(1, 0)	(0, 4)
B	(3, 0)	(2, 3)

Pareto points

Not all Pareto points are equally plausible . . .

	L	R
T	(1, 0)	(0, 4)
B	(3, 0)	(2, 3)

Nash equilibria

$$
\begin{aligned}
&\mathit{Nash}' : \mathit{Play} \to \mathit{Set}\\
&\mathit{Nash}'\ p{*} =\\
&\quad \forall\,(a : \mathit{Agent}) \to \forall(p : \mathit{Play}) \to\\
&\qquad \neg\,(\mathit{prefs}\ a \vdash (\mathit{change}\ p * \mathit{at}\ a\ \mathit{to}\ (p\ a)) > p)
\end{aligned}
$$

$$
\begin{aligned}
\mathit{Nash} &: \mathit{Play} \to \mathit{Set}\\
\mathit{Nash}\ p* &= \forall\,(a : \mathit{Agent}) \to \forall(p : \mathit{Play}) \to\\
&\quad\ \mathit{prefs}\ a \vdash p * \geqslant (\mathit{change}\ p * \mathit{at}\ a\ \mathit{to}\ (p\ a))
\end{aligned}
$$

Computing Nash equilibria

For finite games, by enumeration etc.

The prisoner's dilemma

	L	R
T	(3, 3)	(0, 5)
B	(5, 0)	(1, 1)

Pareto points for the prisoner's dilemma

	L	R
T	(3, 3)	(0, 5)
B	(5, 0)	(1, 1)

Nash equilibrium for prisoner's dilemma

	L	R
T	(3, 3)	(0, 5)
B	(5, 0)	(1, 1)

A game with no Nash equilibrium

	L	R
T	(0, 2)	(2, 0)
B	(2, 0)	(0, 2)

Introducing mixed Nash equilibrium

We change the game: agents can choose probability distributions over their strategy sets.

	L	R
T	(0, 2)	(2, 0)
B	(2, 0)	(0, 2)

Mixed Nash equilibrium

$$
\begin{array}{l}
\textbf{module } \mathit{MixedStrategyContext}\ (\mathit{Strategy} : \mathit{Agent} \to \mathit{Set};\\
\quad \mathit{payoff} : (a : \mathit{Agent}) \to ((a : \mathit{Agent}) \to \mathit{Strategy}\ a) \to \mathit{Float})\\
\quad \textbf{where}
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{MixedStrategy} & : & (a : \mathit{Agent}) \to \mathit{Set}\\
\mathit{MixedStrategy}\ a & = & \mathit{SimpleProb}\ (\mathit{Strategy}\ a)
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{MixedPlay} & : & \mathit{Set}\\
\mathit{MixedPlay} & = & (a : \mathit{Agent}) \to \mathit{MixedStrategy}\ a
\end{array}
$$

Mixed Nash equilibrium

$$
\begin{aligned}
&\mathit{expected\_payoff} : (a : \mathit{Agent}) \to \mathit{MixedPlay} \to \mathit{Float} \\
&\mathit{expected\_payoff}\ \ a\ \mathit{mp} = \mathit{expected}\ (\mathit{fmap}\ ((\mathit{payoff}\ \ a)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathit{mixedPlayToProbPlay}\ \ \mathit{mp}))
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{prefs} \quad : \ (a : \mathit{Agent}) \to \mathit{TotalPreorder}\ \ \mathit{MixedPlay} \\
&\mathit{prefs}\ a = \mathit{InducedPreorder}\ \ \mathit{floatPreorder}\ (\mathit{expected\_payoff}\ \ a)
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{open}\ \mathit{GameContext}'\ \ \mathit{MixedStrategy} \\
&\mathit{open}\ \mathit{GameContext}\ \mathit{prefs}\ \mathit{renaming}\ (\mathit{Nash}\ \mathit{to}\ \mathit{MixedNash})
\end{aligned}
$$

Computing mixed Nash equilibrium

. . . is hard! In fact, NP-hard (Goldberg and Papadimitriou, 2005).

Mixed Nash equilibrium

$p_L = 0.5,\ p_T = 0.5$

	L	R
T	(0, 2)	(2, 0)
B	(2, 0)	(0, 2)

Nash equilibrium example

Example: BoS

	L	R
T	(2, 1)	(0, 0)
B	(0, 0)	(1, 2)

Nash equilibrium example

Pure Nash equilibria:

	L	R
T	(2, 1)	(0, 0)
B	(0, 0)	(1, 2)

Nash equilibrium example

Problem: Pure equilibria are unfair!

	L	R
T	(2, 1)	(0, 0)
B	(0, 0)	(1, 2)

Nash equilibrium example

$p_T = 2\,/\,3$, $p_R = 2\,/\,3$
Expected payoffs: $2\,/\,3$ each.

	L	R
T	(2, 1)	(0, 0)
B	(0, 0)	(1, 2)

Nash equilibrium example

$p_R = 2\,/\,3$, $p_T = 2\,/\,3$
Expected payoffs: $2\,/\,3$ each: we cannot avoid miscoordination, if we want fairness.

	L	R
T	(2, 1)	(0, 0)
B	(0, 0)	(1, 2)

Introducing correlated equilibrium

We change the game again.

A new object: A simple probability distribution on plays.

Interpretation: A coordinator advises agents how to play.

	L	R
T	P_{TL}, (T, L)	P_{TR}, (T, R)
B	P_{BL}, (B, L)	P_{BR}, (B, R)

Introducing correlated equilibrium

Example: avoiding miscoordination in BoS.

	L	R
T	0.3, (T, L)	0.0, (T, R)
B	0.0, (B, L)	0.7, (B, R)

Introducing correlated equilibrium

Strategies: Agent *a* now chooses a function ϕ *a* from *Strategy a* $\rightarrow$ *Strategy a*.

	L	R
T	0.3, (T, L)	0.0, (T, R)
B	0.0, (B, L)	0.7, (B, R)

Introducing correlated equilibrium

Example: The first agent chooses $id : \{T, B\} \to \{T, B\}$, the second chooses $\neg : \{L, R\} \to \{L, R\}$.

	L	R
T	0.3, (T, R)	0.0, (T, R)
B	0.0, (B, L)	0.7, (B, L)

Introducing correlated equilibrium

Payoffs are computed in the expected way.

	L	R
T	0.3, (T, R)	0.0, (T, R)
B	0.0, (B, L)	0.7, (B, L)

Correlated equilibrium

$$
\begin{array}{l}
\textbf{module } \mathit{CorrelatedStrategyContext}' \; (\mathit{Strategy} : \mathit{Agent} \to \mathit{Set}; \\
\quad \mathit{payoff} : (a : \mathit{Agent}) \to ((a : \mathit{Agent}) \to \mathit{Strategy}\ a) \to \mathit{Float}) \\
\quad \textbf{where}
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{Choice} & : & \mathit{Agent} \to \mathit{Set} \\
\mathit{Choice}\ a & = & \mathit{Strategy}\ a \to \mathit{Strategy}\ a
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{Play}' & : & \mathit{Set} \\
\mathit{Play}' & = & (a : \mathit{Agent}) \to \mathit{Strategy}\ a
\end{array}
$$

$$\mathit{open}\ \mathit{GameContext}'\ \mathit{Choice}$$

Correlated equilibrium

module $\mathit{CorrelatedStrategyContext}\ (\mathit{coord} : \mathit{SimpleProb}\ \mathit{Play}')$
where

$\mathit{expected\_payoff} \quad : \ \mathit{Agent} \to \mathit{Play} \to \mathit{Float}$
$\mathit{expected\_payoff}\ a\ p = \mathit{expected}\ (\mathit{fmap}\ (\mathit{payoff}\ \ a)$
$\quad (\mathit{fmap}\ (\mathit{change}\ p)\ \mathit{coord}))$
where
$\quad \mathit{change} : \mathit{Play} \to \mathit{Play}' \to \mathit{Play}'$
$\quad \mathit{change}\ p\ p'\ a = (p\ a)\ (p'\ a)$

$\mathit{prefs} \quad : \ (a : \mathit{Agent}) \to \mathit{TotalPreorder}\ \ \mathit{Play}$
$\mathit{prefs}\ a = \mathit{InducedPreorder}\ \ \mathit{floatPreorder}\ (\mathit{expected\_payoff}\ \ a)$

$\mathit{open}\ \mathit{GameContext}\ \mathit{prefs}\ \mathit{renaming}\ (\mathit{Nash}\ \mathit{to}\ \mathit{CorrelatedEquilibrium})$

Correlated equilibrium example

$P_{TL} = 1 \,/\, 2$, $P_{BR} = 1 \,/\, 2$
Expected payoffs: $3 \,/\, 2$ each.

	L	R
T	(2, 1)	(0, 0)
B	(0, 0)	(1, 2)

Games of exchange

The quintessential economic situation: exchange of goods.

1. Two agents, two goods, X units of the first good, Y units of the second.
2. Agent i has x_i unit of the first good, and y_i units of the second.
3. A distribution of goods to agents, such as $((x_1, y_1), (x_2, y_2))$ is called an *allocation*. Agents have preferences over allocations.
4. Agents are allowed to exchange their goods in order to find a better allocation: no throwing goods away, and no creation of goods: $x_1 + x_2 = X$, $y_1 + y_2 = Y$.

What is a good allocation?

Example: Cobb-Douglas economy

A typical example is the Cobb-Douglas economy, in which the agents preferences induced by the utility functions

$$u_1(x, y) = x^a y^{(1-a)}$$

$$u_2(x, y) = x^b y^{(1-b)}$$

where $0 < a, b < 1$.

Introducing prices

If goods have prices p_x, p_y then an initial allocation gives each agent a *budget*:

$$B_i = p_x x_i + p_y y_i.$$

An agent has to solve:

maximize $u(x, y)$ such that

$$p_x x + p_2 y = B_i$$

Whether the resulting allocation is feasible depends on the prices.

Walrasian equilibrium

An allocation-price pair $(\mathbf{x}, \mathbf{p})$ is a **Walrasian equilibrium** if (1) the allocation is feasible, and (2) each agent is making an optimal choice from its budget set. In equations:

1. $\sum_{i=1}^{n} \mathbf{x}_i = \sum_{i=1}^{n} \boldsymbol{\omega}_i$
2. If $\mathbf{x'}_i$ is preferred by agent i to $\mathbf{x}_i$, then $\mathbf{px'}_i > \mathbf{p}\boldsymbol{\omega}_i$.

Varian, p. 325

Walrasian equilibria

$$
\begin{aligned}
&\mathit{params}\ (\mathit{omega} : \mathit{Vect}\ (\mathit{Vect}\ \mathit{Float}\ \mathit{nG})\ \mathit{nA},\\
&\quad \mathit{prices} : \mathit{Vect}\ \mathit{Float}\ \mathit{nG},\\
&\quad \mathit{prefs} \qquad\quad : \mathit{Fin}\ \mathit{nA} \to \mathit{TotalPreorder}\ (\mathit{Vect}\ \mathit{Float}\ \mathit{nG}))\ \{
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{Feasible} : \mathit{Vect}\ (\mathit{Vect}\ \mathit{Float}\ \mathit{nG})\ \mathit{nA} \to \mathit{Set};\\
&\mathit{Feasible}\ \mathit{xss} = \mathit{SumCols}\ \mathit{xss} === \mathit{SumCols}\ \mathit{omega};
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{Optimal} : \mathit{Vect}\ (\mathit{Vect}\ \mathit{Float}\ \mathit{nG})\ \mathit{nA} \to \mathit{Set};\\
&\mathit{Optimal}\ \mathit{xss} = \mathit{forall}\ (i : \mathit{Fin}\ \mathit{nA}, \mathit{xss}' : \mathit{Vect}\ (\mathit{Vect}\ \mathit{Float}\ \mathit{nG})\ \mathit{nA}) \to\\
&\quad \mathit{gt}\ (\mathit{prefs}\ i)\ (\mathit{vlookup}\ i\ \mathit{xss}')\\
&\qquad (\mathit{vlookup}\ i\ \mathit{xss}) \to\\
&\quad \mathit{gt}\ \mathit{floatOrder}\ (\mathit{prices}\ .\!*\ (\mathit{vlookup}\ i\ \mathit{xss}'))\\
&\qquad (\mathit{prices}\ .\!*\ (\mathit{vlookup}\ i\ \mathit{xss}));
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{WalrasEq} : \mathit{Vect}\ (\mathit{Vect}\ \mathit{Float}\ \mathit{nG})\ \mathit{nA} \to \mathit{Set};\\
&\mathit{WalrasEq}\ \mathit{xss} = \mathit{And}\ (\mathit{Feasible}\ \mathit{xss})\ (\mathit{Optimal}\ \mathit{xss});
\end{aligned}
$$

Computing Walrasian equilibria

How can we compute Walrasian equilibria?

For the special case of the Cobb-Douglas economy, the solution can be computed analytically:

$$\frac{p_y}{p_x} = \frac{(1-a)x_1+(1-b)x_2}{ay_1+by_2}$$

$$x_1^* = \frac{B_1 a}{p_x}$$

$$y_1^* = \frac{B_1(1-a)}{p_y}$$

In general, however, computing Walrasian equilibria involves a lot of numerical methods (optimization, solving linear systems, etc.).

Conclusions

We continue to develop the formalization of economic theory: local Nash equilibria (P. Flondor), general equilibrium, etc.

Work has begun on a DSL for numerical methods (E. Brady).

Ideally, one would like to have numerical methods implemented in terms of constructive reals, used in a constructive economic theory.