Phase Space Topology of Manageable Dynamical Systems with Desirable States

Jobst Heitzig,
Potsdam Institute for Climate Impact Research (PIK)

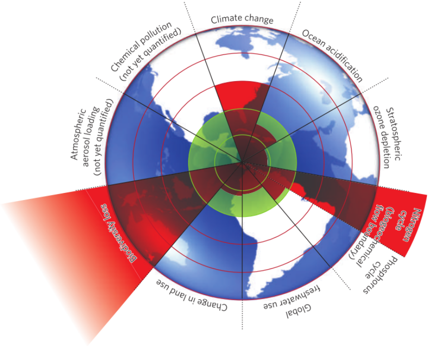
17 November 2016,
Seminar on the Theory of Complex Systems and Neurophysics / Theory of Statistical Physics and Nonlinear Dynamics,
HU Berlin

Potsdam Institute for Climate Impact Research

Table of contents

Motivation

Planetary Boundaries

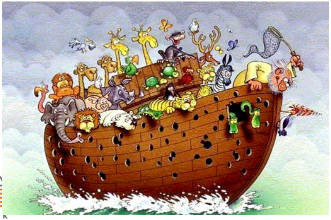


No “control” possible, only “**management**”.

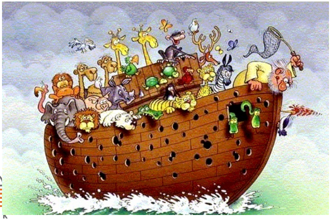
Goal: Not optimization but **sustainability**.

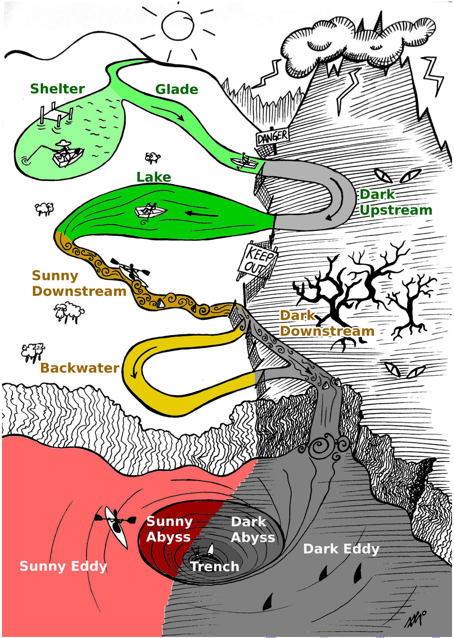
Informal overview

A metaphorical boat ride

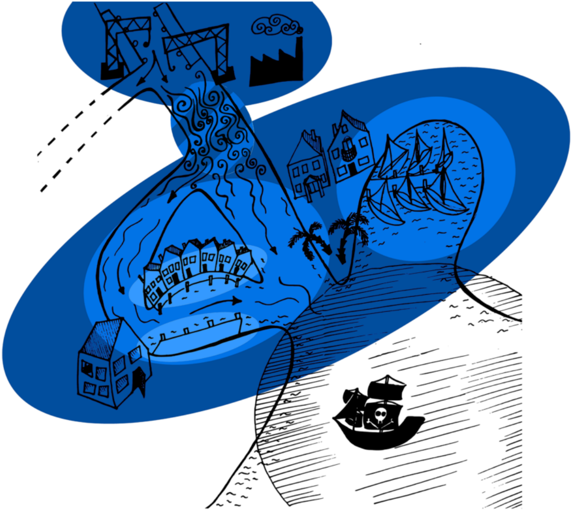


A metaphorical boat ride





A metaphorical boat ride (cont.)



Formal definitions and properties

Def: Manageable Dynamical System w. Desirable States

$\underline{\boldsymbol{X}} = (X, \mathcal{T}, \tau, \mathcal{M}, X^+)$ is a M.D.S.w.D.S (or simply **system**) iff

- $\boldsymbol{X} \neq \emptyset$ is a set (*state space*)
- $\boldsymbol{\mathcal{T}} \subseteq 2^X$ is a Hausdorff topology (set of *open* subsets of X)

Def: Manageable Dynamical System w. Desirable States

$\underline{\boldsymbol{X}} = (X, \mathcal{T}, \tau, \mathcal{M}, X^+)$ is a M.D.S.w.D.S (or simply **system**) iff

- $\boldsymbol{X} \neq \emptyset$ is a set (*state space*)
- $\boldsymbol{\mathcal{T}} \subseteq 2^X$ is a Hausdorff topology (set of *open* subsets of X)
- $\forall\, x \in X; t, t' > 0$:
  - $\boldsymbol{\tau_x} : [0, \infty) \to (X, \mathcal{T})$ continuous (*default forward trajectory* starting at x, e.g. given by a solution to some ODE)
  - $\tau_x(0) = x$, $\tau_{\tau_x(t)}(t') = \tau_x(t + t')$

Def: Manageable Dynamical System w. Desirable States

$\underline{\boldsymbol{X}} = (X, \mathcal{T}, \tau, \mathcal{M}, X^+)$ is a M.D.S.w.D.S (or simply **system**) iff

- $\boldsymbol{X} \neq \emptyset$ is a set (*state space*)
- $\boldsymbol{\mathcal{T}} \subseteq 2^X$ is a Hausdorff topology (set of *open* subsets of X)
- $\forall\, x \in X; t, t' > 0$:
  - $\boldsymbol{\tau_x} : [0, \infty) \to (X, \mathcal{T})$ continuous (*default forward trajectory* starting at x, e.g. given by a solution to some ODE)
  - $\tau_x(0) = x,\ \tau_{\tau_x(t)}(t') = \tau_x(t + t')$
- $\forall\, x \in X$:
  - $\tau_x \in \boldsymbol{\mathcal{M}_x}$ (set of *admissible trajectories* starting at x, e.g. given by a differential inclusion)
  - $\forall\, \boldsymbol{\mu \in \mathcal{M}_x}$: $\mu : [0, \infty) \to (X, \mathcal{T})$ continuous, $\mu(0) = x$
  - if $\mu \in \mathcal{M}_x$, $t > 0$, $x' = \mu(t)$, $\mu' \in \mathcal{M}_{x'}$,
    $\forall\, t'' \in [0, t]$: $\mu''(t'') = \mu(t'')$, and
    $\forall\, t'' > t$: $\mu''(t'') = \mu'(t'' - t)$,
    then $\mu'' \in \mathcal{M}_x$ (closedness under switching at any time)

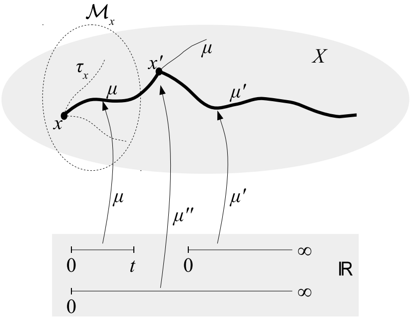
Def: Manageable Dynamical System w. Desirable States

$\underline{X} = (X, \mathcal{T}, \tau, \mathcal{M}, X^+)$ is a M.D.S.w.D.S (or simply **system**) iff

- $X \neq \emptyset$ is a set (*state space*)
- $\mathcal{T} \subseteq 2^X$ is a Hausdorff topology (set of *open* subsets of X)
- $\forall\, x \in X; t, t' > 0$:
  - $\tau_x : [0, \infty) \to (X, \mathcal{T})$ continuous (*default forward trajectory* starting at x, e.g. given by a solution to some ODE)
  - $\tau_x(0) = x$, $\tau_{\tau_x(t)}(t') = \tau_x(t + t')$
- $\forall\, x \in X$:
  - $\tau_x \in \mathcal{M}_x$ (set of *admissible trajectories* starting at x, e.g. given by a differential inclusion)
  - $\forall\, \mu \in \mathcal{M}_x$: $\mu : [0, \infty) \to (X, \mathcal{T})$ continuous, $\mu(0) = x$
  - if $\mu \in \mathcal{M}_x$, $t > 0$, $x' = \mu(t)$, $\mu' \in \mathcal{M}_{x'}$, $\forall\, t'' \in [0, t]$: $\mu''(t'') = \mu(t'')$, and $\forall\, t'' > t$: $\mu''(t'') = \mu'(t'' - t)$, then $\mu'' \in \mathcal{M}_x$ (closedness under switching at any time)
- $X^+ \in \mathcal{T} \setminus \{\emptyset\}$ (*desirable* aka *sunny* region of X)

Switching of admissible trajectories

- $\forall\, x \in X$:
  - $\tau_x \in \mathcal{M}_x$
  - $\forall\, \boldsymbol{\mu} \in \boldsymbol{\mathcal{M}_x}$: $\mu : [0, \infty) \to (X, \mathcal{T})$ continuous, $\mu(0) = x$
  - if $\mu \in \mathcal{M}_x$, $t > 0$, $x' = \mu(t)$, $\mu' \in \mathcal{M}_{x'}$,
    $\forall\, t'' \in [0, t]$: $\mu''(t'') = \mu(t'')$, and
    $\forall\, t'' > t$: $\mu''(t'') = \mu'(t'' - t)$,
    then $\mu'' \in \mathcal{M}_x$



Def: Invariant open kernels, Shelters

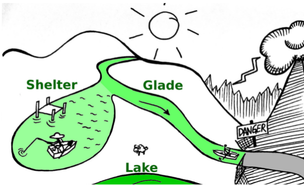
The **invariant open kernel** $A^{\circlearrowleft}$ of $A \subseteq X$ is
the largest open subset K of A
that is forward-invariant under the default flow,
i.e., that has $K \subseteq A$, $K \in \mathcal{T}$ and $\tau_x[[0,\infty)] \subseteq K$ for all $x \in K$.

Def: Invariant open kernels, Shelters

The **invariant open kernel** $A^{\iota\circ}$ of $A \subseteq X$ is
the largest open subset K of A
that is forward-invariant under the default flow,
i.e., that has $K \subseteq A$, $K \in \mathcal{T}$ and $\tau_x[[0,\infty)] \subseteq K$ for all $x \in K$.

Its existence is nontrivial and follows from the fact that
the set of all open and invariant sets is a *kernel system*,
i.e., closed under taking finite and infinite unions.
It may be empty.

Def: Invariant open kernels, Shelters



The **invariant open kernel** A^{io} of $A \subseteq X$ is
the largest open subset K of A
that is forward-invariant under the default flow,
i.e., that has $K \subseteq A$, $K \in \mathcal{T}$ and $\tau_x[[0,\infty)] \subseteq K$ for all $x \in K$.

Its existence is nontrivial and follows from the fact that
the set of all open and invariant sets is a *kernel system*,
i.e., closed under taking finite and infinite unions.
It may be empty.

The system's **shelters** are the set $\boldsymbol{S} := (X^+)^{io}$
= *region where system stays in sun forever by default, even under "infinitesimal" (i.e., positive but sufficiently small) noise.*

Def: Sustainable sets & kernels, Manageable region

$A \in \mathcal{T}$ is *sustainable* iff $\forall\, x \in X\ \exists\, \mu \in \mathcal{M}_x\ \forall\, t \geqslant 0 : \mu(t) \in A$.

Def: Sustainable sets & kernels, Manageable region

$A \in \mathcal{T}$ is *sustainable* iff $\forall\, x \in X\ \exists\, \mu \in \mathcal{M}_x\ \forall\, t \geqslant 0 : \mu(t) \in A$.

The **sustainable kernel** $A^{\mathcal{S}}$ of $A \subseteq X$ is
the largest sustainable open subset of A.

Again, existence follows because
the set of all sustainable sets is a kernel system.

Def: Sustainable sets & kernels, Manageable region

$A \in \mathcal{T}$ is *sustainable* iff $\forall\, x \in X\ \exists\, \mu \in \mathcal{M}_x\ \forall\, t \geqslant 0 : \mu(t) \in A$.

The **sustainable kernel** $A^{\mathcal{S}}$ of $A \subseteq X$ is
the largest sustainable open subset of A.

Again, existence follows because
the set of all sustainable sets is a kernel system.

The system's **manageable region** is $\boldsymbol{M} := (X^+)^{\mathcal{S}}$
= *region where system can be managed to stay in sun forever even under "infinitesimal" noise.*

Def: Sustainable sets & kernels, Manageable region

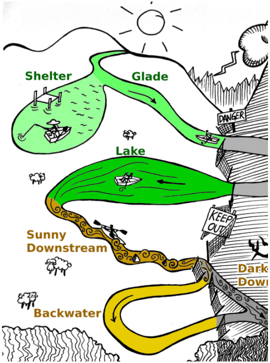
$A \in \mathcal{T}$ is *sustainable* iff $\forall\, x \in X \;\exists\, \mu \in \mathcal{M}_x \;\forall\, t \geqslant 0 : \mu(t) \in A$.

The **sustainable kernel** $A^{\mathcal{S}}$ of $A \subseteq X$ is
the largest sustainable open subset of A.

Again, existence follows because
the set of all sustainable sets is a kernel system.

The system's **manageable region** is $\boldsymbol{M} := (X^+)^{\mathcal{S}}$
= *region where system can be managed to stay in sun forever even under "infinitesimal" noise.*

Every invariant open set is sustainable.
Hence $S \subseteq M$.



Def: Forecourts, Stable reachability

$C \in \mathcal{T}$ is a *forecourt* for $Y \subseteq X$, denoted $C \rightsquigarrow Y$, iff
$\forall\, x \in C\ \exists\, \mu \in \mathcal{M}_x\ \forall\, W \in \mathcal{T}, W \supseteq Y$
$$\exists\, t > 0 : \mu(t) \in W \text{ and } \forall\, t' \in [0, t] : \mu(t') \in C.$$

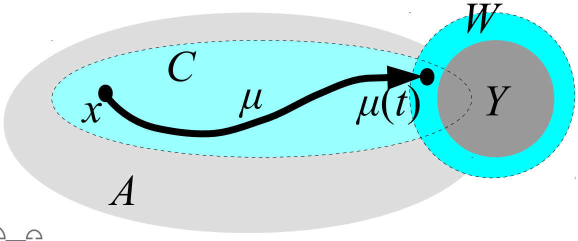
(one can approach Y arbitrarily closely from everywhere in C without leaving C)

Def: Forecourts, Stable reachability

$C \in \mathcal{T}$ is a *forecourt* for $Y \subseteq X$, denoted $C \rightsquigarrow Y$, iff
$\forall\, x \in C\ \exists\, \mu \in \mathcal{M}_x\ \forall\, W \in \mathcal{T}, W \supseteq Y$
$\exists\, t > 0 : \mu(t) \in W$ and $\forall\, t' \in [0, t] : \mu(t') \in C$.

(one can approach Y arbitrarily closely from everywhere in C without leaving C)

$Y \subseteq X$ is **(stably) reachable from** $x \in X$ through $A \subseteq X$, denoted $\boldsymbol{x \rightsquigarrow_A Y}$, iff $\exists$ forecourt $C \subseteq A$ of Y with $x \in C$.



Properties of stable reachability

For all $A, A', C, Y, Z \subseteq X$ and $x, y, z \in X$:

1. If Y is open: (i) $C \rightsquigarrow Y$ iff $\forall x \in C$
   $\exists \mu \in \mathcal{M}_x, t > 0 \; \forall t' \in [0, t] : \mu(t) \in Y$ and $\mu(t') \in C$;
   (ii) $x \rightsquigarrow_A Y$ iff $\exists C \in \mathcal{T}, x \in C \subseteq A \; \forall x' \in C$
   $\exists \mu \in \mathcal{M}_{x'}, t > 0 \; \forall t' \in [0, t] : \mu(t) \in Y$ and $\mu(t') \in C$.
2. Each set of the form $(\rightsquigarrow_A Y) := \{x \in X : x \rightsquigarrow_A Y\}$ is open.
3. **Transitivity:**

$$x \rightsquigarrow_A y \rightsquigarrow_{A'} Z \Longrightarrow x \rightsquigarrow_{A+A'} Z,$$

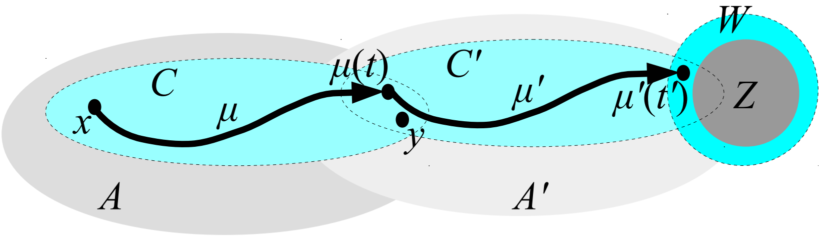
$$x \rightsquigarrow_A y \rightsquigarrow_{A'} z \Longrightarrow x \rightsquigarrow_{A+A'} z.$$

   (But note that not always $x \rightsquigarrow_A x$, e.g. for unstable fixed points)
4. If A is open, it is stably reachable from each of its elements.

Proof sketch for transitivity

- Forecourt: $\forall\, x \in C\; \exists\, \mu \in \mathcal{M}_x\; \forall\, W \in \mathcal{T}, W \supseteq Y$
  $$\exists\, t > 0 : \mu(t) \in W \text{ and } \forall\, t' \in [0,t] : \mu(t') \in C.$$
- Transitivity:

$$x \rightsquigarrow_A y \rightsquigarrow_{A'} Z \Longrightarrow x \rightsquigarrow_{A+A'} Z.$$



Def: Upstream, Downstream, Trenches, Glades, Lakes, Backwaters

- Shelters $S = (X^+)^{\circlearrowleft}$ (will stay in sun by default)
- Manageable region $M = (X^+)^S$ (can stay in sun by management)

- **Upstream** $\boldsymbol{U} := (\rightsquigarrow_X S) \supseteq S$ (can reach shelter)
- **Downstream** $\boldsymbol{D} := (\rightsquigarrow_X M) - (\rightsquigarrow_X S) = (\rightsquigarrow_X M) - U \supseteq M - U$ (can stably reach M but not S)
- **Trenches** $\boldsymbol{\Theta} := X - (\rightsquigarrow_X X^+)$ (cannot reach sun at all)

Def: Upstream, Downstream, Trenches, Glades, Lakes, Backwaters

- Shelters $S = (X^+)^{\iota\circ}$ (will stay in sun by default)
- Manageable region $M = (X^+)^{\mathcal{S}}$ (can stay in sun by management)

- **Upstream** $\boldsymbol{U} := (\rightsquigarrow_X S) \supseteq S$ (can reach shelter)
- **Downstream** $\boldsymbol{D} := (\rightsquigarrow_X M) - (\rightsquigarrow_X S) = (\rightsquigarrow_X M) - U \supseteq M - U$ (can stably reach M but not S)
- **Trenches** $\boldsymbol{\Theta} := X - (\rightsquigarrow_X X^+)$ (cannot reach sun at all)
- **Glades** $\boldsymbol{G} := (\rightsquigarrow_{X^+} S) - S \subseteq U$ (can reach shelter without visiting dark)
- **Lakes** $\boldsymbol{L} := M \cap U - (\rightsquigarrow_{X^+} S) = M \cap U - S - G \subseteq U$ (can avoid dark and can reach shelter, but not both)
- **Backwaters** $\boldsymbol{W} := M \cap D = M - U \subseteq D$ (can avoid dark but cannot reach shelter)

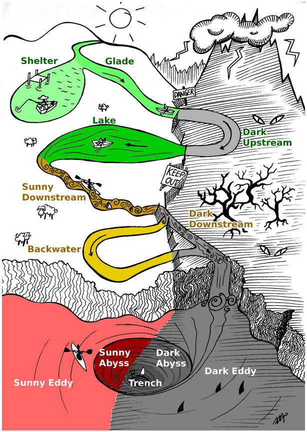
Def: Abysses, Eddies, Main cascade

- Upstream $U := (\rightsquigarrow_X S) \supseteq S$ (can reach shelter)
- Downstream $D := (\rightsquigarrow_X M) - (\rightsquigarrow_X S) = (\rightsquigarrow_X M) - U \supseteq M - U$
  (can stably reach M but not S)
- Trenches $\Theta := X - (\rightsquigarrow_X X^+)$
  (cannot reach sun at all)

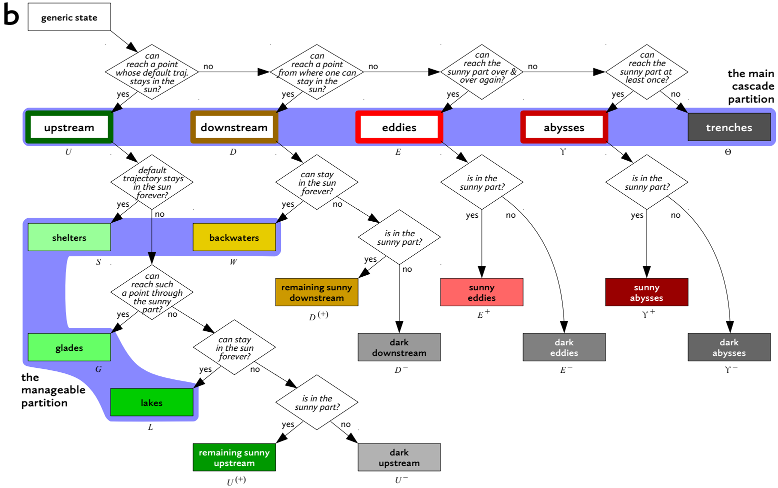
- **Abysses** $\Upsilon :=$
  $\overline{\{x \in X \mid \forall \mu \in \mathcal{M}_x \, \exists t \geqslant 0 : \mu(t) \in \Theta\}} - \Theta$
  (cannot avoid staying in dark eventually)
- **Eddies** $\boldsymbol{E} := X - U - D - \Theta - \Upsilon$

Def: Abysses, Eddies, Main cascade

- Upstream $U := (\rightsquigarrow_X S) \supseteq S$ (can reach shelter)
- Downstream $D := (\rightsquigarrow_X M) - (\rightsquigarrow_X S) = (\rightsquigarrow_X M) - U \supseteq M - U$ (can stably reach M but not S)
- Trenches $\Theta := X - (\rightsquigarrow_X X^+)$ (cannot reach sun at all)
- **Abysses** $\Upsilon :=$ $\overline{\{x \in X \mid \forall \mu \in \mathcal{M}_x \, \exists t \geqslant 0 : \mu(t) \in \Theta\}} - \Theta$ (cannot avoid staying in dark eventually)
- **Eddies** $E := X - U - D - \Theta - \Upsilon$
- **Main cascade** $C := \{U, D, E, \Upsilon, \Theta\}$ (partition)
- $U \not\leftrightsquigarrow D \not\leftrightsquigarrow E \not\leftrightsquigarrow \Upsilon \not\leftrightsquigarrow \Theta$

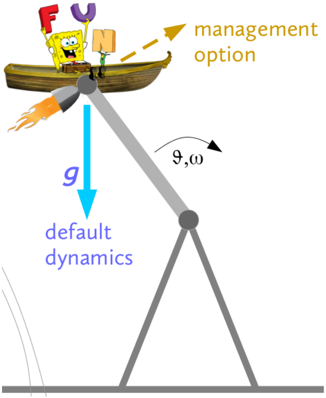


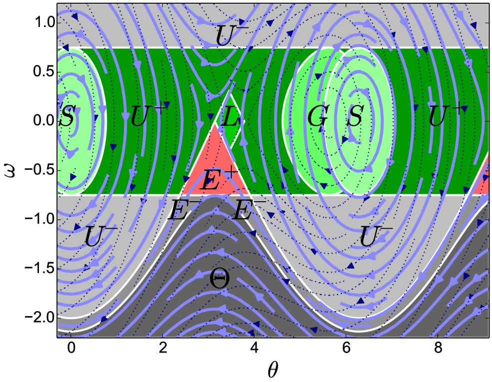
Decision tree representation, Colour scheme



Examples

Example: gravity pendulum fun ride





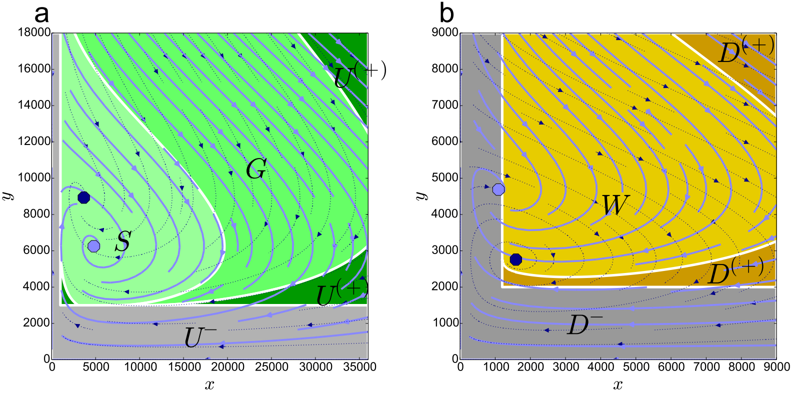
Lake dilemma:

keep thrill forever by repeated bursts, or get to safety risking sickness?

Example: logistic predator-prey model

x = Easter Island population, $\dot{x} = \delta x + \varphi\gamma x y$
y = Easter Island vegetation, $\dot{y} = r y(1 - y/\kappa) - \gamma x y$

Different parameters lead to completely different topologies:



[Brander and Taylor 1998]

Example: Carbon Cycle with Planetary Boundaries

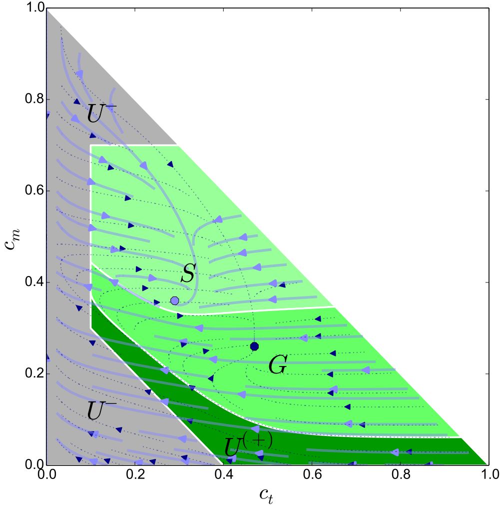
[Anderies et al. 2013]

c_t = terrestrial carbon share (vegetation and soil)

c_m = maritime carbon share (upper oceans)

$c_a = 1 - c_m - c_t$ = atmospheric carbon share

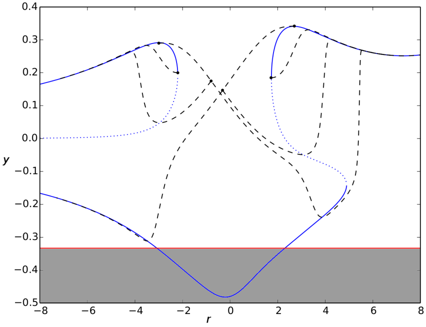
Glade dilemma:
keep high c_t, risking almost extinction when management breaks away?



Example: management by smooth parameter change

$\dot{y} = -(4 + r^2)^3 y^3 + (2r^2 - 1)(4 + r^2)y + e^r - 10,$
default $\dot{r} = 0$, management $\dot{r} \in [-100, 100]$.

Bifurcation diagram with some trajectories for $\dot{r} = \pm 100$:



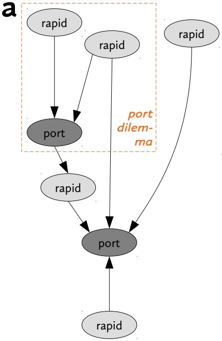
Further definitions and properties

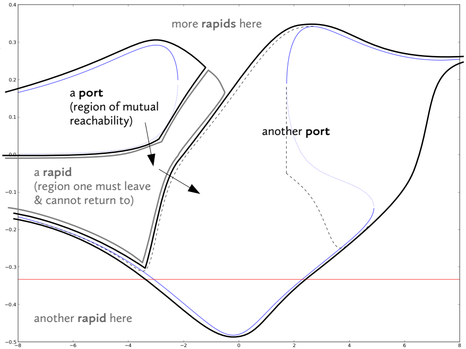
Def: Network of Ports and Rapids

Port = maximal set of mutually reachable states,

Rapid = equivalence class of non-port states w.r.t. which ports they can reach and from which they can be reached.

Network of Ports, rapids, and their reachability:



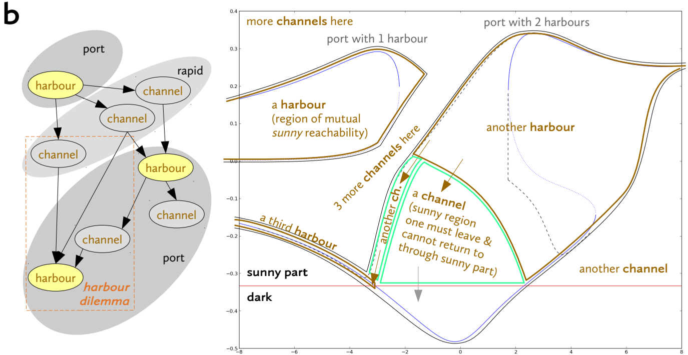


Def: Network of Harbours and Channels

Harbour = maximal set of states mutually reachable **through** X^+,

Channel = equivalence class of non-harbour states w.r.t. **sunny** reachability of harbours.

Network of Ports, rapids, and their reachability:



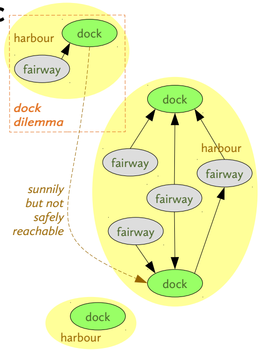
Def: Network of Docks and Fairways

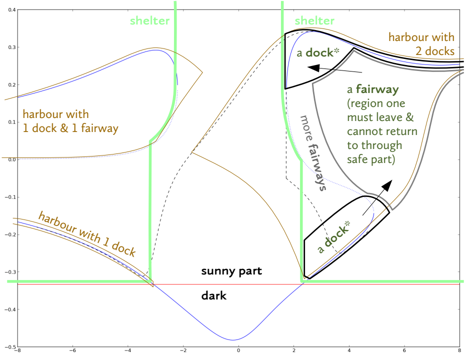
Dock = maximal set of states mutually reachable **through** S,

Fairway = equivalence class of non-harbour states
w.r.t. **safe** reachability of docks.

Network of Docks, fairways, and their reachability:

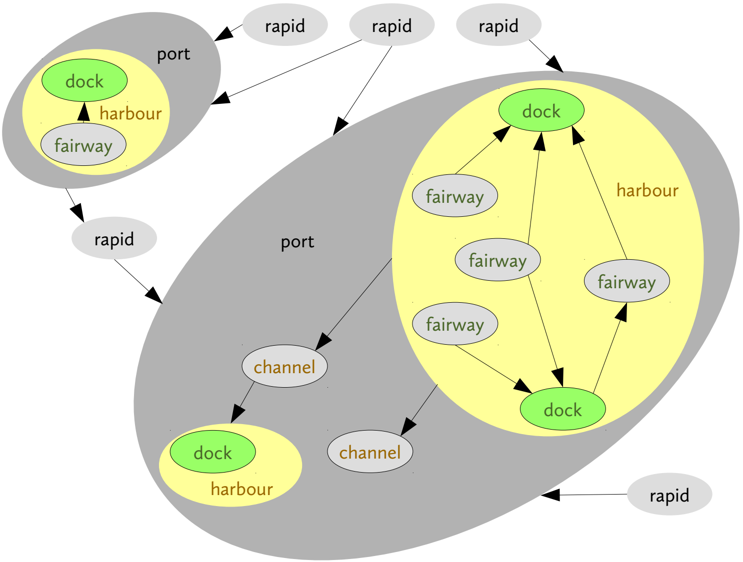
C





*(region of mutual *safe* reachability)

Three-layer Reachability Network of Networks



Properties

Let $X^- = X - X^+$, $\Upsilon^\pm = \Upsilon \cap X^\pm$, $U^\pm = U \cap X^\pm$, etc.

Proposition:

1. Each two ports [harbours, docks] are disjoint.
2. Each port lies completely in one of $U, D, E, \Upsilon^-, \Theta$, no port intersects Υ^+.
3. Each harbour [dock] lies completely in one port [harbour].
4. Each channel [fairway] lies completely in one port or rapid [one harbour or channel].
5. If a harbour H intersects some of the regions $S + G$, L, U^+, W, or D^+, it is already completely contained in that region.

Guess which part is nontrivial!
(My proof even requires the Axiom of Choice. . .)

To show that a port $P \subseteq \Upsilon$ is already in Υ^-, assume $x \in P \cap \Upsilon^+ \subseteq X^+ \in \mathcal{T}$. We will now construct a contradiction by constructing an admissible trajectory from x that avoids Θ forever. Since $x \rightsquigarrow_X x$ and X^+ is open, there is an open set $A \subseteq X^+$ with $y \rightsquigarrow_X x$ for all $y \in A$. Since τ_x is continuous and A open, we find $t_0 > 0$ with $\tau_x(t) \in A$ for all $t \in [0, t_0]$. Let $y = \tau_x(t_0)$ and pick a $\mu \in \mathcal{M}_y$ that returns arbitrarily closely to x. Let $\mathcal{A}$ be the set of all open $A \subseteq X^+$ with $x \in A$, and choose a $t_A > 0$ with $\mu(t_A) \in A$ for all $A \in \mathcal{A}$ (this requires the Axiom of Choice which we will assume here). Let $t_1 = \inf_{A \in \mathcal{A}} \sup_{B \in \mathcal{A}, B \subseteq A} t_B \geqslant 0$. Since $y \in \Upsilon + \Theta$, there is $t' > 0$ with $\mu(t'') \in \Theta$ for all $t'' > t'$, hence $t_A \leqslant t'$ for all $A \in \mathcal{A}$ and thus $t_1 \leqslant t'$. Next we show that $\mu(t_1) = x$. If $\mu(t_1) = y \neq x$, one can choose $A \in \mathcal{A}$ and $C \in \mathcal{T}$ with $y \in C$ and $A \cap C = \emptyset$ (this is the only point where we need the Hausdorff property). Since μ is continuous, there are $t_l < t_1$ and $t_u > t_1$ with $\mu(t') \in C$ for all $t' \in [t_l, t_u]$. By definition of t_1, there is $A' \in \mathcal{A}$ with $\sup_{B \in \mathcal{A}, B \subseteq A'} t_B \in [t_1, t_u]$. Putting $A'' = A \cap A' \in \mathcal{A}$, we then also have $\sup_{B \in \mathcal{A}, B \subseteq A''} t_B \in [t_1, t_u]$, hence there is $B \in \mathcal{A}$ with $B \subseteq A'' \subseteq A$ and $t_B \geqslant t_l$ and hence $\mu(t_B) \in C$ by choice of t_l. But $\mu(t_B) \in B \subseteq A$ by choice of t_B. Hence $\mu(t_B) \in A \cap C = \emptyset$, a contradiction. So $\mu(t_1) = x$ after all. Finally we concatenate $\tau_x[0, t_0]$ and $\mu[0, t_1]$ infinitely many times and get an admissible trajectory from x that avoids Θ forever.

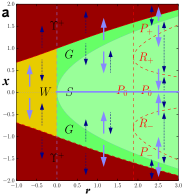
Summary of Dilemmas

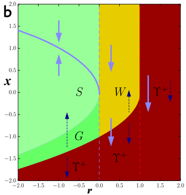
Name	Option 1	Option 2	Possible example
“Glade” dilemma	higher desirability/flexibility	safety	adaptation/mitigation
“Lake” dilemma	uninterrupted desirability	eventual safety	great transformation
“Port” dilemma	higher flexibility	higher desirability	land-use change
“Harbour” dilemma	uninterrupted desirability	eventually higher desirability/flexibility	space colonization
“Dock” dilemma	uninterrupted safety	eventually higher desirability/flexibility	new technologies

“Topological” bifurcations

x = variable, r = parameter, management can change $\dot{x}$ by ± 1.

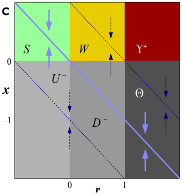
backwater/glade bifurcation and later port pitchfork bifurcation caused by a subcritical pitchfork bifurcation of the default flow (similar in the supercritical case)

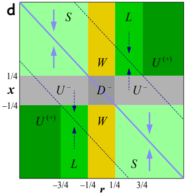




glade/backwater/abyss transition caused by a saddle-node bifurcation, with the second critical value marked in red

shelter/backwater/abyss transition caused by the transition of a stable fixed point into the deep dark





shelter/backwater/lake/upstream transition caused by the transition of a stable fixed point through a dark strip

Thanks!

Earth Syst. Dynam., 7, 1–30, 2016
www.earth-syst-dynam.net/7/1/2016/
doi:10.5194/esd-7-1-2016

Earth System Dynamics
Open Access
EGU

Topology of sustainable management of dynamical systems with desirable states: from defining planetary boundaries to safe operating spaces in the Earth system

J. Heitzig[1], T. Kittel[1,2], J. F. Donges[1,3], and N. Molkenthin[4]

[1]Research Domains Transdisciplinary Concepts & Methods and Earth System Analysis, Potsdam Institute for Climate Impact Research, P.O. Box 60 12 13, 14412 Potsdam, Germany
[2]Department of Physics, Humboldt University, Newtonstr. 15, 12489 Berlin, Germany
[3]Stockholm Resilience Centre, Stockholm University, Kräftriket 2B, 114 19 Stockholm, Sweden
[4]Department for Nonlinear Dynamics & and Network Dynamics Group, Max Planck Institute for Dynamics and Self-Organization, Bunsenstraße 10, 37073 Göttingen, Germany

Correspondence to: J. Heitzig (heitzig@pik-potsdam.de)

Received: 13 February 2015 – Published in Earth Syst. Dynam. Discuss.: 12 March 2015
Revised: 9 November 2015 – Accepted: 26 November 2015 – Published:

Questions? Discussion!