A von Liebig Model for Water and Nitrogen Crop Response

Sadi S. Grimm, Quirino Paris, and William A. Williams

The century-old "law of the minimum" proposed by von Liebig was tested using five independent sets of crop response data on wheat, corn, cotton, silage, and sugar beets. The rival models were polynomial functions reported in the literature as the most suitable models for interpreting those data. Overall, the von Liebig model performed very well. While the nonnested hypothesis test was inconclusive with regard to silage and sugar beets, the von Liebig model rejected the polynomial specifications for wheat, corn and cotton.

Key words: crop response, nonnested hypothesis, nonsubstitution, von Liebig model.

Attempts to develop production functions for irrigated crops have been numerous and can be classified in two main categories depending upon whether or not the timing of input application is explicitly considered in the model formulation. In most empirical studies water production functions have been estimated on the basis of data obtained from field experiments using small plots or lysimeters and follow some specific criterion to determine timing of irrigation applications. Hence, the extrapolation of the results is conditioned on following the same criterion: whenever the soil moisture tension rises to a certain level, sufficient water must be applied to restore soil moisture to field capacity in the entire root zone.

In contrast, the scope of this paper is limited to water production functions where timing of irrigation is not explicitly considered in the model.[1] From an economic viewpoint, the significant explanatory variable is applied water because it is the resource over which farmers exercise direct control and whose unit cost can easily be assessed. Because it does not represent the amount of water actually used by the plant, researchers have adopted some variations of its measure such as the amount obtained by adding up the water applied through irrigation, the rainfall, and the difference between the soil water content at planting time (usually at field capacity) and harvest time.

Cobb-Douglas, Mitscherlich and polynomial functions of varying degree (quadratic, three-halves, and square root) have been most often used to specify water production functions. Invariably, the polynomial forms have been selected as the most adequate. Hexem, Sposito, and Heady, for example, recognized that the Mitscherlich specification is relatively complex to estimate when two or more explanatory variables are included, and the polynomial specifications fit as well or better according to the results of experiments with corn conducted in Colorado and Kansas. Koster and Whittlesey rejected the Cobb-Douglas specification for describing wheat response to irrigation water and nitrogen because it is unable to represent negative marginal productivity and a maximum yield is not defined. Polynomial forms are appealing because they are easy to manipulate allowing specification of the joint effect of water and other inputs as well as for negative marginal productivity. Examples of studies using polynomial forms are those by

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[1] This cumulation of water applications is not analytically different from the practice of aggregating all service flows of labor, from planting to harvest, into a single, timeless variable.

Koster and Whittlesey, and Hexem and Heady in the United States; Eckert, Chaudhry, and Qureshi in Pakistan; Yaron in Israel; Stutler et al., in El Salvador; Iruthayaraj and Morachan in India.

In 1978, Hexem and Heady published *Water Production Functions for Irrigated Agriculture* to introduce agronomists to the economic principles of water allocation and production function estimation. In a literature review about crop-water production functions, Vaux and Pruitt write that "The work of Hexem and Heady represents the single most important contribution of empirical studies" (p. 81). Heady and Hexem illustrated their proposed methodology using experimental data on various combinations of irrigation water and nitrogen fertilizer involving different sites, year, and crops. The functional forms of the response were restricted to quadratic, three-halves, and square root polynomial functions. Hexem and Heady interpreted the fitted production functions as if substitution between water and nitrogen were possible. In contrast, the hypothesis of this paper attempts to model the agronomic principle according to which major nutrients (including water and nitrogen) are essential and follow von Liebig's law of the minimum. Nutrients are, thus, complements and not substitutes.

The objective of this paper is to put to a rigorous statistical test the century-old idea that crop response can be modeled following the limiting nutrient principle for all macronutrients, including water. For this purpose, nonnested hypothesis tests will be applied using as rival hypotheses the most relevant specifications formulated to date, namely, the polynomial response models. Following a brief description of the sample information used in this study, the specification of the von Liebig model and its unfamiliar estimation requirements are discussed. Nonnested hypotheses testing and the construction of two relevant statistics are then presented in some detail. The discussion of the empirical results and suggestions for further work conclude the paper.

Data Source

The data set for this research consists of field experiment results used by Hexem and Heady who presented the sample information in microfiche form as an appendix to their book. These experiments, conducted in several western states during the period 1969–72 and spanning a variety of soil and climate conditions, were designed to estimate water and nitrogen response functions for corn, corn silage, wheat, cotton lint, and sugar beets. One experiment for each crop was selected for this research. The corn experiment (1971) involved the Prairie Valley 40-S hybrid on Keith silt loam soil at the Colby Branch Experiment Station in Kansas. The corn silage experiment (1970) involved the Funks G-711-AA variety on Laveen clay loam soil at Mesa, Arizona. The winter wheat experiment (1971–72) was carried out on Glendale silty clay loam soil at Yuma Valley, Arizona. The cotton experiment (1969) involved Alcala SJ-1 and was conducted on Panoche clay loam soil at the West Side Field Station, in Fresno County, California. The sugar beets experiment (1970–71) involved the monogerm variety S301-H8 on Laveen clay loam soil at the Mesa Branch Experiment Station in Arizona. A detailed description of each experiment is found in Hexem and Heady.

The design of the various experiments follows the incomplete factorial specification which is particularly suitable for estimating polynomial responses. In the analysis carried out in this paper, therefore, the initial advantage is given to the rival polynomial models. A good performance of the von Liebig hypothesis under these conditions would indicate the robustness of the model under different scenarios.

Model Specification

In formulating the "law of the minimum" von Liebig assumed a linear crop response to the limiting nutrient until a maximum plateau is reached and another factor becomes limiting. This proportionality concept was criticized by Mitscherlich and many soil scientists who assumed a response with diminishing marginal productivity. Boyd, however, studied several fertilization experiments with sugar beets, wheat, barley, and potatoes and concluded that in most instances crop responses to nitrogen, phosphorus, and potassium can be characterized by a linear-plateau model. Waugh *et al.*; Anderson and Nelson; Waggoner and Norvell; and Ackello-Ogutu, Paris, and Williams arrived at similar conclusions. Hence, in this research, it is assumed that crop response to nitrogen and irrigation water follows von Liebig's principle of a linear response to the limiting nutrient, with a sharp transition to a plateau maximum. This assumption will be tested against the polynomial specifications selected by Hexem and Heady by means of nonnested hypotheses procedures.

A von Liebig (two-factor) crop response model can be expressed as

$$(1) \qquad Y = \min[f_W(W, \theta_W), f_N(N, \theta_N)] + u,$$

where Y is the observed crop yield, W is applied irrigation water, N is applied nitrogen, θ_W and θ_N are vectors of parameters to be estimated, and u is a Gaussian disturbance. The response functions f_W and f_N can assume any functional form compatible with production technology and theory. A salient feature of the von Liebig model is the absence of factor substitution.

Von Liebig's original specification of the response functions assumes that f_W and f_N are linear in water and nitrogen, respectively. The crop yield is regulated by a plateau maximum, m, which depends on various growth factors such as genetic load, other nutrients, etc. Under this formulation, (1) specializes to the following model

$$(2) \qquad Y = \min\{\alpha_0 + \alpha_1 W, \beta_0 + \beta_1 N, m\} + u,$$

where α_i, β_i, $i = 0$, 1 and m are the unknown parameters. The combination of water and nitrogen which maximizes crop yield is given by

$$(3) \qquad \alpha_0 + \alpha_1 W_K = \beta_0 + \beta_1 N_K = m,$$

where W_K and N_K are the optimal quantities of water and nitrogen, usually referred to in the literature as knots.

The polynomial response functions considered in this paper are the quadratic, the three-halves and the square root specifications written as

$$(4) \qquad Y = \alpha_0 + \alpha_1 W + \alpha_2 N + \alpha_{11} W^2 + \alpha_{22} N^2 + \alpha_{12} WN + u,$$

$$(5) \qquad Y = \alpha_0 + \alpha_1 W + \alpha_2 N + \alpha_{11} W^{1.5} + \alpha_{22} N^{1.5} + \alpha_{12} WN + u,$$

$$(6) \qquad Y = \alpha_0 + \alpha_1 W + \alpha_2 N + \alpha_{11} W^{0.5} + \alpha_{22} N^{0.5} + \alpha_{12}(WN)^{0.5} + u,$$

where the symbols are as defined above. These specific polynomial forms are those chosen by Hexem and Heady to represent response to water and nitrogen for the selected crops. In this study, the experimental design's advantage (incomplete factorial design) is entirely in favor of the polynomial specifications. Hence, a good performance of the von Liebig model under these circumstances would indicate a strong reliability for it.

Estimation

The estimation of the von Liebig model as expressed in (2) was carried out within a maximum likelihood framework. Recent extensions of asymptotic theory have allowed the derivation of maximum likelihood (asymptotically efficient) estimators for models that deviate from traditional specifications (Bates and White). One important aspect concerning the von Liebig model (2) is that the corresponding likelihood function is not differentiable at the knots W_K and N_K, where the transition from a response to a plateau occurs. In other words, the likelihood function does not possess first and second derivatives with respect to the parameters at one specific point or, technically, it is not differentiable on a set of measure zero.

Bates and White have developed the theory of maximum likelihood estimators which are also asympotically normal and efficient for the case where the likelihood function is "almost surely" (a.s.), differentiable. Hence, the a.s. assumption includes the von Liebig model. That the lack of differentiability at one point is not crucial can be illustrated in at least two ways. First of all, the probability that farmers will select the combination of water and nitrogen which exactly corresponds to the knots is zero. Second, the von Liebig model must be considered an approximation to the appropriate response function. Hence, the following approximation is also admissible: it is always possible to select a small interval around the knots and to join its end points with a cubic spline guaranteeing the existence of first and second derivatives of the likelihood function everywhere.

Maximum likelihood estimates of the von Liebig models' parameters and their corresponding asymptotic variances were obtained by following the assumptions of Bates and White, as discussed above. The final reparameterization of model (2) adopted in the estimation procedure exploited the conditions specified in (3) to produce

$$(7) \qquad Y_i = \min[m + \alpha_1(W_i - W_K)D_1; m + \beta_1(N_i - N_K)D_2] + u_i,$$

where

$D_1 = 1$ if $0 \leq W_i < W_K$
$= 0$ if $W_K \leq W_i$
$D_2 = 1$ if $0 \leq N_i < N_K$
$= 0$ if $N_K \leq N_i$.

In the specification of (7), the plateau, m, assumes the role of a common intercept for the water and nitrogen regimes. The $\alpha_1(W_i - W_K)D_1$ and $\beta_1(N_i - N_K)D_2$ terms always represent nonpositive quantities which reduce the response level measured from the plateau (intercept), m. The individual intercept proper of the water and nitrogen regimes can easily be recovered by means of the identities $\alpha_0 = m - \alpha_1 W_K$ and $\beta_0 = m - \beta_1 N_K$, implied by (3).

The linear and plateau model as formulated in (7) was estimated by solving the following nonlinear programming model:

$$\text{(8)} \qquad \text{Minimize} \sum_{i=1}^{T} u_i^2 + Pen\left(\sum_{i=1}^{T} Sw_i Sn_i\right),$$

subject to

$$Y_i = m + \alpha_1(W_i - W_K)D_1 - Sw_i + u_i$$
$$Y_i = m + \beta_1(N_i - N_K)D_2 - Sn_i + u_i$$
$$Sw_i \geq 0,\ Sn_i \geq 0,\ m \geq 0,$$
$$\alpha_1,\ \beta_1, \text{ and } u_i \text{ free variables.}$$

Notice that for each observation the slack variable (Sw_i or Sn_i) for the limiting factor should be equal to zero, while the slack variable for the nonlimiting factor will assume a nonnegative value. This condition is achieved by introducing a sufficiently high penalty (*Pen*) associated with this sum of products of slack variables.

The problem represented in (8) was solved using a nonlinear programming algorithm developed by Murtagh and Saunders (MINOS/Augmented, Version 4, simply, MINOSV4). Asymptotic standard errors of the parameter estimates in model (8) (including the knots) were computed from the inverse of the negative expectation of the information matrix.

Hypothesis Testing

Researchers are constantly faced with the problem of choosing among models. By far, the most popular procedure has been to select the model which minimizes the mean square error (*MSE*) or maximizes the multiple determination coefficient (R^2). It is known that the use of the residual variance as a choice criterion gives rise "on the average" to the correct choice, provided that one of the alternative models considered is the "true" model. The requirement that the "true" model is known is rather stringent and often unrealistic. Furthermore, the choice of a functional form for approximating the "true" model, performed on the basis of its relative goodness to fit, cannot avoid an exercise in subjectivity. In spite of such shortcomings, this and other informal decision rules have often been used for discriminating among models when the objective was to obtain the "best" mathematical specification of a given relationship. These decision rules do not imply an hypothesis test, where the disregarded models are declared "false," but they represent only a subjective judgment as to the "best" approximation of some "true" model for the specific sample under investigation.

In this research, however, the interest is in hypothesis testing rather than discrimination because the objective is to determine which model is correct rather than selecting the model that better fits the sample data. Under the classical framework, the null hypothesis (H_O) is tested against an alternative hypothesis (H_1) and H_0 is either rejected or not rejected at a predetermined probability level of a type I error. Because the decision rule is restricted to only two possibilities (the truth of one hypothesis means the falsity of the other), it implies that one of the models is the true specification. This approach seems inappropriate when the true form of the relationship being tested is unknown and it might be the case that none of the specifications tested corresponds to the true model.

The objective of this research, therefore, is to contrast the von Liebig model (2) for each of the five experiments against the polynomial form selected for each by Hexem and Heady. In order to achieve this goal, methods for testing nonnested hypotheses must be applied. In the context of regression analysis, two hypotheses are said to be nonnested when the corresponding models belong to separate parametric families and one model cannot be obtained from the other as a limiting process.

Let the hypotheses being tested be represented by

$$\text{(9)} \qquad H_0\colon f(X, \alpha) + u_0 = X\alpha + u_0,$$
$$\text{(10)} \qquad H_1\colon g(Z, \beta) + u_1 = Z\beta + u_1,$$

where f and g are crop response functions represented by two nonnested models, X and Z

Table 1. Results for the Polynomial Response Functions

Coefficient	Corn Quadratic	Silage Three-Halves	Wheat Quadratic	Cotton Square-Root	Sugar Beets Quadratic
α_0	−1,337.7	−54,709.0	−10,530.0	−1,751.6	6.0151
	(1,098.4)[a]	(34,177.0)	(5,163.3)	(309.7)	(8.9437)
α_1	430.47	5,291.9	850.41	−80.261	.8004
	(147.97)	(2,213.6)	(394.25)	(12.345)	(.4733)
α_2	40.025	156.07	11.255	−1.458	.0695
	(3.551)	(48.45)	(6.462)	(.448)	(.0256)
α_{11}	−10.868	−510.84	−12.944	912.29	−.0109
	(4.597)	(214.81)	(7.499)	(125.92)	(.0066)
α_{22}	−.0834	−9.4479	−.0322	16.463	−.00019
	(.0085)	(2.0201)	(.0130)	(11.966)	(.00004)
α_{12}	.3737	1.3940	.1062	4.712	.00095
	(.1711)	(.6555)	(.2188)	(1.656)	(.00065)
R^2	.935	.758	.761	.934	.616
Observations	44	44	66	26	44

[a] Numbers in parentheses are standard errors.

are the matrices of explanatory variables in a linear specification of f and g; and α and β are the parameter vectors of the two models. The hypothesis specified in (9) and (10) will be tested by means of two statistics known in the literature under the names of the *CP* test, and the *W* test.

The original approach for testing nonnested hypotheses is due to Cox, who derived the asymptotic distribution of a test statistic based on the Neyman-Pearson likelihood ratio. Later, Cox's procedure was elaborated by Pesaran for linear regression models and by Pesaran and Deaton for nonlinear regression models. The Cox-Pesaran (*CP*) approach encompasses the possibility of rejecting both hypotheses under consideration. Each alternative is taken as the null hypothesis in succession and, therefore, each model is on an equal footing.

In a Monte-Carlo analysis, Pesaran showed that when the sample size is as small as 20, the *CP* test tends to reject H_0 far more frequently than it should and that this overrejection of the null hypothesis becomes increasingly more serious as the number of variables increases relative to the sample size. To correct this unfavorable small sample feature of the *CP* test, Godfrey and Pesaran derived the *W* test, which is an adjusted Cox-type statistic in closer agreement with small sample and asymptotic significant levels. For the definition and construction of the *CP* and *W* tests the reader is referred to Godfrey and Pesaran.

The statistics CP_0 and W_0 are asymptotically distributed as a standardized normal variate when H_0 is true, and are only valid for testing the truth of H_0. The procedure to test the truth of H_1 is to reverse the roles of H_0 and H_1 and carry out the tests again. The new statistics are denoted CP_1, and W_1, indicating that now the previous alternative hypothesis is assumed to be the null model. For a given level of significance, say $\alpha = .05$, these tests can lead to four possible outcomes:

(*a*) Accept H_0 and reject H_1 whenever

$$|T_0| < 1.96 \text{ and } |T_1| \geq 1.96$$

(*b*) Reject H_0 and accept H_1 whenever

$$|T_0| \geq 1.96 \text{ and } |T_1| < 1.96$$

(*c*) Reject both H_0 and H_1 whenever

$$|T_0| \geq 1.96 \text{ and } |T_1| \geq 1.96$$

(*d*) Accept both H_0 and H_1 whenever

$$|T_0| < 1.96 \text{ and } |T_1| < 1.96,$$

where T_0 and T_1 stand for either CP_0, W_0 and CP_1, W_1, respectively.

Results and Discussion

The estimated polynomial forms representing yield-water-nitrogen relationships for corn, corn silage, wheat, cotton, and sugar beets are presented in table 1. They correspond, respectively, to equations (6.1), (6.31), (7.1), (8.6), and (9.8) in *Water Production Functions for Irrigated Agriculture* by Hexem and Heady. The parameter estimates and respective standard errors, as well as the R^2s, presented in

Table 2. Results for the von Liebig Model

Coefficient	Corn	Silage	Wheat	Cotton	Sugar Beets
m	9,046.55 (181.38)[a]	47,477.30 (1,823.66)	5,140.78 (95.65)	1,146.64 (16.93)	35.46 (1.14)
α_1	453.38 (83.27)	445.10 (167.47)	274.09 (37.30)	56.29 (3.95)	.32 (.13)
β_1	50.66 (4.69)	174.14 (37.23)	15.54 (4.38)	3.66 (.53)	.07 (.02)
W_K	15.00 (.92)	54.00 (6.25)	29.00 (.88)	23.00 (.73)	45.00 (5.65)
N_K	140.00 (12.31)	120.00 (24.94)	140.00 (22.32)	105.00 (13.34)	197.00 (48.42)
α_0	2,245.79 (929.02)	23,441.74 (7,482.10)	−2,807.71 (881.06)	−148.03 (59.84)	20.99 (5.01)
β_0	1,954.39 (204.40)	26,580.62 (1,926.55)	2,964.90 (340.90)	762.68 (28.29)	21.11 (2.29)
R^2	.949	.713	.763	.962	.615
Observations	44	44	66	26	44

[a] Numbers in parentheses are asymptotic standard errors.

table 1 closely correspond to those reported by the authors.

The estimated regression coefficients for the quadratic function describing corn response to irrigation and nitrogen fertilization are all significant at the 5% level, except for the intercept. Similar results are observed in the corn silage experiment, where a three-halves polynomial was fitted. In the case of wheat, however, only the intercept and the regression coefficients for W and N^2 of the quadratic function are significant at the 5% level. The square root function fitted for the cotton experiment reveals all estimated coefficients significant at the 5% level, except the coefficient associated with $N^{.5}$. In the case of sugar beets, only the regression coefficients associated with N and N^2 in the quadratic production function are significant at the 5% level. First-order interaction terms are significant at a 95% confidence level for corn and silage and at a 99% level for cotton.[2]

The estimated von Liebig production functions for corn, corn silage, wheat, cotton, and sugar beets are presented in table 2. Recall that W_K and N_K are the knots linking the ascending linear response to the plateau; that is, they represent the level of irrigated water and applied nitrogen at which the maximum yield (m) is reached when both production factors are not limiting output.

The estimates of α_0 and β_0 are derived from those of the five primary parameters α_1, β_1, W_K, N_K, and m, according to the relationships specified in a previous section. Their standard errors are computed using the familiar formula by Bohrnstedt and Goldberger. For example, the variance of $\hat{\alpha}_0$ was computed as

$$V(\hat{\alpha}_0) = V(\hat{m}) + V(\hat{\alpha}_1)\hat{W}_K^2 + V(\hat{W}_K)\hat{\alpha}_1^2 + \text{cov}(\hat{\alpha}_1, \hat{W}_K)^2 - 2\,\text{cov}(\hat{m}, \hat{W}_K)\hat{\alpha}_1 - 2\,\text{cov}(\hat{m}, \hat{\alpha}_1)\hat{W}_K.$$

The linear-plateau functions estimated for the five experiments present all regression coefficients significant at the 1% level, indicating a clear response for all the five crops to irrigation water and applied nitrogen.

In contrast to the polynomial forms, the linear-plateau model possesses an intercept for each production factor. The intercept for water (α_0) represents the expected crop yield in absence of irrigation and rainfall (except in the case of the corn experiment where rainfall is not included in the W variable), when water availability in the soil is the most limiting production factor. The intercept for nitrogen (β_0) indicates the expected yield for the different crops when nitrogen is the most limiting factor and none is added to the soil, given that water is fixed at the lowest treatment level.

Notice that the R^2s are rather similar for each pair of corresponding specifications in tables 1 and 2, making the choice of either model

[2] Alternative specifications with second-order interaction terms were estimated. In all cases this type of interaction was barely "accepted" at a 5% level of significance and rejected at a 1% level.

Table 3. Water and Nitrogen Levels for Maximum Yield

Crop	Form	Polynomial Model Water (acre-inches)	Polynomial Model Nitrogen (lbs./acre)	Polynomial Model Yield	von Liebig Model Water (acre-inches)	von Liebig Model Nitrogen (lbs./acre)	von Liebig Model Yield
Corn	Quadratic	24.9	296	9,936	15	140	9,047
Silage	Three-halves	54.7	269	48,905	54	120	47,477
Wheat	Quadratic	33.8	231	5,139	29	140	5,141
Cotton	Square root	37.7	242	1,177	23	105	1,147
Sugar beets[a]	Quadratic	50.2	308	36.80	45	200	35.51

[a] Tons per acre (root yield adjusted to 15% sucrose content).

rather difficult without a sharper criterion. It is, however, important to point out that these similar levels of fit are obtained by the two models with a different number of parameters: the two-input polynomial models have six parameters (an intercept and five slope coefficients) while the two-input von Liebig model has five parameters (a common plateau, two slopes, and two knots). Therefore, the von Liebig model is parsimonious as well as more agronomically meaningful.

The objective of fitting crop production functions is not only to describe crop response to inputs but also to estimate the optimum input levels, based on some optimization criterion. The levels of water and nitrogen necessary to maximize the yield of the five different crops according to the polynomial and von Liebig models are presented in table 3. There is a sharp difference between the two sets of results, especially with respect to nitrogen where a double amount would be necessary for maximizing yields if the polynomial models were used. The differences are relatively smaller for water, but the optimal levels are consistently higher for the polynomial model. As reported by Boyd; Anderson and Nelson; Waugh et al.; Sanchez and Salinas; and Ackello-Ogutu, Paris, and Williams, these results confirm the tendency of the polynomial model to overestimate the optimal input levels.

The levels of water and nitrogen necessary to maximize profits according to the polynomial forms are presented in table 4 with nitrogen priced at $0.33 per pound and water at $1.98 per acre-inch. For the given output and input prices, the input levels for profit maximization of the von Liebig response functions correspond to the knots for each production factor for all the five crops studied (table 3). The comparison between the two models points to the fact that the adoption of polynomial crop responses would lead to higher input utilization in all cases, except for water in corn silage. These differences are especially large in the case of water for corn and cotton, and in the case of nitrogen for corn, corn silage, and sugar beets.

A graphical representation of the sample data, the von Liebig and the polynomial models (based on tables 1 and 2) for the five crops is given in figure 1. The number of scatter points does not correspond to the reported number of observations because water and nitrogen treatments were replicated. It is interesting to observe that some scatter diagrams (corn and cotton, for example) exhibit a clear plateau discernible also by inspection. For the other crops the detection of such a plateau by inspection is more difficult and one has to rely on a more objective procedure. The important fact to underscore is that the measure of fit, R^2, for both the polynomial and von Liebig models is very similar for all crops under consideration. This fact confirms the inappropriateness of the R^2 statistic as a criterion for selecting functional forms describing crop response. For this goal, a more formal criterion such as the nonnested hypotheses tests described above is required. The results of such tests are presented in table 5.

Hypotheses Tests

The focus of the analysis is on the *W* test as the more appropriate criterion, as suggested by Godfrey and Pesaran. The original Cox-Pesaran (*CP*) test is also reported for comparison. According to the *W* test, the von Liebig model is not rejected for any of the five crops. On the contrary, the polynomial model is clearly rejected for corn, wheat, and cotton, while it is

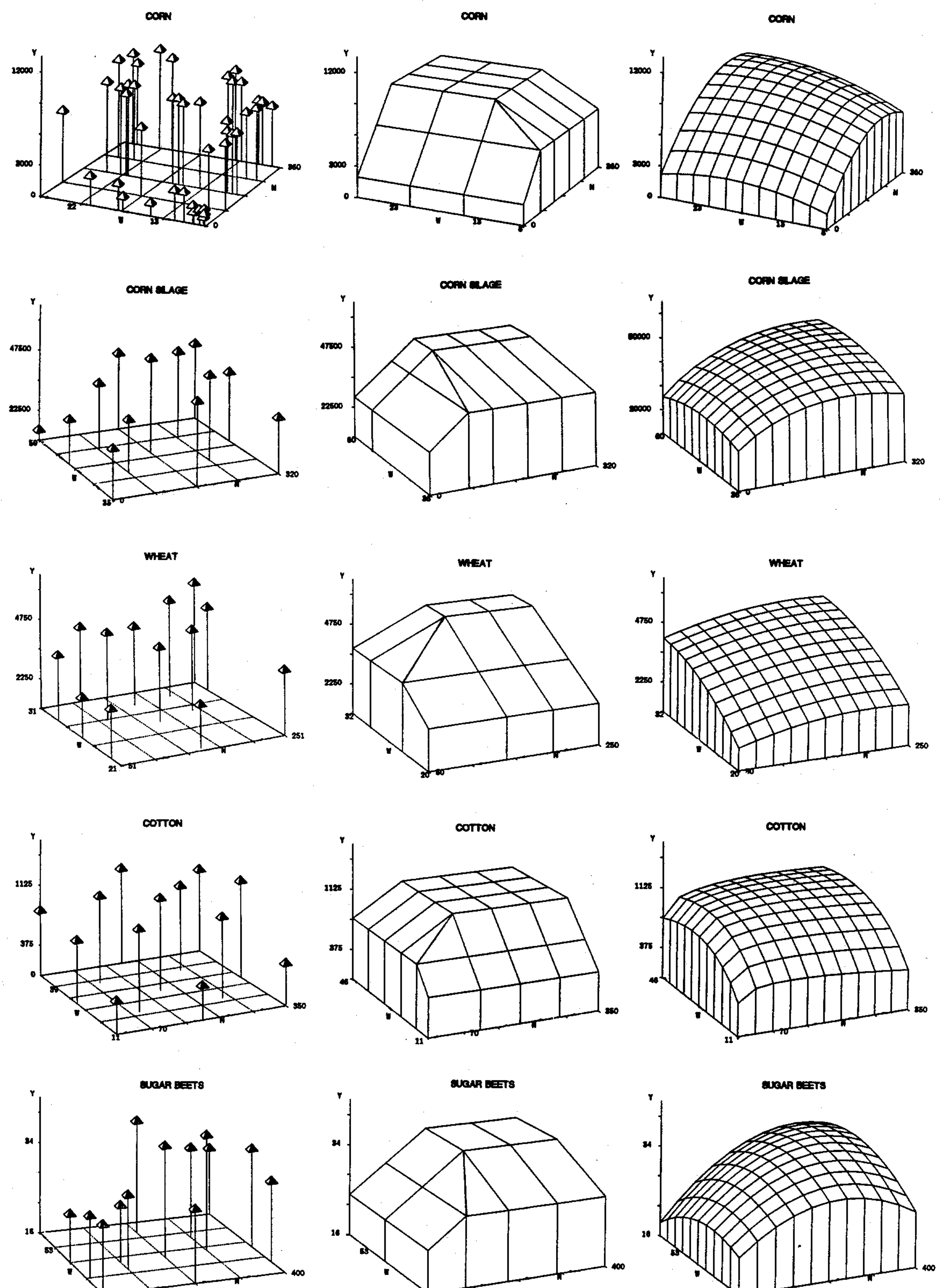

Figure 1. Scatter diagrams, von Liebig and polynomial models

Table 4. Water and Nitrogen Levels for Maximum Profits

Crop	Crop Price	Polynomial Model	Water	Nitrogen	Yield
	($/lb.)				
Corn	.054	Quadratic	22.5	254	9,764
Silage	.010	Three-halves	48.7	182	46,958
Wheat	.060	Quadratic	32.2	143	4,869
Cotton	.600	Square-root	33.2	118	1,142
	($/ton)				
Sugar beets	38.0	Quadratic	46.4	275	36.56

not rejected for corn silage and sugar beets. The size of the *CP* test is considerably higher than that of the *W* test, indicating the possibility of overrejection of the null hypothesis, as discussed by Godfrey and Pesaran. Overall, the von Liebig model, as specified above, outperforms the polynomial functions in three out of the five cases while the sample information of the silage and sugar beets experiments is insufficient for choosing among the rival models.

Conclusions

This study of yield response to water and nitrogen has confirmed that the von Liebig model, based upon the limiting factor and the nonsubstitution hypotheses, is a strong candidate for representing crop response to macro nutrients in a homogenous setting of soil and climate conditions. Conjectures as to why the von Liebig model failed to reject the polynomial response (although it was not rejected by it) to corn silage and sugar beets, can range from lack of sufficient sample information to the more interesting one according to which the von Liebig model might be suitable for yields that do not include the entire plant, to the fact that the experimental design favored the polynomial specifications. Another conjecture may be based on the notion of second-order interaction between nutrients. Let us recall that a von Liebig specification as stated in (1) and (7) implies a first-order interaction between water and nitrogen. A second-order interaction, then, could be specified as

$$(11) \quad Y = \min[f_{W/N}(W, \theta_W), f_{N/W}(N, \theta_N)] + u,$$

where the water and nitrogen response functions are now conditioned on the level of the other nutrient. This second-order interaction can take many forms, and further research is needed to assess the validity of this conjecture.

The above analysis has dealt with purely agronomic data. If the von Liebig model represents a preferred specification for crop response to macronutrients, how can economic choices of inputs such as labor and capital, for example, be integrated in it? One suggestion is contained in the following specification. Assuming the researcher knows the aggregate

Table 5. Results of the Nonnested Hypotheses Tests

Crop (Polynomial)	Hypothesis Test	H_0: von Liebig H_1: Polynomial	H_0: Polynomial H_1: von Liebig
Corn	*CP*-test	−1.83	−6.45*[a]
(Quadratic)	*W*-test	−1.36	−3.91*
Corn silage	*CP*-test	−3.33*	.75
(Three-halves)	*W*-test	−1.87	.55
Wheat	*CP*-test	−1.22	−5.22*
(Quadratic)	*W*-test	−.96	−3.85*
Cotton	*CP*-test	.79	−5.99*
(Square root)	*W*-test	.61	−3.01*
Sugar beets	*CP*-test	.37	−2.39*
(Quadratic)	*W*-test	.23	−1.60

[a] Asterisk indicates significant at the 1% level for a two-tail test.

quantities of labor (L) and capital (C) applied to the crop, then

$$Y = \min[f_W(W, L, C), f_N(N, L, C)] + u \quad (12)$$

is a plausible specification which maintains the von Liebig hypothesis of nonsubstitution between macronutrients but allows substitution between nonnutrient inputs. In other words, labor and capital inputs affect yield only via the nutrient functions which are subject to the law of the minimum. A specification of the von Liebig model such as (12) is suitable for using data generated by behavioral choices. Lack of suitable information has prevented so far a verification of this framework.

The von Liebig hypothesis about crop response, widely known as "the law of minimum," has often been paid lip service but rarely taken seriously in its analytical and economic implications concerning the fertilization problem. Its deceptively simple formulation has appeared implausible to many researchers and, over the century, it has been pushed aside without a rigorous verification. The necessary statistical procedures for such a test became available only in recent times. The fate of the von Liebig hypothesis is indeed intriguing. Originally formulated for explaining a limited biological phenomenon, it was rejected for its naivete and alleged analytical rigidity in representing crop response to macronutrients. It ended up embraced by economists almost one hundred years later in a more rigid specification known as the Leontief model, which has been widely applied as a research and policy tool. The results of this study are interesting because they show that a 130-year old conjecture can be reintroduced in its original field since it is capable of explaining crop response at least as well as, and often better than, the most regarded specification.

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