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Quantum dot laser tolerance to optical feedback

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6.1
Introduction

In optical fiber networks, the semiconductor laser source may be perturbed by unavoidable optical feedback from fiber pigtails or fiber connectors unless expensive optical isolators are used. Analytical expressions for the stable operation of laser diodes are highly desirable and have been a constant preoccupation of researchers in the field [1]. Mork et al. [2] investigated the Lang and Kobayashi equations describing a quantum well (QW) semiconductor laser subject to delayed optical feedback and derived an approximation of the stability boundary in terms of the feedback parameter k. $k^2 \equiv P_r/P_i$ is defined as the ratio of the reflected power P_r and emitted power P_i. Mathematically, this stability boundary corresponds to the lowest possible value of the first Hopf bifurcation of an external cavity mode. The external cavity modes (ECMs) are the basic solutions of a laser subject to optical feedback from a distant mirror. In the weak feedback limit, there exists only one mode which is determined by the feedback phase $C = \omega_0\tau_{ec}$, in first approximation (ω_0 is the angular frequency of the solitary laser and τ_{ec} is the round-trip time). The stability condition derived by Mork et al. [2] is given by

$$k < k_c \equiv \frac{\Gamma^{QW}}{\sqrt{1+\alpha^2}} \tag{6.1}$$

where α is the linewidth enhancement factor and Γ^{QW} is defined as the damping rate of the relaxation oscillations (ROs) multiplied by the photon lifetime τ_p, so that k_c is dimensionless. Because of a possible confusion with a different definition of the damping rate used by Mork et al. [2], we derive the expressions of the ROs frequency ω^{QW} and damping rate Γ^{QW} from their rate equations in Appendix A. As noted by Mork et al. [2], Eq. (6.1) was previously suggested by Helms and Petermann [3] as a simple analytical criterion for tolerance with respect to optical feedback. Helms and Petermann [3] evaluate the validity of Eq. (6.1) by analyzing numerically the stability of the minimum linewidth ECM. They noted that this approximation gives a good

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description of the critical feedback level as long as the linewidth enhancement factor α is significantly larger than unity. They then proposed an empirical law given by

$$k_c = \Gamma^{QW} \frac{\sqrt{1+\alpha^2}}{\alpha^2}. \tag{6.2}$$

Both Eqs. (6.1) and (6.2) are used in current experimental studies of quantum dot (QD) lasers subject to optical feedback. Specifically, Eq. (6.1) is used in Ref. [4] and Eq. (6.2) is used in Refs. [5, 6, 7]. Note that the minimum linewidth mode appears as the first ECM in the weak feedback limit for the feedback phase $C = -\arctan(\alpha)$. For the minimum linewidth mode the minimum value of the feedback strength of the first Hopf bifurcation, which marks the critical feedback strength below that the laser is guaranteed to be stable, is given by [8]

$$k_c = \Gamma^{QW} \frac{\sqrt{1+\alpha^2}}{\alpha^2 - 1}. \tag{6.3}$$

The approximation of the first Hopf bifurcation in terms of an arbitrary phases and thus for arbitrary ECMs is derived in [9]. Substituting the expression for the frequency of the minimum linewidth mode $\Delta \simeq C = -\arctan(\alpha)$ then leads to Eq. (6.3). The denominator in (6.3) is different from the denominator of (6.2) which explains the numerically observed singularity as $\alpha \to 1^+$ [3]. The expression (6.1) follows from analytic considerations of the first Hopf bifurcation at a feedback phase $C = \pi + \arctan(\alpha)$, which provides the lowest possible value of k_c. The inequality (6.1) is based on a series of approximations ($k \ll 1$, $\omega^{QW}\tau_{ec}/\tau_p \gg 1$, $\alpha > 1$) which may or may not be appropriate. Asymptotic techniques were later used to determine systematic approximations for a variety of cases (pump parameter close to threshold, short external cavity) [9]. In this approach, all small or large dimensionless parameters appearing in the rate equations are scaled with respect to a unique parameter γ defined as the ratio of the photon and carrier lifetimes ($\gamma \equiv \tau_p/\tau_s \sim 10^{-2} - 10^{-3}$) [10]. Different scalings lead to different limits. We shall use the same strategy for two different rate equation models that are currently used in order to determine useful information on the dynamics of QD lasers. As we shall demonstrate, the stability condition can still be formulated by Eq. (6.1) but with different expressions for the damping rate Γ.

Both, models with one carrier type and electron-hole models have been successfully used to describe turn-on experiments [11, 12, 13, 14], gain recovery dynamics [15, 16, 17], optical injection [18] and optical feedback [19, 20]. The rate equation models with only one carrier type assume the same scattering rates for electrons and holes. They allow the derivation of simple analytical expressions which are useful when examining experimental data. Electron-hole rate equation models are taking into account the fact that the thermal redistribution occurs on different time scales for electrons and holes. These models aim to bridge the gap between a microscopic description and the simpler rate equation models but are too complicated for direct analysis.

In QD semiconductor devices, the carriers are first injected into a two dimensional carrier reservoir, i.e. a quantum well, before being captured by a dot. The minimal way to describe this process is to formulate three rate equations for the electrical field in the cavity, the carrier density in the reservoir, and the occupation probability of a dot [21, 22]. These equations were analyzed using the asymptotic limit $\gamma \to 0$ in Ref. [23] and we shall apply the same analysis for the laser subject to optical feedback. Our main result is described in Section 6.2. The electron-hole rate equation model that we consider next involves five independent variables for the charge carrier densities in the QD, the charge carrier densities in the reservoir, and the photon density in the cavity and it involves microscopically calculated scattering rates that are strongly nonlinear functions of the carrier densities in the reservoir [24, 25, 11, 12] . (Please see Chapter 1 of this book for a review on the microscopic modeling). We recently showed that these equations can be simplified by taking advantage of the limit $\gamma \to 0$ [26]. Although coefficients of the reduced equations need to be computed numerically, distinct scaling laws can be extracted for the RO frequency and RO damping rate. We plan to use the same analysis here for the case of a laser subject to optical feedback [19]. The main results are summarized in Section 6.3. The asymptotic studies of the two problems are long and tedious. For clarity, the detailed computations are relegated to Appendix B and C, and in the following we only concentrate on the final results.

6.2
QD laser model with one carrier type

The rate equations for a QD laser subject to optical feedback formulated by O'Brien et al. [27] consist of three equations for the amplitude of the normalized laser field in the cavity E, the occupation probability ρ of a QD in the laser, and the number n of carriers in the reservoir per QD. The dimensionless equations are derived in Appendix B and are of the form

$$E' = \frac{1}{2}\left[-1 + g(2\rho - 1)\right](1 + i\alpha)E + k\, e^{-iC} E(t' - \tau), \tag{6.4}$$

$$\rho' = \gamma\left[Bn(1-\rho) - \rho - (2\rho - 1)|E|^2\right], \tag{6.5}$$

$$n' = \gamma\left[J - n - 2Bn(1-\rho)\right] \tag{6.6}$$

where prime means differentiation with respect to the dimensionless time $t' = t/\tau_p$. The factor 2 in Eq. (6.6) accounts for the twofold spin degeneracy in the quantum dot energy levels. A similar factor 2 is included in the definition of the differential gain factor g in Eq. (6.4) [28]. The parameter $\gamma \equiv \tau_p/\tau_s$ is the ratio of the photon lifetime and the carrier recombination time. The relaxation rates of ρ and n are assumed equal for mathematical simplicity. J is the electrically injected pump current per dot, and it is the control parameter. The nonlinear term $Bn(1-\rho)$ describes the carrier exchange rate between the reservoir and the dots. $B \equiv \tau_s/\tau_{cap} \sim 10^2 - 10^3$ is the dimensionless capture rate and $1-\rho$ is the Pauli blocking factor. The three parameters B, γ, and $g-1$ control the time-dependent response of the solitary QD laser. The last term in Eq. (6.4) represents the contribution of the delayed optical feedback. k and τ are the dimensionless feedback rate and round-trip time laser-mirror-laser, respectively, and C is the feedback phase.

As for the conventional laser, our objective is to determine the minimal value of the feedback rate below which a stable operation can be guaranteed. We shall consider

γ as our order parameter because it does not appear in the expressions of the steady states and scale B and $g-1$ with respect to γ. Several cases are possible depending on their respective scalings. The physically most interesting case considers the relation $B(1-\rho) = O(\gamma^{-1/2})$ [23], which basically assumes the carrier capture process into the QDs to be much faster than the radiative recombination time of the carriers in the QDs. The first Hopf bifurcation point k^H is determined by Eq. (6.69) (see Appendix B for the asymptotic analysis), and its lowest possible value and thus the lower bound for the critical feedback strength is given by the same expression as Eq. (6.1) but with a different dimensionless damping rate named Γ_2^{QD}:

$$\Gamma_2^{QD} \equiv \frac{\gamma}{2I^* + B_1^2}\left[2I^*\frac{1+I^*}{1-g^{-1}} + \frac{B_1^2}{2}(1+2I^*)\right] \tag{6.7}$$

with $B_1 \equiv \gamma^{1/2}B(1-g^{-1})$, and the steady state intensity of the solitary laser I^* (see Appendix B). In the limit $\gamma \to 0$, I^* is given by

$$I^* \equiv \frac{g}{2}(J - J_{th}) \tag{6.8}$$

where $J_{th} \equiv 1 + g^{-1}$ is the threshold current in the limit $\gamma \to 0$. The expression for the RO frequency in units of τ_p is $\omega^{QD} \equiv \sqrt{2\gamma I^*}$ and is the same as the one for the QW laser (ω^{QW} is given by Eq. (6.37)). If the damping rate given in Eq. (6.7) is explored in the limits $B_1^2 \to \infty$ (fast capture) or $I^* \to 0$ (close to threshold), the value decreases and approaches the much lower RO damping rate of QW lasers

$$\Gamma^{QW} \equiv \frac{\gamma(1+2I^*)}{2} \tag{6.9}$$

(see Appendix A, Eq. (6.38)), thus in this limits $\Gamma_2^{QD} \to \Gamma^{QW}$.

However, if $B_1^2 = O(1)$ and/or g is close to 1, Γ_2^{QD} is much larger than Γ^{QW}. This can be demonstrated by rewriting Γ_2^{QD} as

$$\Gamma_2^{QD} = \Gamma^{QW} + \frac{\gamma I^*}{2I^* + B_1^2}\frac{g+1+2I^*}{g-1} \tag{6.10}$$

where the correction term clearly indicates the effect of $g-1$ if g is close to 1.

6.3 Electron-hole model for QD laser

The microscopically-based rate-equation model for a solitary QD laser that separately treats electron and hole dynamics has been formulated and further investigated in [24, 12, 14] (see Chapter 1 of this book for a review). Supplemented by the optical feedback term [19] and formulated with dimensionless quantities [20] it describes the evolution of the occupation probability of the confined QD levels, N_e and N_h, the number of carriers in the reservoir per QD, W_e and W_h, (e,h stand for electrons and holes, respectively), and the normalized slowly varying amplitude of the laser field inside the cavity $E = \sqrt{I}\exp(-i\phi)$ with the normalized intensity I and the phase ϕ.

$$E' = \frac{1}{2}\Big[-1 + g(N_e + N_h - 1)\Big](1 - i\alpha)E + k\, e^{iC}E(t' - \tau), \tag{6.11}$$

$$N_e' = \gamma\Big[s_e^{in}(1 - N_e) + s_e^{out}N_e - (N_e + N_h - 1)|E|^2 - N_eN_h\Big], \tag{6.12}$$

$$N_h' = \gamma\Big[s_h^{in}(1 - N_h) + s_h^{out}N_h - (N_e + N_h - 1)|E|^2 - N_eN_h\Big], \tag{6.13}$$

$$W_e' = \gamma\Big[J + (s_e^{in} + s_e^{out})N_e - s_e^{in} - cW_eW_h\Big], \tag{6.14}$$

$$W_h' = \gamma\Big[J + (s_h^{in} + s_h^{out})N_h - s_h^{in} - cW_eW_h\Big]. \tag{6.15}$$

In the above equations prime means differentiation with respect to the dimensionless time $t' = t/\tau_p$ (with the photon lifetime τ_p). As before k, C and τ are the dimensionless feedback rate, the feedback phase and the external round-trip time, respectively and g is the linear gain parameter. The parameter c accounts for spontaneous and non-radiative losses in the reservoir and J is the dimensionless electrically injected pump current per QD. Further s_e^{in}, s_e^{out}, s_h^{in}, s_h^{out} represent dimensionless scattering rates that are rescaled by $s_{e,h}^{in,out} = W^{-1}S_{e,h}^{in,out}$ with W^{-1} being the carrier lifetime due to radiative recombination inside a QD that corresponds to τ_s in the QW- and in the QD model with one carrier type. They are computed numerically from a microscopic theory of carrier-carrier scattering events between QD and reservoir [24, 12]. The scattering times $\tau_e \equiv (S_e^{in} + S_e^{out})^{-1}$ and $\tau_h \equiv \left(S_h^{in} + S_h^{out}\right)^{-1}$ are our reference time scales.

By introducing a rescaled time $s = \gamma^{1/2}t'$, reformulating the above equations in terms of deviations from the steady state and taking advantage of the small value of $\gamma = \tau_p/\tau_s \to 0$, we showed in Ref. [26] that the rate equations can be reduced to four equations that are given in Appendix C. Note that this rescaling of time is suggested by the fact that the RO frequency is proportional to $\gamma^{1/2}$ as $\gamma \to 0$.

As we shall demonstrate, valuable information can be extracted from these equations on the basis of simple scaling assumptions. Three cases were explored in [26] which we now review.

6.3.1 Similar scattering times τ_e and τ_h

At first, one case will be discussed that assumes the scattering times of both carrier types to be on the same timescale. Further, this case assumes $s_e^{in} + s_e^{out}$ and $s_h^{in} + s_h^{out}$ to be $O(1)$ quantities compared to $\gamma^{1/2}$. We find that the expression for the critical feedback strength k_c is the same as Eq. (6.1) but with a different damping rate given by

$$\Gamma^S \equiv \frac{\gamma}{2}\left[\frac{s_e^{in} + s_e^{out}}{2} + 2I^* + N_h^* + N_e^* + \frac{s_h^{in} + s_h^{out}}{2}\right] \tag{6.16}$$

where I^*, N_e^*, and N_h^* are the dimensionless steady state values for the solitary laser of the light intensity, the electron, and the hole occupation probability in the QDs, respectively, that need to be computed numerically. In [26], we noted that

$N_h^* + N_e^* = 1 + g^{-1}$ where $g = O(1)$ is the dimensionless gain coefficient. Eq. (6.16) then simplifies as

$$\Gamma^S = \frac{\gamma}{2}\left[\frac{s_e^{in} + s_e^{out}}{2} + 2I^* + 1 + g^{-1} + \frac{s_h^{in} + s_h^{out}}{2}\right]. \tag{6.17}$$

Eq. (6.17) can be reformulated as

$$\Gamma^S = \Gamma^{QW} + \frac{1}{2}\left[\frac{s_e^{in} + s_e^{out}}{2} + \frac{s_h^{in} + s_h^{out}}{2}\right] \tag{6.18}$$

where

$$\Gamma^{QW} \equiv \frac{\gamma}{2}\left[1 + g^{-1} + 2I^*\right] \tag{6.19}$$

has the same format as Eq. (6.9) and can be considered as the contribution of the conventional QW laser.

6.3.2
Different scattering times τ_e and τ_h

The microscopic calculations predict very large scattering rates for the holes [12] due to their much larger effective mass. Consequently this section aims to discuss the effect of holes if they are much faster than the electrons. For the asymptotic analysis we introduce the dimensionless parameter a as a measure for the hole scattering rates

$$a \equiv \sqrt{\frac{\gamma}{2I^*}}(s_h^{in} + s_h^{out}) \tag{6.20}$$

where I^* is assumed to be an $O(1)$ quantity.

Small scattering lifetime of the holes $a = O(1)$

To this end, we assume that $s_e^{in} + s_e^{out} = O(1)$ while $s_h^{in} + s_h^{out} = O(\gamma^{-1/2})$ which then implies from Eq. (6.20) that $a = O(1)$. Note that this is different to Section 6.3.1 where the scaling $a = O(\gamma^{1/2})$ was discussed. The leading order equation for the growth rate is the same as for the solitary laser [26] and does not contain any contribution of the feedback. In other words, the amplitude of the feedback k is too weak ($k = O(\gamma)$). We would need to consider the case $k = O(\gamma^{1/2})$ in order to find the feedback parameter in the leading equation for the growth rate, but this problem has not been solved analytically yet.

Very small scattering lifetime of the holes $a = O(\gamma^{-1/2})$

For the case were the hole scattering time is on the order of pico-seconds another scaling has to be introduced. Thus, for this case we assume that $a = O(\gamma^{-1/2})$ while $s_e^{in} + s_e^{out} = O(1)$. Compared to the case of similar scattering times the RO frequency is slightly reduced by a factor of $1/\sqrt{1/2}$. The expression for the critical feedback

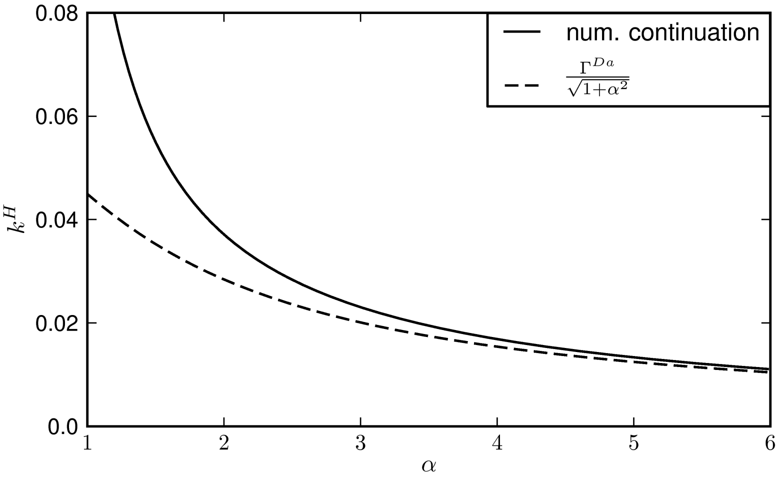

Fig. 6.1 Solid line shows the first Hopf bifurcation point $k = k^H$ as a function of α as obtained numerically from the original rate equations using a continuation method [20] ($C = \pi + \arctan(\alpha)$, $\tau = 80$).The broken line represents its analytical approximation given by Eq. (6.1). As α decreases towards zero, k^H increases and the analytical approximation that assumes $k \ll 1$ begins to fail.

strength is the same as Eq. (6.1) but with a different dimensionless damping rate given by

$$\Gamma^{Da} \equiv \frac{\gamma}{2}\left[\frac{I^*}{\gamma}\frac{1}{s_h^{in} + s_h^{out}} + s_e^{in} + s_e^{out} + I^* + N_h^*\right]. \tag{6.21}$$

In Fig. 6.1, we compare numerical and analytical predictions for a laser subject to a long external cavity. The numerical determination of the Hopf bifurcation point k^H has been obtained by using a continuation technique (DDE-Biftool [29]) applied to the original electron-hole rate equation model [26, 19] and not from the reduced equations (6.87)-(6.90). Details on the numerical simulations and parameter values are documented in Ref. [20]. The broken line in Fig. 6.1 represents the analytical approximation given by Eq. (6.1). As α decreases towards zero, k^H increases and the analytical approximation that assumes $k \ll 1$ begins to fail, while a very good agreement with the analytic results is found for larger α.

6.4 Summary

The expression for the critical feedback strength from Eq. (6.1) provides a sufficient condition for stable operation of a quantum well laser subject to optical feedback. The critical feedback rate above which pulsating instabilities are observed is determined as a function of the linewidth enhancement factor α and the damping rate of the ROs. Its simplicity has encouraged experimental studies of QD lasers subject to optical feedback. It is shown that Eq. (6.1) is also a good approximation for QD lasers provided that their much larger damping rate of the relaxation oscillations is considered. The damping rate is generally obtained by fitting data. In this review we examine two different rate equations models for QD lasers and derive the stability condition with analytical expressions for the damping rate. These expressions allow us to anticipate the effect of specific parameters, e.g. the carrier scattering rates and the differential gain coefficient, and design lasers with a larger tolerance to optical feedback.

6.4.1 Acknowledgement

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6.5 Appendix A: rate equations for quantum well lasers

The rate equations for a solitary quantum well laser used by Mork et al. [2] are given by

$$\frac{d\mathcal{E}}{dt} = \frac{1}{2}\left[G_N(N-N_0) - \frac{1}{\tau_p}\right]\mathcal{E}, \tag{6.22}$$

$$\frac{dN}{dt} = J - \frac{N}{\tau_s} - G_N(N-N_0)\mathcal{E}^2. \tag{6.23}$$

Here $\mathcal{E}$ is the amplitude of the electrical field and N is the carrier density. The linear gain coefficient is denoted by G_N, N_0 is the transparency density of carriers, J is the pumping current and τ_p and τ_s are the photon and carrier lifetimes, respectively. The non-zero intensity steady state is

$$N^* = N_0 + \frac{1}{G_N\tau_p}, \tag{6.24}$$

$$\mathcal{E}^{*2} = \frac{1}{G_N(N-N_0)}(J - \frac{N}{\tau_s}). \tag{6.25}$$

From the linearized equations, we then determine the characteristic equation for the growth rate λ

$$\lambda^2 + (\frac{1}{\tau_s} + G_N\mathcal{E}^{*2})\lambda + \frac{1}{\tau_p}G_N\mathcal{E}^{*2} = 0. \tag{6.26}$$

In order to properly define the relaxation oscillation frequency and its damping rate, we take advantage of the fact that $\tau_p << \tau_s$. The roots of the quadratic equation then take the form

$$\lambda = -\Gamma_{R0}^{QW} \pm i\omega_{RO}^{QW} \tag{6.27}$$

where

$$\omega_{R0}^{QW} \equiv \sqrt{G_N \frac{1}{\tau_p}\mathcal{E}^{*2} - \frac{1}{4}(\frac{1}{\tau_s} + G_N\mathcal{E}^{*2})^2} \simeq \sqrt{G_N \frac{1}{\tau_p}\mathcal{E}^{*2}}, \tag{6.28}$$

$$\Gamma_{R0}^{QW} \equiv \frac{1}{2}(\frac{1}{\tau_s} + G_N\mathcal{E}^{*2}) = \frac{1}{2}(\frac{1}{\tau_s} + \tau_p\omega_{R0}^2) \tag{6.29}$$

are defined as the RO frequency and RO damping rate of the solitary laser, respectively. They are the quantities that are measured experimentally. Mork et al. [2] introduced the RO damping rate as "τ_R^{-1}" which equals $2\Gamma_{R0}^{QW}$ but could wrongly be interpreted as Γ_{R0}^{QW}.

In order to determine asymptotic approximations, we need to reformulate the rate equations in dimensionless form. The simplest way is to measure time in units of the photon lifetime by introducing

$$t' \equiv t/\tau_p. \tag{6.30}$$

Furthermore, introducing the new dimensionless dependent variables E and Z defined by

$$E \equiv \sqrt{\frac{G_N\tau_s}{2}}\mathcal{E} \text{ and } Z \equiv \frac{1}{2}\left[G_N(N - N_0)\tau_p - 1\right] \tag{6.31}$$

allows to reduce the number of parameters. Inserting Eqs. (6.30) and (6.31) into Eqs. (6.22) and (6.23), we find

$$\frac{dE}{dt'} = ZE, \tag{6.32}$$

$$\frac{dZ}{dt'} = \gamma\left[P - Z - (1 + 2Z)E^2\right] \tag{6.33}$$

where γ and P are defined by

$$\gamma \equiv \frac{\tau_p}{\tau_s},\ P \equiv \frac{G_N\tau_p\tau_s}{2}(J - J_{th}),\ \text{with } J_{th} \equiv \frac{N_0}{\tau_s} + \frac{1}{G_N\tau_p\tau_s}. \tag{6.34}$$

The non-zero intensity steady state is

$$Z^* = 0 \text{ and } I^* = E^{*2} = P \tag{6.35}$$

and the characteristic equation for the growth rate σ is given by

$$\sigma^2 + \gamma(1 + 2I^*)\sigma + 2I^*\gamma = 0. \tag{6.36}$$

Provided γ is sufficiently small, the roots of Eq. (6.36) are complex-conjugated. The dimensionless RO frequency and RO damping rate (in units of time t') are then defined from the imaginary and real part of these roots. We obtain

$$\omega^{QW} \equiv \sqrt{2\gamma I^* - \frac{\gamma^2}{4}(1 + 2I^*)^2} \simeq \sqrt{2\gamma I^*} \text{ as } \gamma \to 0 \quad \text{and} \tag{6.37}$$

$$\Gamma^{QW} \equiv \frac{\gamma(1 + 2I^*)}{2}. \tag{6.38}$$

In our analysis of the QD rate equations, we use the same dimensionless time $t' \equiv t/\tau_p$ and reformulate the dynamic equations so that the same γ multiply the right hand side of the carrier equations.

6.6
Appendix B: asymptotic analysis for QD laser model with one carrier type

The equations examined by O'Brien et al. [27] are the following three equations for the amplitude of the laser field in the cavity, $\mathcal{E}$, the number of carriers in the reservoir per dot, N and the occupation probability of the dots in the laser ρ

$$\frac{d\mathcal{E}}{dt} = \frac{1}{2}\left[-\frac{1}{\tau_p} + g_0\theta(2\rho-1)\right]\mathcal{E} + \frac{i\delta\omega}{2}\mathcal{E} + \frac{\eta}{2}\mathcal{E}(t-\tau_{ec}), \tag{6.39}$$

$$\frac{d\rho}{dt} = -\frac{\rho}{\tau_s} - g_0(2\rho-1)|\mathcal{E}|^2 + \tilde{F}(N,\rho), \tag{6.40}$$

$$\frac{dN}{dt} = -\frac{N}{\tau_s} + \tilde{J} - 2N^{QD}\tilde{F}(N,\rho). \tag{6.41}$$

For the definition of the various parameters, see Ref. [27]. The capture rate is described by the term $\tilde{F} = \tilde{C}N^2(1-\rho)$ in [27] and is proportional to the number of carriers present as well as the probability to find a dot. As in [23], we shall consider $\tilde{F} = \tilde{B}N(1-\rho)$ instead of $\tilde{F} = \tilde{C}N^2(1-\rho)$. Here the carrier phonon and the Auger carrier capture rates are denoted by $\tilde{B}$ and $\tilde{C}$, respectively. We define $\delta\omega = \alpha/\tau_p$ where α is the linewidth enhancement factor. N^{QD} is the two dimensional density of quantum dots. In our analysis, we introduce the α factor in the traditional way i.e., by the term $(1+i\alpha)$ multiplying the full square bracket in (6.39). Moreover, we take into account the feedback phase $C = \omega_0\tau_{ec}$ where ω_0 is the angular frequency of the solitary laser ($C = 0 \bmod 2\pi$ in [27]). Our starting equations are then given by

$$\frac{d\mathcal{E}}{dt} = \frac{1}{2}\left[-\frac{1}{\tau_p} + g_0\theta(2\rho-1)\right](1+i\alpha)\mathcal{E} + \frac{\eta}{2}e^{-iC}\mathcal{E}(t-\tau_{ec}), \tag{6.42}$$

$$\frac{d\rho}{dt} = -\frac{\rho}{\tau_s} - g_0(2\rho-1)|\mathcal{E}|^2 + \tilde{B}N(1-\rho), \tag{6.43}$$

$$\frac{dN}{dt} = -\frac{N}{\tau_s} + \tilde{J} - 2N^{QD}\tilde{B}N(1-\rho). \tag{6.44}$$

By introducing a dimensionless time t', the number of carriers in the reservoir per QD n, and a normalized field E , according to

$$t' \equiv t/\tau_p, \qquad n \equiv N/N^{QD}, \qquad E \equiv \sqrt{g_0\tau_s}\mathcal{E}, \tag{6.45}$$

the Eqs. (6.42)-(6.44) simplify as

$$\frac{dE}{dt'} = \frac{1}{2}\left[-1 + g(2\rho-1)\right](1+i\alpha)E + ke^{-iC}E(t'-\tau), \tag{6.46}$$

$$\frac{d\rho}{dt'} = \gamma\left[-\rho - (2\rho-1)|E|^2 + Bn(1-\rho)\right], \tag{6.47}$$

$$\frac{dn}{dt'} = \gamma\left[-n + J - 2Bn(1-\rho)\right] \tag{6.48}$$

where

$$\gamma \equiv \frac{\tau_p}{\tau_s},\ g \equiv g_0\theta\tau_p,\ k \equiv \frac{\eta}{2}\tau_p,\ \tau \equiv \tau_{ec}/\tau_p,\ J \equiv \frac{\tilde{J}\tau_s}{N^{QD}},\ \text{and } B \equiv \tilde{B}N^{QD}\tau_s. \tag{6.49}$$

If we consider the rate equations (6.46)-(6.48), in terms of the normalized intensity I and the phase ϕ of the field $E = \sqrt{I}\exp(i\phi)$, the equations can be rewritten as

$$I' = [-1 + g(2\rho - 1)]\, I + 2k\sqrt{I(t'-\tau)I(t')}\cos(-C + \phi(t'-\tau) - \phi), \tag{6.50}$$

$$\phi' = \frac{1}{2}\,[-1 + g(2\rho - 1)]\,\alpha + k\sqrt{\frac{I(t'-\tau)}{I(t')}}\sin(-C + \phi(t'-\tau) - \phi), \tag{6.51}$$

$$\rho' = \gamma\,[Bn(1-\rho) - \rho - (2\rho - 1)I]\,, \tag{6.52}$$

$$n' = \gamma\,[J - n - 2Bn(1-\rho)]\,. \tag{6.53}$$

6.6.1
External cavity modes

The basic solutions of the feedback problem are the external cavity modes (ECMs). They are defined as solutions with constant field intensity and carrier numbers, i.e. $I = I_s$, $\rho = \rho_s$, $n = n_s$, and a phase of the field that varies linearly in time given by $\phi = \phi_s = -C\frac{t'}{\tau} + \Delta\frac{t'}{\tau}$ with the ECM frequency Δ. For clarity the index s is omitted in the following equations.

From Eqs. (6.50)-(6.53), the basic solution satisfy the following conditions

$$\begin{aligned} \frac{1}{2}\,[-1 + g(2\rho - 1)] &= -k\cos(\Delta), \\ \Delta &= C - k\tau\,[\alpha\cos(\Delta) + \sin(\Delta)]\,, \\ n &= \frac{J}{1 + 2B(1-\rho)}, \\ I &= \frac{Bn(1-\rho) - \rho}{2\rho - 1}. \end{aligned}$$

Solving for ρ, we obtain

$$\rho = \frac{1}{2}(1 + g^{-1}) - \frac{k}{g}\cos(\Delta), \tag{6.54}$$

$$n = \frac{J}{1 + 2B(1-\rho)}, \tag{6.55}$$

$$I = \frac{B(1-\rho)}{1 + 2B(1-\rho)}\,\frac{J - J_{th}}{2\rho - 1} \tag{6.56}$$

where

$$J_{th} \equiv \frac{\rho(1 + 2B(1-\rho))}{B(1-\rho)}. \tag{6.57}$$

We note the following relations which will be useful when we eliminate n from the coefficients of the characteristic equation:

$$Bn + 1 + 2I = \frac{1+I}{1-\rho},$$
$$\left[\begin{array}{c} (1+2B(1-\rho))\,(Bn+1+2I) \\ -2B^2 n(1-\rho) \end{array}\right] = \frac{1+I}{1-\rho} + 2B(1-\rho)(1+2I).$$

6.6.2 Stability

From the linearized equations, we determine the following condition for the growth rate σ:

$$\left[\begin{array}{cccc} \begin{pmatrix} k\cos(\Delta)F \\ -\sigma \end{pmatrix} & k\sqrt{I}\sin(\Delta)F & g\sqrt{I} & 0 \\ \frac{-k}{\sqrt{I}}F\sin(\Delta) & \begin{pmatrix} \cos(\Delta)F \\ -\sigma \end{pmatrix} & g\alpha & 0 \\ -2\gamma(2\rho-1)\sqrt{I} & 0 & \begin{pmatrix} -\gamma\begin{pmatrix} Bn+1 \\ +2I \end{pmatrix} \\ -\sigma \end{pmatrix} & \gamma B(1-\rho) \\ 0 & 0 & 2\gamma Bn & \begin{pmatrix} -\gamma\begin{pmatrix} 1 \\ +2B(1-\rho) \end{pmatrix} \\ -\sigma \end{pmatrix} \end{array}\right] = 0 \tag{6.58}$$

where

$$F \equiv \exp(-\sigma\tau) - 1. \tag{6.59}$$

Expanding the determinant (6.58), we find the following characteristic equation for the growth rate σ

$$\begin{aligned} 0 = \sigma^4 &+ \sigma^3\left[\gamma\left(1+2B(1-\rho)+\frac{1+I}{1-\rho}\right) - 2k\cos(\Delta)F\right] \\ &+ \sigma^2\left[\begin{array}{c} 2\gamma(2\rho-1)gI + \gamma^2\left(\frac{1+I}{1-\rho} + 2B(1-\rho)(1+2I)\right) + k^2F^2 \\ -\gamma 2k\cos(\Delta)F\left(2B(1-\rho) + \frac{2+I-\rho}{1-\rho}\right) \end{array}\right] \\ &+ \sigma\left[\begin{array}{c} 2\gamma(2\rho-1)gI\left[\gamma\left(1+2B(1-\rho)\right) + k(\alpha\sin(\Delta) - \cos(\Delta))F\right] \\ -\gamma^2 2k\cos(\Delta)F\left(\frac{1+I}{1-\rho} + 2B(1-\rho)(1+2I)\right) \\ +\gamma k^2F^2\left(2B(1-\rho) + \frac{2+I-\rho}{1-\rho}\right) \end{array}\right] \\ &+ \left[\begin{array}{c} \gamma^2 k^2 F^2\left(\frac{1+I}{1-\rho} + 2B(1-\rho)(1+2I)\right) \\ +\gamma^2 k 2(2\rho-1)gI\left(1+2B(1-\rho)\right)(\alpha\sin(\Delta) - \cos\Delta))F \end{array}\right]. \end{aligned} \tag{6.60}$$

We next investigate two cases that depend on the size of parameter B.

6.6.3 $B(1-\rho) = O(1)$

We solve Eq. (6.60) by seeking a solution of the form

$$\sigma = \gamma^{1/2}\sigma_0 + \gamma\sigma_1 + \ldots,$$
$$k = \gamma k_1 + \ldots. \tag{6.61}$$

From the Eqs. (6.54)-(6.56) we note the following scalings:

$$I = I^* + O(\gamma),\ \rho = \rho^* + O(\gamma),\ \text{and}\ \Delta = \Delta_0 + O(\gamma) \tag{6.62}$$

where

$$I^* = \frac{B(1-\rho^*)}{(1+2B(1-\rho^*))}\frac{(J-J_{th,0})}{(2\rho^*-1)} = \frac{g}{2}\frac{B(1-g^{-1})}{1+B(1-g^{-1})}(J-J_{th,0}),$$
$$\rho^* = \frac{1}{2}(1+g^{-1}), \qquad \text{and} \qquad \Delta_0 = C. \tag{6.63}$$

Here I^* and ρ^* denote intensity and occupation probability of the QDs for the solitary laser, respectively and the threshold current of the solitary laser is given by [23]

$$J_{th,0} \equiv \frac{\rho^*(1+2B(1-\rho^*))}{(1-\rho^*)B} = \frac{1+B(1-g^{-1})}{B(1-g^{-1})}(1+g^{-1}).$$

We find from Eq. (6.60) the following sequence of problems

$$O(\gamma^2): 0 = \sigma_0^4 + \sigma_0^2 2I^*, \tag{6.64}$$
$$\begin{aligned} O(\gamma^{5/2}): 0 = &\left(4\sigma_0^2 + 4I^*\right)\sigma_0\sigma_1 \\ &+ \sigma_0^3\left[1+2B(1-\rho^*) + \frac{1+I^*}{1-\rho^*} - 2k_1\cos(\Delta_0)F_0\right] \\ &+ 2I^*\sigma_0\left[k_1(\alpha\sin(\Delta_0) - \cos(\Delta_0))F_0\right. \\ &\left.+1+2B(1-\rho^*)\right] \end{aligned} \tag{6.65}$$

where

$$F_0 \equiv \exp(-\gamma^{1/2}\sigma_0\tau) - 1. \tag{6.66}$$

From Eq. (6.64), we determine σ_0 as

$$\sigma_0 = i\sqrt{2I^*}$$

and from Eq. (6.65) with

$$\omega^{QD} \equiv \sqrt{2\gamma I^*}, \tag{6.67}$$

we find σ_1 as

$$\sigma_1 = -\Gamma + \frac{k_1}{2}(\alpha\sin(\Delta_0) + \cos(\Delta_0))\left[(\cos(\omega^{QD}\tau) - 1) - i\sin(\omega^{QD}\tau)\right]$$

where

$$\Gamma \equiv \frac{1+I^*}{1-g^{-1}}.$$

The real part of σ_1 then is

$$\mathrm{Re}(\sigma_1) = -\Gamma - k_1(\alpha\sin(\Delta_0)+\cos(\Delta_0))\sin^2(\frac{\omega^{QD}\tau}{2}) \tag{6.68}$$

which implies stability for all values of k_1 if $(\alpha\sin(\Delta_0)+\cos(\Delta_0)) > 0$ or provided that

$$\begin{aligned} k_1 < k_1^H &\equiv \frac{-\Gamma}{(\alpha\sin(\Delta_0)+\cos(\Delta_0))\sin^2(\frac{\omega^{QD}\tau}{2})} \\ &= \frac{-2}{(1-\cos(\omega^{QD}\tau)(\cos(\Delta_0-\arctan(\alpha))}\frac{\Gamma}{\sqrt{1+\alpha^2}} \end{aligned} \tag{6.69}$$

if $\alpha\sin(\Delta_0)+\cos(\Delta_0) < 0$. From Eq. (6.69) we see that the lowest possible value for k_1^H is for

$$\Delta_0 = C = \pi + \arctan(\alpha) \text{ and } \omega^{QD}\tau = \pi(\mathrm{mod}\,2\pi).$$

It is given by

$$k_{1c} \equiv \frac{\Gamma}{\sqrt{1+\alpha^2}}. \tag{6.70}$$

In terms of the original parameters, the stability condition (6.70) implies that

$$k < k_c \equiv \frac{\Gamma_1^{QD}}{\sqrt{1+\alpha^2}} \tag{6.71}$$

where $\Gamma_1^{QD} \equiv \gamma\Gamma$, or equivalently,

$$\Gamma_1^{QD} = \gamma\frac{1+R_0^2}{1-g^{-1}}. \tag{6.72}$$

6.6.4 $B(1-\rho) = O(\gamma^{-1/2})$

Taking into account that $B(1-\rho) = O(\gamma^{-1/2})$ we introduce a $O(1)$ quantity B_1 as $B_1 \equiv \gamma^{1/2}2B(1-\rho^*)$. With the scaling of $\rho = \rho^* + O(\gamma)$ (see Eq. (6.62)) we may expand J_{th} from Eq. (6.57) and I from Eq. (6.56) in powers of $\gamma^{1/2}$, which yields

$$\begin{aligned} J_{th} &= 2\rho^* + \frac{1}{B(1-\rho^*)}\rho^* + O(\gamma) = 2\rho^* + \gamma^{1/2}2\rho^* B_1^{-1} + O(\gamma), \\ I &= I^* + \gamma^{1/2}I_1 + O(\gamma) \end{aligned} \tag{6.73}$$

where we have defined the steady state intensity of the solitary laser I^* in the limit $\gamma \to 0$ and its first order correction I_1

$$I^* = \frac{1}{2}\frac{1}{2\rho^*-1}(J-2\rho^*) = \frac{g}{2}(J-(1+g^{-1})), \tag{6.74}$$

$$I_1 = -\frac{g}{2}B_1^{-1}J. \tag{6.75}$$

Inserting Eq. (6.61) and Eq. (6.73) into the characteristic equation (6.60) we find the following problems for σ_0 and σ_1

$$O(\gamma^2): \quad \sigma_0^4 + \sigma_0^3 B_1 + \sigma_0^2 2I^* + \sigma_0 2I^* B_1 = 0, \tag{6.76}$$

$$\begin{aligned} O(\gamma^{5/2}): \quad & \left[4\sigma_0^3 + 3\sigma_0^2 B_1 + 2\sigma_0 2I^* + 2I^* B_1\right]\sigma_1 \\ & = -(\sigma_0^2 + \sigma_0 B_1)2I_1 \\ & \quad - \sigma_0^3 \left[1 + 2\frac{1+I^*}{1-g^{-1}} - 2k_1 \cos(\Delta_0) F_0\right] \\ & \quad - \sigma_0^2 \left[B_1(1+2I^*) - 2k_1 \cos(\Delta_0) F_0 B_1\right] \\ & \quad - \sigma_0 2I^* \left[1 + k_1(\alpha \sin(\Delta_0) - \cos(\Delta_0))F_0\right] \\ & \quad - \left[2I^* B_1 k_1 (\alpha \sin(\Delta_0) - \cos(\Delta_0)) F_0\right] \end{aligned} \tag{6.77}$$

where F_0 is defined by (6.66). Equation (6.76) admits the solution

$$\sigma_0^2 = -2I^* \tag{6.78}$$

and from (6.77) with (6.78) and (6.67), we find

$$\begin{aligned} \sigma_1 = & \frac{2\sigma_0 I_1}{4I^*} + \frac{k_1(\alpha \sin(\Delta_0) + \cos(\Delta_0))\left[(\cos(\omega^{QD}\tau) - 1) - i \sin(\omega^{QD}\tau)\right]}{2} \\ & - \frac{1}{4I^*(\sigma_0 + B_1)}\left[\sigma_0 4I^* \frac{1+I^*}{1-g^{-1}} + 2I^* B_1(1+2I^*)\right]. \end{aligned} \tag{6.79}$$

Equation (6.79) implies that

$$\mathrm{Re}(\sigma_1) = -\Gamma + \frac{k_1(\alpha \sin(\Delta_0) + \cos(\Delta_0))(\cos(\omega^{QD}\tau) - 1)}{2} \tag{6.80}$$

where

$$\Gamma \equiv \frac{1}{2I^* + B_1^2}\left[2I^* \frac{1+I^*}{1-g^{-1}} + \frac{B_1^2}{2}(1+2I^*)\right] \tag{6.81}$$

is the damping rate of the solitary laser [9]. Our stability conditions are now similar to those of Eqs. (6.71)-(6.72) with Γ_2^{QD} replacing Γ_1^{QD} where

$$\Gamma_2^{QD} \equiv \gamma\Gamma = \frac{\gamma}{2I^* + \gamma B^2(1-g^{-1})}\left[2I^* \frac{1+I^*}{1-g^{-1}} + \frac{\gamma B^2(1-g^{-1})}{2}(1+2I^*)\right]. \tag{6.82}$$

6.7
Appendix C: asymptotic analysis for a QD laser model with electrons and holes

The microscopically based electron-hole rate equation model describe the evolution of the charge carrier densities in the QD (n_e and n_h), the carrier densities in the reservoir (w_e and w_h) (e,h stand for electrons and holes, respectively), and the photon density

n_{ph}. Please see Chapter 1 of this book for the equations with dimensions while the dimensionless form is given in Eqs. (6.11)-(6.15). To reformulate the equations we introduced the new dimensionless variables

$$I \equiv n_{ph}A, \quad N_{e/h} \equiv n_{e/h}/N^{QD}, \quad W_{e/h} \equiv w_{e/h}/N^{sum} \text{and} \quad t' \equiv t/\tau_p, \tag{6.83}$$

and the dimensionless parameters:

$$g \equiv \frac{\Gamma W A N^{QD}}{2\kappa}, \quad \gamma \equiv \frac{W}{2\kappa}, \qquad k \equiv \frac{1}{2\kappa}\frac{K}{\tau_{in}}, \qquad \tau \equiv 2\kappa\tau_{ec}, \tag{6.84}$$

$$s_{e/h}^{\text{in/out}} \equiv \frac{1}{W} S_{e/h}^{\text{in/out}}, \qquad c \equiv \frac{BN^{sum}}{W}, \quad J \equiv \frac{j}{e_0 N^{sum} W}. \tag{6.85}$$

By formulating dimensionless equations in terms of deviations from the steady state and by taking advantage of the small value of $\gamma \to 0$, we showed in Ref. [26] that the five rate equations without feedback can be reduced to three equations. Supplemented by the optical feedback term [19], they consist of four equations for the deviation of the intensity from its steady state, y, the phase of the electrical field ϕ, and the deviations $u_{e/h}$ of the QD occupation probabilities from their steady state values. Specifically the new dynamic variables y, u_e and u_h are defined via

$$I = I^*(1+y) \qquad \text{and} \qquad N_{e/h} = N_{e/h}^* + \sqrt{\gamma}\omega g^{-1} u_{e/h} \tag{6.86}$$

where the superscript * denotes the steady state values of the solitary laser.

The new set of rate equations is given by

$$\begin{aligned} y' =& (u_e + u_h)(1+y) \\ &+ 2\varepsilon\zeta\sqrt{(1+y)(1+y(s-s_c))}\cos(C - \phi(s-s_c) + \phi), \end{aligned} \tag{6.87}$$

$$\begin{aligned} \phi' =& \alpha\frac{1}{2}(u_e + u_h) \\ &- \varepsilon\zeta\sqrt{\frac{1+y(s-s_c)}{1+y}}\sin(C - \phi(s-s_c) + \phi), \end{aligned} \tag{6.88}$$

$$\begin{aligned} u_e' =& -\frac{1}{2}y - \varepsilon(s_e^{in} + s_e^{out})u_e \\ &- \varepsilon(u_e + u_h)I^* - \varepsilon(u_e N_h^* + N_e^* u_h) + O(\gamma), \end{aligned} \tag{6.89}$$

$$\begin{aligned} u_h' =& -\frac{1}{2}y - a u_h \\ &- \varepsilon(u_e + u_h)I^* - \varepsilon(u_e N_h^* + N_e^* u_h) + O(\gamma) \end{aligned} \tag{6.90}$$

where prime means differentiation with respect to the dimensionless time $s \equiv \omega t' = \omega t/\tau_p$ and

$$\omega \equiv \sqrt{2\gamma I^*} \tag{6.91}$$

is the RO frequency of the solitary laser. Equation (6.91) is identical to ω^{QW} given by (6.37) and ω^{QD} given by (6.67). I^*, N_e^*, N_h^* are dimensionless steady state values of the solitary laser that need to be computed numerically. The new feedback amplitude $\zeta = O(1)$, the delay s_c the small parameter ε, and a are defined by

$$\zeta \equiv \frac{k}{\gamma},\ s_c \equiv \omega\tau, \varepsilon \equiv \sqrt{\frac{\gamma}{2I^*}},\ \text{and} \tag{6.92}$$

$$a \equiv \varepsilon(s_h^{in} + s_h^{out}). \tag{6.93}$$

The dimensionless scattering rates that also need to be computed numerically are denoted by s_e^{in}, s_e^{out}, s_h^{in}, s_h^{out}. As we shall now demonstrate, valuable information can be extracted from these equations on the basis of simple scaling assumptions.

6.7.1
External cavity modes

The basic solutions are the external cavity modes (ECMs). Analog to Sec. 6.6.1 they are defined as the steady state solutions for y, u_e, u_h, and a phase that changes linearly in time

$$\phi = -C\frac{s}{s_c} + \Delta\frac{s}{s_c} \tag{6.94}$$

with ECM frequency $\Delta \equiv \sigma s_c$. From (6.87) and (6.88), we find that Δ satisfies the following transcendental equation

$$\Delta = C - \varepsilon\zeta s_c\,(\alpha\cos(\Delta) + \sin(\Delta)) \tag{6.95}$$

which implies that $\Delta \simeq C$ as $\varepsilon \to 0$, i.e. Δ is independent of the feedback amplitude ζ, in first approximation. For the subsequent asymptotics we write

$$\Delta = \Delta_0 + O(\epsilon) \tag{6.96}$$

with $\Delta_0 = C$. From (6.87), we also note that

$$u_e + u_h = -2\varepsilon\zeta\cos(\Delta) \tag{6.97}$$

which indicates that both u_e and u_h are $O(\varepsilon)$ small. From Eq. (6.89), we then find that y is $O(\varepsilon^2)$ small. These scaling laws for u_e, u_h, and y are useful when we reorganize the coefficients of the characteristic equation in powers of ε. Three cases were explored in [26] which we now examine.

6.7.2

Stability

From the linearized equations, we determine the following condition for the growth rate μ:

$$\begin{vmatrix} \begin{pmatrix} -\varepsilon\zeta\cos(\Delta)F \\ -\mu \end{pmatrix} & \begin{pmatrix} -2\varepsilon\zeta(1+y) \\ \times\sin(\Delta)F \end{pmatrix} & 1+y & 1+y \\ \varepsilon\zeta\frac{\sin(\Delta)}{2(1+y)}F) & \begin{pmatrix} -\varepsilon\zeta\cos(\Delta)F \\ -\mu \end{pmatrix} & \frac{\alpha}{2} & \frac{\alpha}{2} \\ -\frac{1}{2} & 0 & \begin{pmatrix} -\epsilon\begin{pmatrix} s_e^{in}+s_e^{out} \\ +I^*+N_h^* \end{pmatrix} \\ -\mu \end{pmatrix} & -\varepsilon\left(I^*+N_e^*\right) \\ -\frac{1}{2} & 0 & -\varepsilon\left(I^*+N_h^*\right) & \begin{pmatrix} -a \\ -\varepsilon(I^*+N_e^*) \\ -\mu \end{pmatrix} \end{vmatrix} = 0 \tag{6.98}$$

where

$$F \equiv 1 - e^{-\mu s_c}. \tag{6.99}$$

Expanding the determinant, we obtain

$$\begin{aligned} &\mu^4 + \mu^3\left[(s_e^{in}+s_e^{out}+2I^*+N_h^*+N_e^*)\varepsilon + a + 2\varepsilon\zeta\cos(\Delta)F\right] \\ &-\mu^2\left[-(1+y) - \left[\begin{array}{c} \varepsilon\left(s_e^{in}+s_e^{out}+I^*+N_h^*\right)\left(a+\varepsilon(I^*+N_e^*)\right) \\ -\varepsilon^2\left(I^*+N_h^*\right)\left(I^*+N_e^*\right)+\varepsilon^2\zeta^2F^2 \\ +2\varepsilon\zeta\cos(\Delta)F\left(a+\varepsilon(2I^*+N_e^*+N_h^*+s_e^{in}+s_e^{out})\right) \end{array}\right]\right] \\ &+\mu\left[\begin{array}{c} \varepsilon^2\zeta^2F^2\left(a+\varepsilon(2I^*+N_e^*+s_e^{in}+s_e^{out}+N_h^*)\right) \\ +2\varepsilon\zeta\cos(\Delta)F\left[\varepsilon\begin{pmatrix} s_e^{in}+s_e^{out} \\ +I^*+N_h^* \end{pmatrix}\begin{pmatrix} a \\ +\varepsilon(I^*+N_e^*) \\ -\varepsilon^2(I^*+N_h^*)\left(I^*+N_e^*\right) \end{pmatrix}\right] \\ -\varepsilon\zeta(1+y)\sin(\Delta)F\alpha \\ +2\varepsilon\zeta\cos(\Delta)F\frac{1}{2}(1+y)+\frac{1}{2}(1+y)\left[a+(s_e^{in}+s_e^{out})\varepsilon\right] \end{array}\right] \\ &+\varepsilon^2\zeta^2F^2\left[\varepsilon\begin{pmatrix} s_e^{in}+s_e^{out} \\ +I^*+N_h^* \end{pmatrix}\begin{pmatrix} a+\varepsilon I^* \\ +\varepsilon N_e^* \end{pmatrix} - \varepsilon^2\left(I^*+N_h^*\right)\left(I^*+N_e^*\right)\right] \\ &\qquad -2\varepsilon\zeta(1+y)\sin(\Delta)F\frac{\alpha}{4}\left[a+(s_e^{in}+s_e^{out})\varepsilon\right] \\ &\qquad +\varepsilon\zeta\cos(\Delta)F\frac{1}{2}(1+y)\left[a+(s_e^{in}+s_e^{out})\varepsilon\right]. \end{aligned} \tag{6.100}$$

6.7.3
Similar carrier lifetimes τ_e and τ_h (case S)

We seek a solution of the form

$$\mu = \mu_0 + \varepsilon\mu_1 + \ldots \tag{6.101}$$

and assume that $s_h^{in} + s_h^{out} = O(1)$. Inserting (6.93), (6.96) and (6.101) into (6.100), we obtain the following sequence of problems for μ_0 and μ_1

$$O(1) : \mu_0^4 + \mu_0^2 = 0, \tag{6.102}$$

$$\begin{aligned} O(\varepsilon) : & 4\mu_0^3\mu_1 + 2\mu_0\mu_1 \\ & + \mu_0^3 \left[s_e^{in} + s_e^{out} + 2I^* + N_h^* + N_e^* + s_h^{in} + s_h^{out} + 2\zeta\cos(\Delta_0)F_0 \right] \\ & + \mu_0 \left[\begin{array}{c} -\zeta\sin(\Delta_0)F_0\alpha \\ +\zeta\cos(\Delta_0)F_0 + \frac{1}{2}(s_h^{in} + s_h^{out} + s_e^{in} + s_e^{out}) \end{array} \right] = 0, \end{aligned} \tag{6.103}$$

where we have introduced

$$F_0 \equiv 1 - e^{-\mu_0 s_c}. \tag{6.104}$$

The solution of Eq. (6.102) is

$$\mu_0^2 = -1$$

and from (6.103), we then obtain

$$\mu_1 = -\Gamma - \frac{1}{2}\zeta F_0(\cos(\Delta_0) + \sin(\Delta_0)\alpha) \tag{6.105}$$

where

$$\Gamma \equiv \frac{1}{2}\left[\frac{s_e^{in} + s_e^{out}}{2} + 2I^* + N_h^* + N_e^* + \frac{s_h^{in} + s_h^{out}}{2}\right] \tag{6.106}$$

is the damping rate of the solitary laser [26]. Using (6.99) and $\lambda_0 = i$, (6.105) then implies that

$$\begin{aligned} \mathrm{Re}(\mu_1) &= -\Gamma - \frac{1}{2}\zeta(1 - \cos(s_c))(\cos(\Delta_0) + \sin(\Delta_0)\alpha) \\ &= -\Gamma - \zeta\sin^2(\frac{s_c}{2})(\cos(\Delta_0) + \sin(\Delta_0)\alpha). \end{aligned} \tag{6.107}$$

The stability condition now is

$$\zeta < \zeta_H \equiv -\frac{\Gamma}{\sin^2(\frac{s_c}{2})(\cos(\Delta_0) + \sin(\Delta_0)\alpha)}, \tag{6.108}$$

if $\alpha\sin(\Delta_0) + \cos(\Delta_0) < 0$. The lowest possible value for ζ_H is for

$$\Delta_0 = C = \pi + \arctan(\alpha) \text{ and } s_c = \pi (\mathrm{mod}\, 2\pi).$$

It is given by

$$\zeta_c \equiv \frac{\Gamma}{\sqrt{1+\alpha^2}}. \tag{6.109}$$

In terms of the original parameters, the stability condition is the same as for (6.71)-(6.72) with Γ^S replacing Γ_1^{QD} where

$$\Gamma^S \equiv \frac{\gamma}{2}\left[\frac{s_e^{in} + s_e^{out}}{2} + 2I^* + N_h^* + N_e^* + \frac{s_h^{in} + s_h^{out}}{2}\right]. \tag{6.110}$$

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6.7.4 Different carrier lifetimes τ_e and τ_h and $a = O(1)$

Assuming that $s_h^{in} + s_h^{out} = O(\varepsilon^{-1})$ or equivalently $a = O(1)$, we now find from (6.100) that μ_0 satisfies

$$O(1) : \mu_0^4 + \mu_0^3 a + \mu_0^2 + \mu_0 \frac{a}{2} = 0 \tag{6.111}$$

which is analyzed in [26]. We note that ζ does not appear in (6.111) meaning that the feedback is too weak ($k = O(\gamma)$) to have an effect in this case.

6.7.5 Very small scattering lifetime of the holes ($a = O(\epsilon^{-1})$)

We next assume that

$$a = \frac{a_1}{\varepsilon} \tag{6.112}$$

where $a_1 \equiv \varepsilon^2(s_h^{in} + s_h^{out}) = O(1)$. Inserting (6.112) into (6.100), we now find the following problems for μ_0 and μ_1

$$O(\varepsilon^{-1}) : a_1\mu_0^3 + \mu_0 \frac{a_1}{2} = 0 \tag{6.113}$$

$$\begin{aligned} O(1) : {} & 3a_1\mu_0^2\mu_1 + \mu_1 \frac{a_1}{2} + \mu_0^4 \\ & - \lambda_0^2 \left[-1 - \left(s_e^{in} + s_e^{out} + I^* + N_h^*\right) a_1 - 2\zeta \cos(\Delta_0) F_0 a_1\right] \\ & - 2\zeta \sin(\Delta_0) F_0 \frac{\alpha}{4} a_1 + \zeta \cos(\Delta_0) F_0 \frac{1}{2} a_1 = 0 \end{aligned} \tag{6.114}$$

where F_0 is defined in (6.104). The solution of (6.113) is

$$\mu_0^2 = -\frac{1}{2}$$

and from (6.114), we then obtain

$$\mu_1 = -\Gamma - \frac{\zeta}{2} F_0(\cos(\Delta_0) + \sin(\Delta_0)\alpha) \tag{6.115}$$

where

$$\Gamma \equiv \frac{1}{2}\left[\frac{1}{2a_1} + s_e^{in} + s_e^{out} + I^* + N_h^*\right] \tag{6.116}$$

is the damping rate of the ROs for the solitary laser [26]. Using (6.99) and $\mu_0 = i/\sqrt{2}$, (6.115) implies

$$\begin{aligned} \mathrm{Re}(\mu_1) &= -\Gamma - \frac{1}{2}\zeta(1 - \cos(s_c))(\cos(\Delta_0) + \sin(\Delta_0)\alpha) \\ &= -\Gamma - \zeta \sin^2(\frac{s_c}{2\sqrt{2}})(\cos(\Delta_0) + \sin(\Delta_0)\alpha). \end{aligned} \tag{6.117}$$

The stability conditions are the same as for (6.71)-(6.72) with Γ^{Da} replacing Γ_1^{QD} where

$$\Gamma^{Da} \equiv \frac{\gamma}{2}\left[\frac{1}{2a_1} + s_e^{in} + s_e^{out} + I^* + N_h^*\right]. \tag{6.118}$$

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