Dry atmosphere asymptotics

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Abstract

Understanding and computing motions in the atmosphere is particularly challenging because of the multitude of physical processes and of the different space and time scales involved. Nonetheless, atmospheric motion takes place in a very special regime: The Mach number M, the ratio of the wind speed to the speed of sound, varies from zero at rest to 0.3 in fast jet streams, hurricanes and tornadoes. Unfortunately, in the $M \to 0$ limit, the fully compressible fluid equations which govern atmospheric motion become singular. Because of this, numerical schemes for these equations exhibit a dramatic breakdown in efficiency and accuracy in the low Mach number regime.

We propose multiple scale low Mach number / low Froude number asymptotic analysis as a general framework for understanding the motion in the atmosphere on space scales ranging from a few meters to thousands of kilometers. The theory provides a consistent picture of slow atmospheric flows and turns out to be the natural framework in which popular approximations, traditionally obtained on the basis of simplifying assumptions or ad hoc scaling arguments, can be derived.

The analysis yields strong implications for discrete methods aiming at the numerical computation of atmosphere motions as, e.g., in numerical weather forecasting or climate modeling. There are two kinds of implications. On one hand one finds constraints on parameterizations of unresolved physical processes such as those for turbulent heat or for for velocity boundary conditions. These constraints apply to the the continuous as well as the discrete equations. On the other hand the asymptotic analysis suggests how to design efficient methods for computing low speed flows in the atmosphere. The theory provides guidelines for consistently "filtering" the equations in the various regimes of length and time scales and shows that the "dynamic range problem" can be overcome by introducing suitable multiple variables that mimic the asymptotic representation of the field variables.

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Contents

1 Introduction

Understanding and computing motions in the atmosphere is particularly challenging because of the multitude of physical processes and of the different space and time scales involved. Very little is known of the interactions which take place between the different scales and whether, how and in which measure small scale motion affects the large scale dynamics and vice versa.

This knowledge, however, is crucial for the construction of accurate and efficient numerical methods for climate research, weather forecasting, regional modeling, accident simulation and other applications.

In spite of its physical complexity and variety, atmospheric motion is confined in a very special regime. The Mach number M, the ratio of the wind speed to the speed of sound, varies from zero at rest to 0.3 in fast jet streams, hurricanes and tornadoes. Unfortunately, in the $M \to 0$ limit, the fully compressible fluid equations which govern atmospheric motion become singular. These equations are essentially conservation laws for mass,momentum and energy for a multi-phase compressible flow. They are equipped with some empirical or semi-empirical description of the macroscopical effects of unresolved physical processes (the so-called "parameterizations") like turbulent heat transfer.

It is because of the singularity of the governing equations in the limit $M \to 0$ that pressure differences as small as M^2 can generate $O(1)$ accelerations of the horizontal winds. The impact on numerical methods is dramatic. In fact the problem of extracting the "right" net effect out of an almost perfect balance is a very common one in weather forecasting, see, e.g., [12] page 187. Indeed, both theoretical and numerical investigations of atmospheric motion make use on approximate *models* of the governing equations. Approximate models allow a simplification of the governing equations by assuming some kind of balance: *hydrostatic*, *geostrophic*, *Boussinesq*, *anelastic*, *pseudo-incompressible* are popular key-words used to describe simplifying assumptions, subsets of the governing equations, numerical models or, in general, a particular regime or kind of motion. For most models no rigorous derivation from the governing equations is known. Therefore, neither quantitative knowledge of the range of validity of the models nor error estimates are available.

In spite of the outstanding role of approximate models, modern computational approaches in numerical weather forecasting turn back to discrete forms of the original conservation laws, see [2]. Some reasons are:

1. The governing equations are believed to allow a better description of real motions than approximate models do.

2. One would like to design numerical methods which are valid on a wide range of scales. Such methods could then be coupled with local mesh refinement techniques for regional weather forecasting. Since very little is known about the range of validity of approximate models one prefers to use the original conservation laws which are believed to hold on a very wide range of scales.

3. In the numerical computation of atmospheric motions efficiency is a crucial issue. Both numerical weather forecasting and climate research need fast algorithms. If one knows how to “filter” the full equations to avoid time step restrictions due to fast modes, the computational cost of integrating the full three dimensional Navier-Stokes equations with explicit or semi-implicit methods is significantly lower than the cost of numerically solving the equation of, e.g. an anelastic model. The last approach would require the solution of a three-dimensional elliptic problem at each time step. The advantages of explicit methods are particularly evident if computations are performed on massive parallel distributed memory architectures.

When attempting to compute numerical solutions of the original conservation laws on different scales, however, one has to face two major problems. The first problem is a consequence of the above mentioned singularity of the $M \to 0$ limit. It is well known that numerical methods designed for the full compressible governing equations exhibit a breakdown of both accuracy and efficiency in the low Mach number regime, see [15], [9], [8], [14], [13]. This “dynamic range problem” is probably the most severe obstacle in using numerical methods for the full governing equations in the computation of atmospheric motions: the above mentioned “filtering” problem is, in fact, far from having been solved.

The second problem is that as the physical scales change one has to adjust the *parameterizations* of the sub-cell processes according to the new scaling. Often the models used to represent unresolved processes like turbulent heat transport, condensation/evaporation, radiative heat transfer and chemical reactions are heavily based on empirical relations and data fitting ([2], [11]). Such parameterizations have been designed (and are believed to be valid) for a given scaling or even for a well defined grid size. Tautologically, there is very little knowledge on how parameterizations should depend on the grid size. It has been observed in climate research and weather forecasting that, for a given model, a simple grid refinement by a factor of two (a common practice in the experimental estimation of the accuracy of a numerical method) may have disastrous consequences on the solution.

In the scientific community there is a broad awareness of the importance of these two problems. Nevertheless we do not know any systematic attempt to obtain a quantitative picture of the relevance of dynamic range problems in e.g. numerical weather forecasting. Nor are we aware of any attempt to understand what are the implications of the $M \to 0$ limit on parameterizations. We believe that multi scale asymptotics can be usefully applied to answer these questions.

We propose multiple scale low Mach number / low Froude number asymptotic analysis as a general framework for understanding the motion in the atmosphere on space scales ranging from a few meters to thousands of kilometers. The theory provides a consistent picture of slow atmospheric flows and turns out to be the natural framework in which popular approximations, traditionally obtained on the basis of simplifying assumptions or ad hoc scaling arguments, can be derived. The analysis yields strong implications for discrete methods aiming at the numerical computation of atmospheric motions as, e.g., in numerical weather prediction or climate modeling. There are two kinds of implications. On one hand one finds side constraints on parameterizations of unresolved physical processes such as turbulent heat transport or velocity boundary conditions. These constraints apply to the continuous as well as the discrete equations. On the other hand the asymptotic analysis suggests how to design efficient and accurate methods for computing low speed flows in the atmosphere. The theory provides guidelines for consistently "filtering" the equations in the various regimes of length and time scales and suggests that the "dynamic range problem" can be overcome by introducing suitable multiple variables that mimic the asymptotic representation of the field variables. These applications will be addressed in a follow up publication.

This report is organized as follows: in the rest of this section the governing equations are introduced. We discuss a few general assumptions and introduce the basic concepts and the notation used in the analysis. In section 2 we shortly review two classical approaches used in the derivation of approximate theories and models of atmospheric motion. Sections 3,4 and 5 are focused on the asymptotic analysis of atmospheric motions on micro, meso and synoptic scales. In section 6 we draw some conclusions and outline future work.

1.1 Dimensional equations

The equations governing a dry atmosphere can be written as follows

$$\begin{aligned}
&\rho'_{t'} + \nabla' \cdot (\rho' \boldsymbol{v}') = 0 \\
&(\rho' \boldsymbol{v}')_{t'} + \nabla' \cdot (\rho' \boldsymbol{v}' \circ \boldsymbol{v}') + \nabla' p' + \rho' g' \boldsymbol{k} + \rho' 2\Omega' \boldsymbol{\Omega} \times \boldsymbol{v}' = \boldsymbol{D}'_{\rho' \boldsymbol{v}'} \\
&(\rho' e')_{t'} + \nabla' \cdot ((\rho' e' + p') \boldsymbol{v}') = D'_{\rho' e'} \\
&p' = (\gamma - 1)(\rho' e' - \frac{1}{2} \rho' \boldsymbol{v}' \cdot \boldsymbol{v}' - \rho' g' z')
\end{aligned} \tag{1}$$

where ρ, $\boldsymbol{v}$, p and e represent density, velocity, pressure and internal energy, respectively and γ is the ratio of specific heats. Here primes denote dimensional variables. We will consider a simplified geometrical setup in which the acceleration of gravity, g, acts in the vertical direction $\boldsymbol{k}$ of a Cartesian frame of reference of coordinates x, y, z and unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. This frame of reference rotates with constant angular velocity Ω' around the axis defined by the unit vector $\boldsymbol{\Omega} := \boldsymbol{j}\cos\varphi + \boldsymbol{k}\sin\varphi$. $\boldsymbol{D}'_{\rho'\boldsymbol{v}'}$ represents the effects of microscopical transport of momentum. $D'_{\rho' e'}$ accounts for both microscopical transport of energy and for diabatic heating (chemical reactions and radiation). These processes are usually *parameterized* by means of empirical relationships or models.

1.2 Hydrostatic balance, scale height

An atmosphere is said to be in hydrostatic balance when the vertical pressure gradient balances the force of gravity

$$\frac{\partial p'}{\partial z'} = -\rho' g' \tag{2}$$

Let p'_{ref} be a reference pressure (e.g. some mean sea level pressure). The scale height h'_{scale}

$$h'_{\text{scale}} := \frac{p'_{\text{ref}}}{\rho'_{\text{ref}} g'} \tag{3}$$

represents the height in which pressure changes of magnitude p'_{ref} would occur in an atmosphere in hydrostatic balance and of constant density ρ'_{ref}. Generally, the scale height is a proper measure of the vertical distances over which relative pressure variations of order unity arise.

1.3 Stability and entropy stratification

For $\boldsymbol{v}' = 0$ one has $\boldsymbol{D}'_{\rho'\boldsymbol{v}'} = 0$. Thus an atmosphere at rest is in hydrostatic balance. This balance may be stable, neutral or unstable depending on whether

$$\frac{\partial \rho'}{\partial z'} - \left.\frac{\partial \rho'}{\partial z'}\right|_s \tag{4}$$

is less than, equal to or greater than zero. Here s is the entropy. The second term of (4) represents the isentropic (i.e. adiabatic *and* reversible) rate of change of density with height. The rate of change of entropy with height is

$$\frac{\partial s'}{\partial z'} = c_v' \left(\frac{1}{p'} \frac{\partial p'}{\partial z'} - \gamma \frac{1}{\rho'} \frac{\partial \rho'}{\partial z'} \right) \tag{5}$$

Thus the isentropic rate of change of density with height reads

$$\left. \frac{\partial \rho'}{\partial z'} \right|_s = \frac{\rho'}{\gamma p'} \frac{\partial p'}{\partial z'} \tag{6}$$

Using (6) in (4) one has

$$\frac{\partial \rho'}{\partial z'} - \left. \frac{\partial \rho'}{\partial z'} \right|_s = \frac{\partial \rho'}{\partial z'} - \frac{\rho'}{\gamma p'} \frac{\partial p'}{\partial z'} = -\frac{\rho'}{\gamma} \left(\frac{1}{p'} \frac{\partial p'}{\partial z'} - \frac{\gamma}{\rho'} \frac{\partial \rho'}{\partial z'} \right) = -\frac{\rho'}{\gamma c_v} \frac{\partial s'}{\partial z'} \tag{7}$$

The last equation shows that the stability of the atmosphere depends on the sign of $\partial s'/\partial z'$. One has

$$\begin{aligned} \partial s'/\partial z' > 0 \quad &\Rightarrow \quad \text{stable} \\ \partial s'/\partial z' = 0 \quad &\Rightarrow \quad \text{neutral} \\ \partial s'/\partial z' < 0 \quad &\Rightarrow \quad \text{unstable} \end{aligned}$$

For a stable atmosphere a widely used stability measure is the buoyancy (Brunt-Väisälä) frequency

$$N'^2 := -g' \left(\frac{1}{\rho'} \frac{\partial \rho'}{\partial z'} + \frac{g' \rho'}{\gamma p'} \right) \tag{8}$$

From (7) and the hydrostatic balance (2) one has

$$N'^2 = \frac{g'}{\gamma c_v'} \frac{\partial s'}{\partial z'} \tag{9}$$

1.4 Reference values, dimensionless equations

Equations (1) can be made dimensionless by introducing the following set of reference variables

$$\rho'_{\text{ref}}, \; v'_{\text{ref}}, \; p'_{\text{ref}}, \; g'_{\text{ref}}, \; l'_{\text{ref}}, \; t'_{\text{ref}} := l'_{\text{ref}}/v'_{\text{ref}} \tag{10}$$

The resulting system of equations reads

$$\begin{aligned}
&\rho_t + \nabla\cdot(\rho\boldsymbol{v}) = 0\\
&(\rho\boldsymbol{v})_t + \nabla\cdot(\rho\boldsymbol{v}\circ\boldsymbol{v}) + \frac{p'_{\rm ref}}{\rho'_{\rm ref}v'^2_{\rm ref}}\nabla p + \frac{g'_{\rm ref}l'_{\rm ref}}{v'^2_{\rm ref}}\rho g\boldsymbol{k} + \frac{2\Omega' l'_{\rm ref}}{v'_{\rm ref}}\rho\boldsymbol{\Omega}\times\boldsymbol{v} = \boldsymbol{D}_{\rho\boldsymbol{v}}\\
&(\rho e)_t + \nabla\cdot((\rho e + p)\boldsymbol{v}) = D_{\rho e}\\
&p = (\gamma - 1)(\rho e - \frac{1}{2}\frac{v'^2_{\rm ref}\rho'_{\rm ref}}{p'_{\rm ref}}\rho\boldsymbol{v}\cdot\boldsymbol{v} - \frac{\rho'_{\rm ref}g'_{\rm ref}l'_{\rm ref}}{p'_{\rm ref}}\rho g z)
\end{aligned} \tag{11}$$

These equations explicitly depend on the following dimensionless parameters

$$M := \frac{v'_{\rm ref}}{\sqrt{\frac{p'_{\rm ref}}{\rho'_{\rm ref}}}} \qquad Fr := \frac{v'_{\rm ref}}{\sqrt{g'_{\rm ref}l'_{\rm ref}}} \qquad Ro := \frac{v'_{\rm ref}}{2\Omega' l'_{\rm ref}} \tag{12}$$

Other dimensionless parameters (e.g. a Reynolds number) are hidden in the right hand side of (11).

1.5 General assumptions

We analyze (11) in the singular regime $M \to 0$ under the following assumptions.

(A1) The reference length $l'_{\rm ref}$ is equal to the scale height $h'_{\rm scale}$.

(A2) The right hand side of (11) contains terms which are at most $O(1)$ as $M \to 0$ (no singular perturbation in the right hand side).

(A3) The Rossby number Ro is $O(1)$ as $M \to 0$.

The first assumption implies that $Fr = M$ and, taking into account (A3), we are left with a system of equations in only one singular parameter. The first assumption also implies that, for a reference velocity of the order of 10 m/s, the reference time $t'_{\rm ref}$ is of about 20 minutes. This is quite a short time in meteorology. According to [12] it is about two order of magnitude shorter than the characteristic lifetime of meteorological phenomena on the so-called “synoptic scale” and about three order of magnitude shorter than the lifetime of phenomena on the planetary scale.

20 minutes is the characteristic time associated to meteorological phenomena of horizontal and vertical extension of about 10 kilometers such as, e.g., cumulonimbus convection. It is the relevant time scale for numerical

weather prediction in the sense that it can be fully resolved with time step sizes which are or are to be standard in high resolution numerical methods for weather forecasting, see [2].

1.6 Governing equations

Using the dimensionless parameters (12) and the first of the above assumptions the governing equations (11) can be written as

$$\begin{aligned}
&\rho_t + \nabla\cdot(\rho\boldsymbol{v}) = 0\\
&(\rho\boldsymbol{v})_t + \nabla\cdot(\rho\boldsymbol{v}\circ\boldsymbol{v}) + \frac{1}{M^2}\nabla p + \frac{1}{M^2}\rho g\boldsymbol{k} + \frac{1}{Ro}\rho\boldsymbol{\Omega}\times\boldsymbol{v} = \boldsymbol{D}_{\rho\boldsymbol{v}}\\
&(\rho e)_t + \nabla\cdot((\rho e + p)\boldsymbol{v}) = D_{\rho e}\\
&p = (\gamma-1)(\rho e - \frac{1}{2}M^2\rho\boldsymbol{v}\cdot\boldsymbol{v} - \rho g z)
\end{aligned} \tag{13}$$

From (13) one can derive an evolution equation for the entropy

$$s := p\rho^{-\gamma} \tag{14}$$

This equation reads

$$s_t + \boldsymbol{v}\cdot\nabla s = \rho^{-\gamma}(\gamma-1)\left(D_{\rho e} - M^2\boldsymbol{v}\cdot\boldsymbol{D}_{\rho\boldsymbol{v}}\right) \tag{15}$$

1.7 Horizontal and vertical directions

In the analysis of (13)-(15) it is useful to split the momentum equation into a horizontal and a vertical component. With the following notation:

$$\boldsymbol{v} := \boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k}w =: \boldsymbol{u} + \boldsymbol{k}w \tag{16}$$

$$\nabla\cdot\boldsymbol{v} := u_x + v_y + w_z =: \nabla_{\shortparallel}\cdot\boldsymbol{u} + w_z \tag{17}$$

$$\begin{aligned}
\nabla\cdot(\boldsymbol{v}\circ\boldsymbol{v}) := {}& \boldsymbol{i}\nabla\cdot(\boldsymbol{i}u^2 + \boldsymbol{j}uv + \boldsymbol{k}uw) +\\
&\boldsymbol{j}\nabla\cdot(\boldsymbol{i}vu + \boldsymbol{j}v^2 + \boldsymbol{k}vw) +\\
&\boldsymbol{k}\nabla\cdot(\boldsymbol{i}wu + \boldsymbol{j}wv + \boldsymbol{k}w^2) =\\
={}& \boldsymbol{i}\left(\nabla_{\shortparallel}\cdot(\boldsymbol{u}u) + (uw)_z\right) +\\
&\boldsymbol{j}\left(\nabla_{\shortparallel}\cdot(\boldsymbol{u}v) + (vw)_z\right) +\\
&\boldsymbol{k}\left(\nabla_{\shortparallel}\cdot(\boldsymbol{u}w) + (w^2)_z\right) =:\\
=:{}& \nabla_{\shortparallel}\cdot(\boldsymbol{u}\circ\boldsymbol{u}) + (\boldsymbol{u}w)_z + \boldsymbol{k}\left(\nabla_{\shortparallel}\cdot(\boldsymbol{u}w) + (w^2)_z\right)
\end{aligned} \tag{18}$$

$$\nabla p := \boldsymbol{i}p_x + \boldsymbol{j}p_y + \boldsymbol{k}p_z := \nabla_{\shortparallel} p + \boldsymbol{k}p_z \tag{19}$$

$$\begin{aligned}\boldsymbol{\Omega}\times\boldsymbol{v} &= \boldsymbol{i}w\cos\varphi - \boldsymbol{i}v\sin\varphi + \boldsymbol{j}u\sin\varphi - \boldsymbol{k}u\cos\varphi \\ &=: \boldsymbol{u}^{\perp}\sin\varphi + \boldsymbol{i}w\cos\varphi - \boldsymbol{k}u\cos\varphi\end{aligned} \tag{20}$$

$$\boldsymbol{D}_{\rho\boldsymbol{v}} =: \boldsymbol{D}_{\rho\boldsymbol{u}} + \boldsymbol{k}D_{\rho w} \tag{21}$$

and $\mathcal{C}$, $\mathcal{S}$ for $\cos\varphi$, $\sin\varphi$ respectively the system (13) can be written as

$$\begin{aligned}
&\rho_t + \nabla_{\shortparallel}\cdot(\rho\boldsymbol{u}) + (\rho w)_z = 0 \\
&(\rho\boldsymbol{u})_t + \nabla_{\shortparallel}\cdot(\rho\boldsymbol{u}\circ\boldsymbol{u}) + (\rho\boldsymbol{u}w)_z + \frac{1}{M^2}\nabla_{\shortparallel} p + \frac{1}{Ro}\rho\left(\boldsymbol{u}^{\perp}\mathcal{S} + \boldsymbol{i}w\mathcal{C}\right) = \boldsymbol{D}_{\rho\boldsymbol{u}} \\
&(\rho w)_t + \nabla_{\shortparallel}\cdot(\rho\boldsymbol{u}w) + (\rho w^2)_z + \frac{1}{M^2}p_z + \frac{1}{M^2}\rho g - \frac{1}{Ro}\rho u\mathcal{C} = D_{\rho w} \\
&(\rho e)_t + \nabla_{\shortparallel}\cdot((\rho e + p)\boldsymbol{u}) + ((\rho e + p)w)_z = D_{\rho e} \\
&p = (\gamma - 1)(\rho e - \frac{1}{2}M^2\rho\boldsymbol{v}\cdot\boldsymbol{v} - \rho g z)
\end{aligned} \tag{22}$$

Replacing the energy equation with the entropy evolution equation (15) one obtains the following non-conservative form of the governing equations

$$\begin{aligned}
&\rho_t + \nabla_{\shortparallel}\cdot(\rho\boldsymbol{u}) + (\rho w)_z = 0 \\
&(\rho\boldsymbol{u})_t + \nabla_{\shortparallel}\cdot(\rho\boldsymbol{u}\circ\boldsymbol{u}) + (\rho\boldsymbol{u}w)_z + \frac{1}{M^2}\nabla_{\shortparallel} p + \frac{1}{Ro}\rho\left(\boldsymbol{u}^{\perp}\mathcal{S} + \boldsymbol{i}w\mathcal{C}\right) = \boldsymbol{D}_{\rho\boldsymbol{u}} \\
&(\rho w)_t + \nabla_{\shortparallel}\cdot(\rho\boldsymbol{u}w) + (\rho w^2)_z + \frac{1}{M^2}p_z + \frac{1}{M^2}\rho g - \frac{1}{Ro}\rho u\mathcal{C} = D_{\rho w} \\
&s_t + \boldsymbol{u}\cdot\nabla_{\shortparallel} s + w s_z = \rho^{-\gamma}(\gamma - 1)\left(D_{\rho e} - M^2\boldsymbol{v}\cdot\boldsymbol{D}_{\rho\boldsymbol{v}}\right) \\
&p = s\rho^{\gamma}
\end{aligned} \tag{23}$$

1.8 Asymptotic ansatz

Let $\mathcal{U}$ be a shortcut for a solution component or a functional of a solution of the governing equations (23). For instance $\mathcal{U}$ could represent the pressure p or the velocity $\boldsymbol{v}$. In general $\mathcal{U}$ depends on the horizontal coordinate $\boldsymbol{x} := \boldsymbol{i}x + \boldsymbol{j}y$, on the vertical coordinate z, on the time t, on the singular perturbation parameter M and on other dimensionless parameters like Ro, the Reynolds number etc. We focus the attention on the behavior of $\mathcal{U}$ as $M \to 0$ and simply write $\mathcal{U}(\boldsymbol{x}, z, t; M)$ for $\mathcal{U} = \mathcal{U}(\boldsymbol{x}, z, t; M, Ro, \dots)$. We consider special cases of the following asymptotic expansion of $\mathcal{U}$

$$\mathcal{U}(\boldsymbol{x}, z, t; M) := \sum_{i=0} M^i\,\mathcal{U}^{(i)}(\boldsymbol{\eta}, \boldsymbol{x}, \boldsymbol{\xi}, z, \tau, t, \theta) \tag{24}$$

$\boldsymbol{\eta}$ is a new small scale horizontal coordinate e.g. $\boldsymbol{\eta} := \boldsymbol{x}/M$. $\boldsymbol{\xi}$ represents a new large horizontal coordinate. In a similar way τ and θ are new short and long time coordinates. We will not consider any multi-scale ansatz in the vertical direction in this report.

We require the functions $\mathcal{U}^{(i)}$ to satisfy the following *sub-linear growth conditions*

$$\lim_{M\to 0} M\mathcal{U}^{(i)}(\boldsymbol{x}/M, \boldsymbol{x}, M\boldsymbol{x}, z, t/M, t, Mt) = 0 \qquad i = 0, 1, \cdots \tag{25}$$

$$\lim_{M\to 0} M\mathcal{U}^{(i)}(\boldsymbol{\xi}/M^2, \boldsymbol{\xi}/M, \boldsymbol{\xi}, z, \theta/M^2, \theta/M, \theta) = 0 \qquad i = 0, 1, \cdots \tag{26}$$

1.9 Averages, fluctuations

In the analysis of the governing equations (23) we take averages on the domain of definition of the independent variables. The averaged variables are represented by means of over-lining. For instance $\overline{\mathcal{U}^{(i)}}^t$ represents the temporal average of $\mathcal{U}^{(i)}$ on the domain of definition of t. The difference between a variable and its average is called the fluctuation. Thus

$$\widetilde{\mathcal{U}^{(i)}}^t := \mathcal{U}^{(i)} - \overline{\mathcal{U}^{(i)}}^t \tag{27}$$

represents the fluctuation in time of $\mathcal{U}^{(i)}$. We will use the following averages

$$\overline{\mathcal{U}^{(i)}}^{\boldsymbol{x}} = \frac{1}{|\mathcal{D}_{\boldsymbol{x}}|}\int\limits_{\mathcal{D}_{\boldsymbol{x}}} \mathcal{U}^{(i)}\, dx\, dy \qquad |\mathcal{D}_{\boldsymbol{x}}| := \int\limits_{\mathcal{D}_{\boldsymbol{x}}} dx\, dy \tag{28}$$

$$\overline{\mathcal{U}^{(i)}}^{z} = \frac{1}{|\mathcal{D}_{z}|}\int\limits_{\mathcal{D}_{z}} \mathcal{U}^{(i)}\, dz \qquad |\mathcal{D}_{z}| := \int\limits_{\mathcal{D}_{z}} dz \tag{29}$$

$$\overline{\mathcal{U}^{(i)}}^{t} = \frac{1}{|\mathcal{D}_{t}|}\int\limits_{\mathcal{D}_{t}} \mathcal{U}^{(i)}\, dt \qquad |\mathcal{D}_{t}| := \int\limits_{\mathcal{D}_{t}} dt \tag{30}$$

where $\mathcal{D}_{\boldsymbol{x}} \subset \mathbb{R}^2$, $\mathcal{D}_z \subset \mathbb{R}$ and $\mathcal{D}_t \subset \mathbb{R}$ are the domain of definition of $\boldsymbol{x}$, z and t respectively. Notice that, since we are integrating on the whole domain of dependence, averages of fluctuations are identically zero:

$$\overline{\widetilde{\mathcal{U}^{(i)}}^{\boldsymbol{x}}}^{\boldsymbol{x}} = \overline{\widetilde{\mathcal{U}^{(i)}}^{\boldsymbol{x}}}^{z} = \overline{\widetilde{\mathcal{U}^{(i)}}^{\boldsymbol{x}}}^{t} = 0 \tag{31}$$

2 Classical theories, approximate models

As mentioned in the introduction, theoretical and numerical studies of atmospheric motions are usually based on approximate models of the full compressible governing equations.

2.1 Adjustment problems

One class of theories or approximate models can be derived by considering adjustment or perturbation problems. The approach dates back to Laplace (1749-1827) who first derived an equation for the vertical displacement of a material particle on a free surface. The idea is to perturb a known solution of the governing equations, the so-called basic state, and derive evolution equations for the perturbation. The basic state is often the hydrostatic state of rest ([5], chapters 5-7) or some simple geostrophic motion ([12], chapter 8). The amplitude of the perturbation is assumed to be small enough for the effects of quadratic terms in the perturbation quantities to be negligible.

Introducing ad hoc, physically motivated approximations into the system governing the evolution of the perturbation one can select special classes of responses. These motions, e.g. of an incompressible, stratified fluid, shed some light on the structure of solutions of the full equations in a given special regime: namely that one in which the introduced approximations hold. In many cases, for instance, the motion can be decomposed into traveling wave components and one can derive constraints, e.g. in the form of dispersion relationships, for the motion of these components.

As an example consider the equations which describe the evolution of small perturbations of the rest state in a non-rotating, compressible, inviscid, dry atmosphere. According to [5], pages 169-177, these are:

$$\begin{aligned}\frac{\partial^2 p}{\partial t^2} - c_0^2\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) &= \rho_0 \frac{\partial}{\partial t}\left(gw - c_0^2 \frac{\partial w}{\partial z}\right) \\ c_0^2 \frac{\partial^2 w}{\partial t^2} + N_0^2 c_0^2 w &= -\frac{1}{\rho_0}\frac{\partial}{\partial t}\left(gp + c_0^2 \frac{\partial p}{\partial z}\right)\end{aligned} \tag{32}$$

where N_0, c_0^2 are the buoyancy frequency and the speed of sound of the rest state, respectively

$$N_0^2 := -g\left(\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} + g c_0^2\right) \qquad c_0^2 := \gamma \frac{p_0}{\rho_0} \tag{33}$$

These equations support special solutions consisting of acoustic waves, internal gravity waves and the Lamb wave. In section 5 we will show that a system of equations similar to (32) naturally emerges, in the framework of a two-scales asymptotic analysis of atmospheric motions, as an evolution equation for the large scale motion of first order perturbation quantities. However, in the present framework the leading order solution (the analog to the basic state) is neither at rest nor stationary and has, in general, non trivial structure both on meso and on synoptic scales. Therefore, we find, in contrast to (32), additional forcing terms. These terms account for the effects of rotation, of non-trivial meso scale motion, heat transfer and friction on the large scale motion of the first order perturbation.

2.2 Scale analysis

A second class of models for atmospheric motion is obtained through "scale analysis". In meteorology scale analysis is usually understood as a procedure to derive simplified systems of equations which describe with suitable accuracy atmospheric motions in some particular regime, [12]. The procedure is very straight-forward: the governing equations are re-written in non-dimensional quantities by introducing suitable reference variables. These variables are chosen in such a way that the dimensionless quantities are functions having values about 1. The procedure leads to a certain number of dimensionless characteristic numbers just as the Froude number, the Mach number and the Rossby number in equation (11). The idea is then to consider special regimes in which some of the characteristic numbers are small w.r.t. the others by simply neglecting those terms of the governing equations which are multiplied by the small parameters. As an example consider the horizontal component of the dimensionless momentum equation $(11)_2$:

$$(\rho \boldsymbol{u})_t + \nabla_{\shortparallel} \cdot (\rho \boldsymbol{u} \circ \boldsymbol{u}) + (\rho \boldsymbol{u} w)_z + \frac{p'_{\text{ref}}}{\rho'_{\text{ref}} v'^2_{\text{ref}}} \nabla_{\shortparallel} p + \frac{2\Omega' l'_{\text{ref}}}{v'_{\text{ref}}} \rho \left(\boldsymbol{u}^{\perp} \mathcal{S} + \boldsymbol{i} w \mathcal{C} \right) = \boldsymbol{D}_{\rho \boldsymbol{v}} \tag{34}$$

We already know $M << 1$. For $l'_{\text{ref}} >> v'_{\text{ref}}/2\Omega'$, that is l'_{ref} much bigger than about 100 kilometers, the Rossby number Ro is the ratio between v'_{ref} and $2\Omega' l'_{\text{ref}}$ is much smaller than one as well. Thus, one can derive an approximate horizontal momentum equation by simply neglecting the first three terms and the last term of (34). Transforming back to dimensional variables leads to the following equation:

$$\nabla'_{\shortparallel} p' + \rho' 2\Omega' \left(\boldsymbol{u}'^{\perp} \mathcal{S} + \boldsymbol{i} w' \mathcal{C} \right) = 0 \tag{35}$$

This form sheds some light on the role of pressure gradients in the large scale dynamics of the atmosphere. In this regime pressure forces essentially balance Coriolis forces and the resulting horizontal wind $\boldsymbol{u}'$ blows parallel to the iso-bars. This kind of balance is called geostrophic and models based on (35) are called geostrophic models.

The synoptic scale atmospheric motions responsible for weather dynamics are in an approximate geostrophic balance, see, e.g., [7]. Their temporal evolution is governed by the advection of potential vorticity which occurs on time scales much longer than those considered in the present work, see section 1. The present theory could be extended to describe these long times effects by introducing multiple time scale analysis. This is beyond the scope of the present work and left to subsequent publications.

Another family of familiar approximations is obtained by replacing mass conservation $(1)_1$ with one of the following equations:

$$\nabla' \cdot \boldsymbol{v}' = 0 \tag{36}$$

$$\nabla' \cdot (\overline{\rho'}\boldsymbol{v}') = 0 \tag{37}$$

$$\nabla' \cdot (\overline{\rho'\theta'}\boldsymbol{v}') = \frac{D_{\rho' e'} - \boldsymbol{v}' \cdot \boldsymbol{D}_{\rho' \boldsymbol{v}'}}{\overline{\pi'}} \tag{38}$$

$$T' := \frac{p'}{\rho' R'} \qquad \theta' := T' \left(\frac{p'}{p'_0}\right)^{-\frac{R'}{c'_p}} \qquad \pi' := \left(\frac{p'}{p'_0}\right)^{\frac{R'}{c'_p}} \tag{39}$$

According to [3] $\overline{\rho'}$ is a “vertically varying mean-state density”. Similarly $\overline{\theta'}$, $\overline{\pi'}$ are vertically varying mean-state potential temperature and Exner function.

Equation (36)-(38) are called the Boussinesq, anelastic and pseudo incompressible approximations, respectively. The pseudo-incompressible approximation can be derived from the governing equations (1) under two assumptions. These are, see [3],:

$$|\pi' - \overline{\pi'}| << \overline{\pi'}$$

$$\left(\frac{l_{\text{ref}}}{t_{\text{ref}}}\right)^2 << \gamma \frac{p_{\text{ref}}}{\rho_{\text{ref}}}$$

Notice that if $l_{\text{ref}}/t_{\text{ref}} = v_{\text{ref}}$, which is what we are assuming throughout this report, then the second assumption simply means that we are considering motions at low Mach numbers. Equations (37)-(38) are widely used both in data analysis and in numerical methods for weather forecasting. A discussion on the range of applicability of these equations and a comparison between anelastic and quasi-incompressible approximation is presented in [3]. Here we want to stress the following fact: in those regimes in which both the anelastic and the quasi-incompressible approximations are valid one has

$$w' = \frac{D_{\rho' e'} - \boldsymbol{v}' \cdot \boldsymbol{D}_{\rho' \boldsymbol{v}'}}{\overline{\rho'}\, \overline{\pi'} \dfrac{\partial \overline{\theta'}}{\partial z'}}$$

that is the vertical velocity w' is defined in terms of heating and stratification through a simple algebraic relationship! We will encounter different forms of equation (2.2) throughout this report. This equation suggests that, in those regimes in which it holds, atmospheric motion is not truly three-dimensional or, equivalently, that the fluxes of vertical momentum must satisfy some particular constraint for (2.2) to hold.

3 Micro scales: deep convection.

We consider the following particular case of the asymptotic ansatz (24)

$$\mathcal{U}(\boldsymbol{x}, z, t; M) := \sum_{i=0} M^i \mathcal{U}^{(i)}(\boldsymbol{\eta}, z, \tau) \tag{40}$$

$$\boldsymbol{\eta} := \frac{\boldsymbol{x}}{M} \qquad \tau := \frac{t}{M} \tag{41}$$

This ansatz can be embedded in a multiple scale framework in which the solution has non trivial structure on the meso scale and/or on the synoptic scale. In such framework we expect to see non-trivial interactions between the different scales; in particular, we expect small scale correlations to appear as forcing term in the equations which describe motions on larger scales.

Here, however, we are interested in working out the sole microscale physics and we focus on the above single scale ansatz under the general assumptions (A1)-(A3) and the following additional assumptions:

$$\partial_t s^{(0)} = 0 \tag{42}$$

$$\text{The physical domain } \mathcal{D}_{\boldsymbol{\eta}} \text{ is unbounded} \tag{43}$$

The unboundedness of $\mathcal{D}_{\boldsymbol{\eta}}$ together with the sub-linear growth condition (25) imply that in the limit $M \to 0$

$$\overline{\nabla_{\boldsymbol{\eta}} \mathcal{U}^{(i)}}^{\boldsymbol{\eta}} = \frac{1}{|\mathcal{D}_{\boldsymbol{\eta}}|} \int\limits_{\partial \mathcal{D}_{\boldsymbol{\eta}}} \mathcal{U}^{(i)} \boldsymbol{n}\, dL = 0 \tag{44}$$

as one can see by parameterizing $|\mathcal{D}_{\boldsymbol{\eta}}|$, $|\partial\mathcal{D}_{\boldsymbol{\eta}}|$ through M and using the mean value theorem to express the integral of $\mathcal{U}^{(i)}\boldsymbol{n}$ on the boundary of $\mathcal{D}_{\boldsymbol{\eta}}$.

3.1 Asymptotic equations

Introducing the ansatz (40) into the governing equations (23) one obtains the following cascade of asymptotic equations.

3.1.1 Mass

$$\partial_\tau \rho^{(0)} + \nabla_{\boldsymbol{\eta}} \cdot (\rho \boldsymbol{u})^{(0)} = 0 \qquad (C^{-1})$$

$$\partial_\tau \rho^{(1)} + \nabla_{\boldsymbol{\eta}} \cdot (\rho \boldsymbol{u})^{(1)} + \partial_z (\rho w)^{(0)} = 0 \qquad (C^{0})$$

3.1.2 Horizontal momentum

$$\nabla_{\boldsymbol{\eta}} p^{(0)} = 0 \qquad (H^{-3})$$

$$\nabla_{\boldsymbol{\eta}} p^{(1)} = 0 \qquad (H^{-2})$$

$$\partial_\tau (\rho \boldsymbol{u})^{(0)} + \nabla_{\boldsymbol{\eta}} \cdot (\rho \boldsymbol{u} \circ \boldsymbol{u})^{(0)} + \nabla_{\boldsymbol{\eta}} p^{(2)} = D^{(-1)}_{\rho \boldsymbol{u}} \qquad (H^{-1})$$

3.1.3 Vertical momentum

$$\partial_z p^{(0)} = -\rho^{(0)} g \qquad (V^{-2})$$

$$\partial_\tau (\rho w)^{(0)} + \nabla_{\boldsymbol{\eta}} \cdot (\rho \boldsymbol{u} w)^{(0)} + \partial_z p^{(1)} = -\rho^{(1)} g + D^{(-1)}_{\rho w} \qquad (V^{-1})$$

3.1.4 Entropy

$$\partial_\tau s^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}} s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) D^{(-1)}_{\rho e} \qquad (E^{-1})$$

$$\partial_\tau s^{(1)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}} s^{(1)} + \boldsymbol{u}^{(1)} \cdot \nabla_{\boldsymbol{\eta}} s^{(0)} + w^{(0)} \partial_z s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) \left(D^{(0)}_{\rho e} - \gamma \frac{\rho^{(1)}}{\rho^{(0)}} D^{(-1)}_{\rho e} \right) \qquad (E^{0})$$

3.1.5 State: $p = f(\rho, s)$

$$p^{(0)} = s^{(0)} {\rho^{(0)}}^{\gamma} \qquad (S^{0}_{1})$$

$$p^{(1)} = s^{(1)} {\rho^{(0)}}^{\gamma} + {c^{(0)}}^2 \rho^{(1)} \qquad {c^{(0)}}^2 := \frac{\gamma p^{(0)}}{\rho^{(0)}} \qquad (S^{1}_{1})$$

3.1.6 Energy

$$\partial_\tau(\rho e)^{(0)} + \nabla_{\boldsymbol{\eta}} \cdot ((\rho e + p)\boldsymbol{u})^{(0)} = D_{\rho e}^{(-1)} \tag{En^{-1}}$$

$$\partial_\tau(\rho e)^{(1)} + \nabla_{\boldsymbol{\eta}} \cdot ((\rho e + p)\boldsymbol{u})^{(1)} + \partial_z((\rho e + p)\boldsymbol{u})^{(0)} = D_{\rho e}^{(0)} \tag{En^{0}}$$

3.1.7 State: $p = f(\rho, e, \boldsymbol{v})$

$$p^{(0)} = (\gamma - 1)(\rho^{(0)} e^{(0)} - \rho^{(0)} g z) \tag{S_2^0}$$

$$p^{(1)} = (\gamma - 1)(\rho^{(0)} e^{(1)} + \rho^{(1)} e^{(0)} - \rho^{(0)} g z) \tag{S_2^1}$$

3.2 Analysis

3.2.1 Leading order pressure, density and entropy.

The horizontal momentum equations (H^{-3}), (H^{-2}) imply

$$p^{(0)} = p^{(0)}(z, \tau) \tag{45}$$

$$p^{(1)} = p^{(1)}(z, \tau) \tag{46}$$

Through (V^{-2}) the first equations yields

$$\rho^{(0)} = \rho^{(0)}(z, \tau) \tag{47}$$

The state equation (S_1^0) shows that the entropy $s^{(0)}$ does not depend on $\boldsymbol{\eta}$. Recalling the additional assumption (42) one has

$$s^{(0)} = s^{(0)}(z) \tag{48}$$

The state equation (S_1^0) can be used to express the leading order density on the right hand side of (V^{-2}) by means of $p^{(0)}(z, \tau)$ and $s^{(0)}(z)$. Integrating this equation in z from the (unknown) initial condition $p_b^{(0)}(\tau)$ yields

$$p^{(0)}(z, \tau) = p_b^{(0)}(\tau) + \pi^{(0)}(z) \tag{49}$$

Taking the time derivative of (V^{-2}) and using (49) yields $\partial_\tau \rho^{(0)} = 0$. Thus

$$p^{(0)} = p^{(0)}(z) \tag{50}$$

$$\rho^{(0)} = \rho^{(0)}(z) \tag{51}$$

$$s^{(0)} = s^{(0)}(z) \tag{52}$$

i.e. the leading order thermodynamical state only depends on the vertical coordinate and is known in terms of the initial condition. This fact has two major consequences. The first one is that $D_{\rho e}^{(-1)}$ (microscopical energy transport and radiative heating) has to be zero (equation E^{-1}).

3.2.2 Leading order velocity.

The second consequence is that, because of (C^{-1}), the leading order horizontal velocity $\boldsymbol{u}^{(0)}$ must be divergence-free:

$$\boxed{\nabla_{\boldsymbol{\eta}} \cdot \boldsymbol{u}^{(0)} = 0} \tag{53}$$

Using continuity the horizontal momentum equation (H^{-1}) can be written as

$$\boxed{\partial_\tau \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}} \boldsymbol{u}^{(0)} + \frac{1}{\rho^{(0)}} \nabla_{\boldsymbol{\eta}} p^{(2)} = \frac{1}{\rho^{(0)}} D_{\rho \boldsymbol{u}}^{(-1)}} \tag{54}$$

Equations (53),(54) are the incompressible Navier-Stokes equations in horizontal planes and $\rho^{(0)}$ is a known function (of the vertical coordinate). Thus, one possible solution could be a constant wind

$$\boldsymbol{u}^{(0)} = \text{const.} \tag{55}$$

In the following we concentrate the attention on this particular case. Notice, however, that one may want to consider other kinds of flows in which $\boldsymbol{u}^{(0)}$ has some structure on the $\boldsymbol{\eta}$ scale or, in a more general setting, even on larger scales.

3.2.3 First order perturbation.

Using (55) and the results obtained so far equations (C^0), (V^{-1}) and (E^0) yield

$$\begin{aligned}
&\partial_\tau \rho^{(1)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}} \rho^{(1)} + \rho^{(0)} \nabla_{\boldsymbol{\eta}} \cdot \boldsymbol{u}^{(1)} + \partial_z (\rho w)^{(0)} = 0 \\
&\partial_\tau w^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}} w^{(0)} + \frac{1}{\rho^{(0)}} \partial_z p^{(1)} + \frac{\rho^{(1)}}{\rho^{(0)}} g = \frac{1}{\rho^{(0)}} D_{\rho w}^{(-1)} \\
&\partial_\tau s^{(1)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}} s^{(1)} + w^{(0)} \partial_z s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) D_{\rho e}^{(0)}
\end{aligned} \tag{56}$$

Averaging over $\mathcal{D}_{\boldsymbol{\eta}}$ and taking into account (44) the above system yields

$$\boxed{\begin{aligned}
&\partial_\tau \overline{\rho^{(1)}}^{\boldsymbol{\eta}} + \partial_z (\rho^{(0)} \overline{w^{(0)}}^{\boldsymbol{\eta}}) = 0 \\
&\partial_\tau \overline{w^{(0)}}^{\boldsymbol{\eta}} + \frac{1}{\rho^{(0)}} \partial_z p^{(1)} = -\frac{\overline{\rho^{(1)}}^{\boldsymbol{\eta}}}{\rho^{(0)}} g + \frac{1}{\rho^{(0)}} \overline{D_{\rho w}^{(-1)}}^{\boldsymbol{\eta}} \\
&\partial_\tau \overline{s^{(1)}}^{\boldsymbol{\eta}} + \overline{w^{(0)}}^{\boldsymbol{\eta}} \partial_z s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) \overline{D_{\rho e}^{(0)}}^{\boldsymbol{\eta}}
\end{aligned}} \tag{57}$$

One can use the state equation (S_1^1) to replace the last equation with an evolution equation for the first order pressure $p^{(1)}$:

$$\partial_\tau p^{(1)} + \partial_z(\gamma p^{(0)}\overline{w^{(0)}}^{\eta}) = (\gamma - 1)\left(\overline{D_{\rho e}^{(0)}}^{\eta} - \rho^{(0)} g \overline{w^{(0)}}^{\eta}\right) \tag{58}$$

Equation (58) shows that system (57) supports linear acoustics in the vertical direction. This can be easily seen by taking the τ derivative of (58) and using (57.2) to eliminate $\partial_\tau \partial_z \overline{w^{(0)}}^{\eta}$. Stationary solutions of this system satisfy

$$\begin{aligned} &\partial_z(\rho^{(0)}\overline{w^{(0)}}^{\eta}) = 0 \\ &\partial_z p^{(1)} = -\overline{\rho^{(1)}}^{\eta} g + \overline{D_{\rho w}^{(-1)}}^{\eta} \\ &\partial_z(\gamma p^{(0)}\overline{w^{(0)}}^{\eta}) = (\gamma - 1)\left(\overline{D_{\rho e}^{(0)}}^{\eta} - \rho^{(0)} g \overline{w^{(0)}}^{\eta}\right) \end{aligned} \tag{59}$$

Since $\rho^{(0)} \to 0$ for $z \to \infty$ and $\overline{w^{(0)}}^{x}$ is bounded it must be $\overline{w^{(0)}}^{\eta} = 0$. Thus, in the stationary case, one has

$$\begin{aligned} &\overline{w^{(0)}}^{\eta} = 0 \\ &\partial_z p^{(1)} = -\overline{\rho^{(1)}}^{\eta} g + \overline{D_{\rho w}^{(-1)}}^{\eta} \\ &\overline{D_{\rho e}^{(0)}}^{\eta} = 0 \end{aligned} \tag{60}$$

i.e. the leading order heating must average out over the micro scales!

Subtracting (57.1) form (56.1) and recalling the definition of fluctuations (27) and $\nabla_{\boldsymbol{\eta}} \rho^{(1)} = \nabla_{\boldsymbol{\eta}} \rho^{(1)} - \nabla_{\boldsymbol{\eta}} \overline{\rho^{(1)}}^{\eta} = \nabla_{\boldsymbol{\eta}} \widetilde{\rho^{(1)}}^{\eta}$ one has

$$\partial_\tau \widetilde{\rho^{(1)}}^{\eta} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}} \widetilde{\rho^{(1)}}^{\eta} + \rho^{(0)} \nabla_{\boldsymbol{\eta}} \cdot \boldsymbol{u}^{(1)} + \partial_z(\rho^{(0)} \widetilde{w^{(0)}}^{\eta}) = 0 \tag{61}$$

Equations (57.2) and (56.2) yield

$$\partial_\tau \widetilde{w^{(0)}}^{\eta} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}} \widetilde{w^{(0)}}^{\eta} + \frac{\widetilde{\rho^{(1)}}^{\eta}}{\rho^{(0)}} g = \frac{1}{\rho^{(0)}} \widetilde{D_{\rho w}^{(-1)}}^{\eta} \tag{62}$$

Using the state equation (S_2^1) and the continuity equation (56.1) the energy equation (En^0) can be re-written as

$$\partial_\tau p^{(1)} + \gamma p^{(0)} \nabla_{\boldsymbol{\eta}} \cdot \boldsymbol{u}^{(1)} + \gamma \partial_z(p^{(0)} w^{(0)}) + (\gamma - 1)\rho^{(0)} g w^{(0)} = (\gamma - 1) D_{\rho e}^{(0)} \tag{63}$$

Subtracting (58) from the above equation yields

$$\gamma p^{(0)}\nabla_{\boldsymbol{\eta}}\cdot\boldsymbol{u}^{(1)}+\gamma\partial_z(p^{(0)}\widetilde{w^{(0)}}^{\boldsymbol{\eta}})+(\gamma-1)\rho^{(0)}g\widetilde{w^{(0)}}^{\boldsymbol{\eta}}=(\gamma-1)\widetilde{D_{\rho e}^{(0)}}^{\boldsymbol{\eta}} \tag{64}$$

Equation (64) can now be used to eliminate $\nabla_{\boldsymbol{\eta}}\cdot\boldsymbol{u}^{(1)}$ in the evolution equation for the density fluctuation $\widetilde{\rho^{(1)}}^{\boldsymbol{\eta}}$ (61). One gets

$$\partial_\tau\widetilde{\rho^{(1)}}^{\boldsymbol{\eta}}+\boldsymbol{u}^{(0)}\cdot\nabla_{\boldsymbol{\eta}}\widetilde{\rho^{(1)}}^{\boldsymbol{\eta}}+\widetilde{w^{(0)}}^{\boldsymbol{\eta}}\left(\partial_z\rho^{(0)}-\frac{1}{{c^{(0)}}^2}\partial_z p^{(0)}\right)=-\frac{\gamma-1}{\gamma}\frac{\rho^{(0)}}{p^{(0)}}\widetilde{D_{\rho e}^{(0)}}^{\boldsymbol{\eta}} \tag{65}$$

Introducing

$$\frac{D}{D\tau}:=\partial_\tau+\boldsymbol{u}^{(0)}\cdot\nabla_{\boldsymbol{\eta}} \tag{66}$$

and noticing that

$$\partial_z\rho^{(0)}-\frac{1}{{c^{(0)}}^2}\partial_z p^{(0)}=-{\rho^{(0)}}^{\gamma}\frac{1}{{c^{(0)}}^2}\partial_z s^{(0)} \tag{67}$$

equations (65),(62) can be written in the following form.

$$\boxed{\begin{aligned}{c^{(0)}}^2{\rho^{(0)}}^{-\gamma}\frac{D\widetilde{\rho^{(1)}}^{\boldsymbol{\eta}}}{D\tau}-\partial_z s^{(0)}\widetilde{w^{(0)}}^{\boldsymbol{\eta}}&=-{\rho^{(0)}}^{-\gamma}(\gamma-1)\widetilde{D_{\rho e}^{(0)}}^{\boldsymbol{\eta}}\\ \frac{D\widetilde{w^{(0)}}^{\boldsymbol{\eta}}}{D\tau}+\frac{\widetilde{\rho^{(1)}}^{\boldsymbol{\eta}}}{\rho^{(0)}}g&=\frac{1}{\rho^{(0)}}\widetilde{D_{\rho w}^{(-1)}}^{\boldsymbol{\eta}}\end{aligned}} \tag{68}$$

The systems (57),(68) describe deep convection in a constant horizontal wind $\boldsymbol{u}^{(0)}$. Notice that in the stationary limit (as both density and vertical velocity fluctuations are simply advected by the horizontal wind) the vertical velocity fluctuation becomes a simple function of heating and stratification and buoyancy forces are balanced by turbulent drag:

$$\begin{aligned}\partial_z s^{(0)}\widetilde{w^{(0)}}^{\boldsymbol{\eta}}&={\rho^{(0)}}^{-\gamma}(\gamma-1)\widetilde{D_{\rho e}^{(0)}}^{\boldsymbol{\eta}}\\ \frac{\widetilde{\rho^{(1)}}^{\boldsymbol{\eta}}}{\rho^{(0)}}g&=\frac{1}{\rho^{(0)}}\widetilde{D_{\rho w}^{(-1)}}^{\boldsymbol{\eta}}\end{aligned} \tag{69}$$

The first of the above equations is nothing but equation (2.2) in non-dimensional form. Thus we have found that the vertical velocity in stationary deep convection is the same velocity which is obtained when both the anelastic and the quasi-incompressible approximations hold!

4 Meso scales, single time scale

We now consider the following particular case of the asymptotic ansatz (24) under the general assumptions (A1)-(A3).

$$\mathcal{U}(\boldsymbol{x}, z, t; M) := \sum_{i=0} M^i \mathcal{U}^{(i)}(\boldsymbol{x}, z, t) \tag{70}$$

and the following additional assumption:

$$\partial_t s^{(0)} = 0 \tag{71}$$

4.1 Asymptotic equations

Introducing the ansatz (70) into the governing equations (23) one obtains the following cascade of asymptotic equations

4.1.1 Mass

$$\partial_t \rho^{(0)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u})^{(0)} + \partial_z (\rho w)^{(0)} = 0 \qquad (C^0)$$

$$\partial_t \rho^{(1)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u})^{(1)} + \partial_z (\rho w)^{(1)} = 0 \qquad (C^1)$$

4.1.2 Horizontal momentum

$$\nabla_{\boldsymbol{x}} p^{(0)} = 0 \qquad (H^{-2})$$

$$\nabla_{\boldsymbol{x}} p^{(1)} = 0 \qquad (H^{-1})$$

$$\partial_t (\rho \boldsymbol{u})^{(0)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u} \circ \boldsymbol{u})^{(0)} + \partial_z (\rho \boldsymbol{u} w)^{(0)} + \nabla_{\boldsymbol{x}} p^{(2)} + \frac{1}{Ro} \Big((\rho \boldsymbol{u}^{\perp} \mathcal{S})^{(0)} + \boldsymbol{i} (\rho w \mathcal{C})^{(0)} \Big) = \boldsymbol{D}_{\rho \boldsymbol{u}}^{(0)} \qquad (H^0)$$

4.1.3 Vertical momentum

$$\partial_z p^{(0)} = -\rho^{(0)} g \qquad (V^{-2})$$

$$\partial_z p^{(1)} = -\rho^{(1)} g \qquad (V^{-1})$$

$$\partial_t (\rho w)^{(0)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u} w)^{(0)} + \partial_z (\rho w^2)^{(0)} + \partial_z p^{(2)} + \rho^{(2)} g - \frac{1}{Ro} (\rho u \mathcal{C})^{(0)} = D_{\rho w}^{(0)} \qquad (V^0)$$

4.1.4 Entropy

$$\partial_t s^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} s^{(0)} + w^{(0)} \partial_z s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) D_{\rho e}^{(0)} \qquad (E^0)$$

$$\begin{aligned} &\partial_t s^{(1)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} s^{(1)} + \boldsymbol{u}^{(1)} \cdot \nabla_{\boldsymbol{x}} s^{(0)} + w^{(0)} \partial_z s^{(1)} + w^{(1)} \partial_z s^{(0)} \\ &\qquad = {\rho^{(0)}}^{-\gamma} (\gamma - 1) D_{\rho e}^{(1)} - {\rho^{(0)}}^{-\gamma-1} \rho^{(1)} \gamma (\gamma - 1) D_{\rho e}^{(0)} \qquad (E^1) \end{aligned}$$

4.1.5 State

$$p^{(0)} = s^{(0)} {\rho^{(0)}}^{\gamma} \qquad (S^0)$$

$$p^{(1)} = s^{(1)} {\rho^{(0)}}^{\gamma} + {c^{(0)}}^2 \rho^{(1)} \qquad {c^{(0)}}^2 := \frac{\gamma p^{(0)}}{\rho^{(0)}} \qquad (S^1)$$

Notice that, due to assumption (A2), no dissipative effects appear on the right hand side of the momentum equations of order less than zero and there are no entropy equations of order less than zero.

4.2 Analysis

4.2.1 Leading order pressure, density and entropy.

The horizontal momentum equations (H^{-2}), (H^{-1}) imply

$$p^{(0)} = p^{(0)}(z, t) \qquad (72)$$

$$p^{(1)} = p^{(1)}(z, t) \qquad (73)$$

Together with (V^{-2}), (V^{-1}) the above equations yield

$$\rho^{(0)} = \rho^{(0)}(z, t) \qquad (74)$$

$$\rho^{(1)} = \rho^{(1)}(z, t) \qquad (75)$$

The state equations (S^0), (S^1) show that the entropies $s^{(0)}$ and $s^{(1)}$ do not depend on $\boldsymbol{x}$. Recalling the additional assumption (71) one has

$$s^{(0)} = s^{(0)}(z) \qquad (76)$$

$$s^{(1)} = s^{(1)}(z, t) \qquad (77)$$

The state equation (S^0) can be used to express the leading order density on the right hand side of (V^{-2}) by means of $p^{(0)}(z,t)$ and $s^{(0)}(z)$. Integrating this equation in z from the (unknown) initial condition $p_b^{(0)}(t)$ yields

$$p^{(0)}(z,t) = p_b^{(0)}(t) + \pi^{(0)}(z) \tag{78}$$

Averaging (C^0) over $\mathcal{D}_{\boldsymbol{x}}$, taking into account (74) and using the divergence theorem yields

$$\partial_t \rho^{(0)} + \partial_z (\rho^{(0)} \overline{w^{(0)}}^{\boldsymbol{x}}) = -\rho^{(0)} \frac{1}{|\mathcal{D}_{\boldsymbol{x}}|} \int\limits_{\partial \mathcal{D}_{\boldsymbol{x}}} \boldsymbol{u}^{(0)} \cdot \boldsymbol{n} \, dL \tag{79}$$

Because of the hydrostatic relationship (V^{-2}) this equation can be rewritten as

$$\partial_t \partial_z p^{(0)} - g \partial_z (\rho^{(0)} \overline{w^{(0)}}^{\boldsymbol{x}}) = g \rho^{(0)} \frac{1}{|\mathcal{D}_{\boldsymbol{x}}|} \int\limits_{\partial \mathcal{D}_{\boldsymbol{x}}} \boldsymbol{u}^{(0)} \cdot \boldsymbol{n} \, dL \tag{80}$$

Inserting (78) in the above equation shows that the leading order density and pressure (and, per assumption, the leading order entropy) only depend on the vertical coordinate and are known through the initial condition:

$$p^{(0)} = p^{(0)}(z) \tag{81}$$

$$\rho^{(0)} = \rho^{(0)}(z) \tag{82}$$

$$s^{(0)} = s^{(0)}(z) \tag{83}$$

This result is not surprising. As seen in section 1.5, our reference time is about 20 minutes. On such time scale the leading order thermodynamic quantities are stationary. The point here is that we have not explicitely considered multiple time scales in the analysis so far. This would lead to non-trivial evolution equation for $p^{(0)}$, $\rho^{(0)}$ and $s^{(0)}$ in the slow time coordinate.

Notice that the vertical rate of change of $\rho^{(0)} \overline{w^{(0)}}^{\boldsymbol{x}}$ (or, since $\rho^{(0)}$ is known, the vertical rate of change of $\overline{w^{(0)}}^{\boldsymbol{x}}$) is defined in terms of the mass flux through the boundary of $\mathcal{D}_{\boldsymbol{x}}$. This mass flux is known if boundary conditions for the velocity $\boldsymbol{u}^{(0)}$ are given:

$$\boxed{\partial_z (\rho^{(0)} \overline{w^{(0)}}^{\boldsymbol{x}}) = \rho^{(0)} \frac{1}{|\mathcal{D}_{\boldsymbol{x}}|} \int\limits_{\partial \mathcal{D}_{\boldsymbol{x}}} \boldsymbol{u}^{(0)} \cdot \boldsymbol{n} \, dL} \tag{84}$$

Taking the average of equation (E^0) over $\mathcal{D}_{\boldsymbol{x}}$ yields

$$\boxed{\overline{w^{(0)}}^{\boldsymbol{x}} \partial_z s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) \overline{D^{(0)}_{\rho e}}^{\boldsymbol{x}}} \tag{85}$$

The last two equations show that if $\partial_z s^{(0)} \neq 0$ then the leading order mass flux through $\partial \mathcal{D}_{\boldsymbol{x}}$ and the vertical rate of change of the leading order average heating $\overline{D^{(0)}_{\rho e}}^{\boldsymbol{x}}$ are coupled by a simple relationship. In the special case of zero mass flux equation (84) requires $\partial_z(\rho^{(0)} \overline{w^{(0)}}^{\boldsymbol{x}})$ to be zero. Since $\rho^{(0)} \to 0$ for $\boldsymbol{x} \to \infty$ and $\overline{w}^{\boldsymbol{x}}$ is bounded it must be

$$\overline{w^{(0)}}^{\boldsymbol{x}} = 0 \tag{86}$$

In this case equation (85) shows that the $\boldsymbol{x}$-average of $D^{(0)}_{\rho e}$ (microscopical energy transport and radiative heating) must be zero as well.

4.2.2 Leading order velocity.

Assume the vertical velocity $w^{(0)}$ is known. Then the zero order continuity equation (C^0) can be interpreted as a divergence constraint for the horizontal velocity $\boldsymbol{u}^{(0)}$:

$$\nabla_{\boldsymbol{x}} \cdot (\rho^{(0)} \boldsymbol{u}^{(0)}) = \rho^{(0)} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(0)} = -\partial_z(\rho^{(0)} w^{(0)}) \tag{87}$$

Subtracting ∂_t of the above equation from the $\boldsymbol{x}$-scale divergence of the horizontal momentum equation (H^0) yields a two-dimensional Poisson problem for the second order pressure $p^{(2)}$:

$$\begin{aligned} \Delta_{\boldsymbol{x}} p^{(2)} = {} & \partial_t \partial_z (\rho^{(0)} w^{(0)}) \\ & - \rho^{(0)} \nabla_{\boldsymbol{x}} \cdot \Big(\boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u}^{(0)} + w^{(0)} \partial_z \boldsymbol{u}^{(0)} \Big) \\ & - \frac{1}{Ro} \nabla_{\boldsymbol{x}} \cdot \Big((\rho \boldsymbol{u}^{\perp} \mathcal{S})^{(0)} + \boldsymbol{i} (\rho w \mathcal{C})^{(0)} \Big) \\ & + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{D}^{(0)}_{\rho \boldsymbol{u}} \end{aligned} \tag{88}$$

Given a right hand side and suitable boundary conditions for $p^{(2)}$ equation (88) provides, for each z and t, the second order pressure. The time derivative on the right hand side of (88) points back to the assumption that $w^{(0)}$ being known.

To find the vertical velocity $w^{(0)}$ consider the entropy equation (E^0). Together with (76) this equation yields

$$w^{(0)} \partial_z s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) D^{(0)}_{\rho e} \tag{89}$$

and one has to distinguish between the two cases $\partial_z s^{(0)} \neq 0$ and $\partial_z s^{(0)} = 0$.

Leading order stratification $\partial_{\mathbf{z}}\mathbf{s}^{(\mathbf{0})} \neq \mathbf{0}$. Equation (89) yields the vertical velocity $w^{(0)}$ in terms of the microscopical energy transport and radiative heating:

$$\boxed{w^{(0)} = \frac{{\rho^{(0)}}^{-\gamma}(\gamma-1)D^{(0)}_{\rho e}}{\partial_z s^{(0)}}} \tag{90}$$

If this heating vanishes then $w^{(0)} = 0$ and the divergence constraint (87) becomes

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(0)} = 0 \tag{91}$$

Notice that equation (90) is consistent with the stationary limit of small scales deep convection (69) of density fluctuations. Notice also that (90) is simply equation (2.2) in non-dimensional form.

First order stratification $\partial_{\mathbf{z}}\mathbf{s}^{(\mathbf{0})} = \mathbf{0}$, $\partial_{\mathbf{z}}\mathbf{s}^{(\mathbf{1})} \neq \mathbf{0}$. In this case equation (89) yields

$$D^{(0)}_{\rho e} = 0 \tag{92}$$

i.e. the leading order heating must be zero. Consider the first order entropy equation (E^1). Because of (76), (77), (92) and since $\partial_z s^{(0)} = 0$ this equation becomes

$$\partial_t s^{(1)} + w^{(0)}\partial_z s^{(1)} = {\rho^{(0)}}^{-\gamma}(\gamma-1)D^{(1)}_{\rho e} \tag{93}$$

Averaging over $\mathcal{D}_{\boldsymbol{x}}$ and taking into account that $\rho^{(0)}$, $s^{(1)}$ do not depend on the $\boldsymbol{x}$-coordinate one has

$$\partial_t s^{(1)} + \overline{w^{(0)}}^{\boldsymbol{x}}\partial_z s^{(1)} = {\rho^{(0)}}^{-\gamma}(\gamma-1)\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}} \tag{94}$$

Subtracting equation (94) from (93) yields

$$\widetilde{w^{(0)}}^{\boldsymbol{x}}\partial_z s^{(1)} = {\rho^{(0)}}^{-\gamma}(\gamma-1)\widetilde{D^{(1)}_{\rho e}}^{\boldsymbol{x}} \tag{95}$$

The integral of (84) and equation (95) define the vertical velocity $w^{(0)}$. The last equation shows that if $\partial_z s^{(1)} \neq 0$ the fluctuation of the leading order vertical velocity $w^{(0)}$ is defined in terms of the first order perturbation of the differential heating. If $D^{(1)}_{\rho e}$ has no fluctuation then $\widetilde{w^{(0)}}^{\boldsymbol{x}}$ is identically zero. This means that $w^{(0)} = \overline{w^{(0)}}^{\boldsymbol{x}}$ and the divergence constraint (87) reduces to

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(0)} = -\frac{1}{|\mathcal{D}_{\boldsymbol{x}}|}\int_{\partial\mathcal{D}_{\boldsymbol{x}}} \boldsymbol{u}^{(0)}\cdot\boldsymbol{n}\,dL \tag{96}$$

Higher order stratification $\partial_z s^{(0)} = \partial_z s^{(1)} = 0$. A detailed analysis for very weak stratification with only second or higher order variations of entropy with height is beyond the scope of this report. Preliminary considerations have revealed that the strict coupling between vertical velocity and heating, found in (90) and (95), is weakened. For very weak stratification more complex dynamical balances of vertical momentum are expected to emerge. The elaboration of an appropriate asymptotic analysis is work in progress.

4.2.3 The anelastic and Boussinesq approximations.

Consider the anelastic approximation (37) in non-dimensional form

$$\nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0 \tag{97}$$

where $\overline{\rho}$ is a vertically varying mean-state density i.e. $\overline{\rho}$ coincides with our ρ^0. Thus

$$\nabla \cdot (\overline{\rho} \boldsymbol{v}) = \nabla \cdot (\rho^{(0)} \boldsymbol{v}) = \nabla_{\boldsymbol{x}} \cdot (\rho^{(0)} \boldsymbol{u}^{(0)}) + \partial_z (\rho^{(0)} w^{(0)}) + O(M) = O(M) \tag{98}$$

The last equality follows from (87) and shows that (97) approximates the full continuity equation up to terms $O(M)$ as $M \to 0$. Notice also that the Boussinesq approximation (36):

$$\nabla \cdot \boldsymbol{v} = 0 \tag{99}$$

may be as good as $O(M)$ or as bad as $O(1)$ depending on whether $w^{(0)}$ is equal to zero or not. In case of leading order stratification $w^{(0)}$ is equal to zero if and only if the heating $D_{\rho e}^{(0)}$ is zero.

4.2.4 First order perturbation.

Averaging (C^1), (V^{-1}) and (E^1) over $\mathcal{D}_{\boldsymbol{x}}$ and taking into account the results of the leading order analysis yields

$$
\begin{aligned}
\partial_t \rho^{(1)} + \partial_z(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}) &= -\,\rho^{(0)}\frac{1}{|\mathcal{D}_{\boldsymbol{x}}|}\int\limits_{\partial\mathcal{D}_{\boldsymbol{x}}} \boldsymbol{u}^{(1)}\cdot\boldsymbol{n}\,dL \\
&\quad -\,\rho^{(1)}\frac{1}{|\mathcal{D}_{\boldsymbol{x}}|}\int\limits_{\partial\mathcal{D}_{\boldsymbol{x}}} \boldsymbol{u}^{(0)}\cdot\boldsymbol{n}\,dL \\
\partial_z p^{(1)} &= -\,\rho^{(1)} g \\
\partial_t s^{(1)} + \overline{w^{(0)}}^{\boldsymbol{x}}\partial_z s^{(1)} + \overline{w^{(1)}}^{\boldsymbol{x}}\partial_z s^{(0)} &= +\,{\rho^{(0)}}^{-\gamma}(\gamma-1)\overline{D_{\rho e}^{(1)}}^{\boldsymbol{x}} \\
&\quad -\,{\rho^{(0)}}^{-\gamma}\frac{\rho^{(1)}}{\rho^{(0)}}\gamma(\gamma-1)\overline{D_{\rho e}^{(0)}}^{\boldsymbol{x}}
\end{aligned}
\tag{100}
$$

Consider the time derivative of the state equation (S^1):

$$
\partial_t s^{(1)} = {\rho^{(0)}}^{-\gamma}\left(\partial_t p^{(1)} - {c^{(0)}}^2\partial_t \rho^{(1)}\right) \tag{101}
$$

Replacing $\partial_t s^{(1)}$ in $(100)_3$ with the above expression yields

$$
\begin{aligned}
\partial_t p^{(1)} - {c^{(0)}}^2\partial_t\rho^{(1)} + {\rho^{(0)}}^{\gamma}\overline{w^{(1)}}^{\boldsymbol{x}}\partial_z s^{(0)} &= -\,{\rho^{(0)}}^{\gamma}\overline{w^{(0)}}^{\boldsymbol{x}}\partial_z s^{(1)} + (\gamma-1)\overline{D_{\rho e}^{(1)}}^{\boldsymbol{x}} \\
&\quad -\,\frac{\rho^{(1)}}{\rho^{(0)}}\gamma(\gamma-1)\overline{D_{\rho e}^{(0)}}^{\boldsymbol{x}}
\end{aligned}
\tag{102}
$$

Equations $(100)_1$, $(100)_2$ and (102) can be combined to obtain an ordinary differential equation for the average first order vertical velocity $\overline{w^{(1)}}^{\boldsymbol{x}}$. This is particularly simple for the case of zero leading order mass flux through $\partial\mathcal{D}_{\boldsymbol{x}}$ or, because of (26) in the limit $\mathcal{D}_{\boldsymbol{x}} \to 0$. In these cases $\overline{w^{(0)}}^{\boldsymbol{x}} = \overline{D_{\rho e}^{(0)}}^{\boldsymbol{x}} = 0$ and the above equations reduce to

$$
\begin{aligned}
&\partial_t \rho^{(1)} + \partial_z(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}) = -\rho^{(0)}\frac{1}{|\mathcal{D}_{\boldsymbol{x}}|}\int\limits_{\partial\mathcal{D}_{\boldsymbol{x}}} \boldsymbol{u}^{(1)}\cdot\boldsymbol{n}\,dL \\
&\partial_z p^{(1)} = -\rho^{(1)} g \\
&\partial_t p^{(1)} - {c^{(0)}}^2\partial_t\rho^{(1)} + {\rho^{(0)}}^{\gamma}\overline{w^{(1)}}^{\boldsymbol{x}}\partial_z s^{(0)} = (\gamma-1)\overline{D_{\rho e}^{(1)}}^{\boldsymbol{x}}
\end{aligned}
\tag{103}
$$

Taking the ∂_z-derivative of $(103)_3$ and replacing $\partial_z p^{(1)}$ and $\partial_t \rho^{(1)}$ by means of $(103)_1$ and $(103)_2$ yields the following ODE for $\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}$

$$\begin{aligned}
&\left[{c^{(0)}}^2\right] \partial_{zz} \left(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}\right) + \\
&\left[g + \partial_z({c^{(0)}}^2) + {\rho^{(0)}}^{\gamma}\partial_z s^{(0)}\right] \partial_z \left(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}\right) + \\
&\left[\partial_z({\rho^{(0)}}^{\gamma}\partial_z s^{(0)}) - {\rho^{(0)}}^{\gamma}\partial_z s^{(0)}\partial_z \rho^{(0)}\right] \left(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}\right) = P
\end{aligned} \tag{104}$$

$$P = (\gamma-1)\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}} - \left(\rho^{(0)}g + \partial_z\left(\rho^{(0)}{c^{(0)}}^2\right) + \rho^{(0)}{c^{(0)}}^2\partial_z\right)\frac{1}{|\mathcal{D}_{\boldsymbol{x}}|}\int\limits_{\partial\mathcal{D}_{\boldsymbol{x}}} \boldsymbol{u}^{(1)}\cdot\boldsymbol{n}\, dL$$

5 Meso and synoptic space scales, single time scale.

We consider the following particular case of the asymptotic ansatz (24)

$$\mathcal{U}(\boldsymbol{x}, z, t; M) := \sum_{i=0} M^i\, \mathcal{U}^{(i)}(\boldsymbol{x}, \boldsymbol{\xi}, z, t) \tag{105}$$

$$\boldsymbol{\xi} := M\boldsymbol{x} \tag{106}$$

under the general assumptions (A1)-(A3) and the following additional assumptions:

$$\partial_t s^{(0)} = 0 \tag{107}$$

$$\text{The physical domain } \mathcal{D}_{\boldsymbol{x}} \text{ is unbounded} \tag{108}$$

The unboundedness of $\mathcal{D}_{\boldsymbol{x}}$ together with the sub-linear growth condition (26) imply that in the limit $M \to 0$

$$\overline{\nabla_{\boldsymbol{x}}\mathcal{U}^{(i)}}^{\boldsymbol{x}} = \frac{1}{|\mathcal{D}_{\boldsymbol{x}}|}\int\limits_{\partial\mathcal{D}_{\boldsymbol{x}}} \mathcal{U}^{(i)}\boldsymbol{n}\, dL = 0 \tag{109}$$

as one can see by parameterizing $|\mathcal{D}_{\boldsymbol{x}}|$, $|\partial\mathcal{D}_{\boldsymbol{x}}|$ through M and using the mean value theorem to express the integral of $\mathcal{U}^{(i)}\boldsymbol{n}$ on the boundary of $\mathcal{D}_{\boldsymbol{x}}$.

5.1 Asymptotic equations

Introducing the ansatz (105) into the governing equations (23) one obtains the following cascade of asymptotic equations.

5.1.1 Mass

$$\partial_t \rho^{(0)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u})^{(0)} + \partial_z (\rho w)^{(0)} = 0 \qquad (C^0)$$

$$\partial_t \rho^{(1)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u})^{(1)} + \nabla_{\boldsymbol{\xi}} \cdot (\rho \boldsymbol{u})^{(0)} + \partial_z (\rho w)^{(1)} = 0 \qquad (C^1)$$

5.1.2 Horizontal momentum

$$\nabla_{\boldsymbol{x}} p^{(0)} = 0 \qquad (H^{-2})$$

$$\nabla_{\boldsymbol{x}} p^{(1)} + \nabla_{\boldsymbol{\xi}} p^{(0)} = 0 \qquad (H^{-1})$$

$$\partial_t (\rho \boldsymbol{u})^{(0)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u} \circ \boldsymbol{u})^{(0)} + \partial_z (\rho \boldsymbol{u} w)^{(0)} + \nabla_{\boldsymbol{x}} p^{(2)} + \nabla_{\boldsymbol{\xi}} p^{(1)} + \frac{1}{Ro} \Big((\rho \boldsymbol{u}^{\perp} \mathcal{S})^{(0)} + \boldsymbol{i} (\rho w \mathcal{C})^{(0)} \Big) = \boldsymbol{D}_{\rho \boldsymbol{u}}^{(0)} \qquad (H^0)$$

5.1.3 Vertical momentum

$$\partial_z p^{(0)} = -\rho^{(0)} g \qquad (V^{-2})$$

$$\partial_z p^{(1)} = -\rho^{(1)} g \qquad (V^{-1})$$

$$\partial_t (\rho w)^{(0)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u} w)^{(0)} + \partial_z (\rho w^2)^{(0)} + \partial_z p^{(2)} + \rho^{(2)} g - \frac{1}{Ro} (\rho u \mathcal{C})^{(0)} = D_{\rho w}^{(0)} \qquad (V^0)$$

5.1.4 Entropy

$$\partial_t s^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} s^{(0)} + w^{(0)} \partial_z s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) D_{\rho e}^{(0)} \qquad (E^0)$$

$$\partial_t s^{(1)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} s^{(1)} + \boldsymbol{u}^{(1)} \cdot \nabla_{\boldsymbol{x}} s^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\xi}} s^{(0)} + w^{(0)} \partial_z s^{(1)} + w^{(1)} \partial_z s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) D_{\rho e}^{(1)} - {\rho^{(0)}}^{-\gamma - 1} \rho^{(1)} \gamma (\gamma - 1) D_{\rho e}^{(0)} \qquad (E^1)$$

5.1.5 State

$$p^{(0)} = s^{(0)} \rho^{(0)\gamma} \qquad (S^0)$$

$$p^{(1)} = s^{(1)} \rho^{(0)\gamma} + c^{(0)2} \rho^{(1)} \qquad c^{(0)2} := \frac{\gamma p^{(0)}}{\rho^{(0)}} \qquad (S^1)$$

Notice that, due to assumption (A2), no dissipative effects appear on the right hand side of the momentum equations of order less than zero and there are no entropy equations of order less than zero.

5.2 Analysis

5.2.1 Leading order pressure, density and entropy.

The horizontal momentum equation (H^{-2}) implies $p^{(0)} = p^{(0)}(\boldsymbol{\xi}, z, t)$. Averaging (H^{-1}) over $\mathcal{D}_{\boldsymbol{x}}$ and taking into account that $p^{(0)}$ does not depend on $\boldsymbol{x}$ one has

$$\overline{\nabla_{\boldsymbol{x}} p^{(1)}}^{\boldsymbol{x}} + \nabla_{\boldsymbol{\xi}} p^{(0)} = 0 \tag{110}$$

Recalling (109) the above equation yields $\nabla_{\boldsymbol{\xi}} p^{(0)} = 0$. Using this result in (H^{-1}) again one gets

$$p^{(0)} = p^{(0)}(z, t) \tag{111}$$

$$p^{(1)} = p^{(1)}(\boldsymbol{\xi}, z, t) \tag{112}$$

Through (V^{-2}), (V^{-1}) the above equations imply that

$$\rho^{(0)} = \rho^{(0)}(z, t) \tag{113}$$

$$\rho^{(1)} = \rho^{(1)}(\boldsymbol{\xi}, z, t) \tag{114}$$

The state equations (S^0), (S^1) show that the entropies $s^{(0)}$ and $s^{(1)}$ do not depend on $\boldsymbol{x}, \boldsymbol{\xi}$ and $\boldsymbol{x}$, respectively. Recalling the additional assumption (107) one has

$$s^{(0)} = s^{(0)}(z) \tag{115}$$

$$s^{(1)} = s^{(1)}(\boldsymbol{\xi}, z, t) \tag{116}$$

The state equation (S^0) can be used to express the leading order density on the right hand side of (V^{-2}) by means of $p^{(0)}(z, t)$ and $s^{(0)}(z)$. Integrating this equation in z from the (unknown) initial condition $p_b^{(0)}(t)$ yields

$$p^{(0)}(z, t) = p_b^{(0)}(t) + \pi^{(0)}(z) \tag{117}$$

Averaging (C^0) over $\mathcal{D}_{\boldsymbol{x}}$ and taking into account (113) and (109) yields

$$\partial_t \rho^{(0)} + \partial_z (\rho \overline{w}^{\boldsymbol{x}})^{(0)} = 0 \tag{118}$$

Because of the hydrostatic relationship (V^{-2}) this equation can be rewritten as

$$\partial_t \partial_z p^{(0)} - g \partial_z (\rho \overline{w}^{\boldsymbol{x}})^{(0)} = 0 \tag{119}$$

Inserting (117) in the above equation shows that $\partial_z(\rho^{(0)} \overline{w^{(0)}}^{\boldsymbol{x}})$ is equal to zero. Since $\rho^{(0)} \to 0$ for $z \to \infty$ and $\overline{w^{(0)}}^{\boldsymbol{x}}$ is bounded it must be

$$\boxed{\overline{w^{(0)}}^{\boldsymbol{x}} = 0} \tag{120}$$

Inserting this constraint in (118) one gets $\partial_t \rho^{(0)} = 0$ and, through the state equation (S^0) and the additional assumption (107) $\partial_t p^{(0)} = 0$.

Thus $p^{(0)}$, $\rho^{(0)}$ (and $s^{(0)}$ per assumption) only depend on the vertical coordinate and are known through the initial condition:

$$p^{(0)} = p^{(0)}(z) \tag{121}$$

$$\rho^{(0)} = \rho^{(0)}(z) \tag{122}$$

$$s^{(0)} = s^{(0)}(z) \tag{123}$$

Moreover the $\boldsymbol{x}$-average of $D_{\rho e}^{(0)}$ (microscopical energy transport and radiative heating) must be zero as one can see by averaging (E^0) over $\mathcal{D}_{\boldsymbol{x}}$ and taking into account (120):

$$\boxed{\overline{D_{\rho e}^{(0)}}^{\boldsymbol{x}} = 0} \tag{124}$$

5.2.2 Leading order velocity.

Assume the vertical velocity $w^{(0)}$ is known. Then the zero order continuity equation (C^0) can be interpreted as a divergence constraint for the horizontal velocity $\boldsymbol{u}^{(0)}$:

$$\nabla_{\boldsymbol{x}} \cdot (\rho^{(0)} \boldsymbol{u}^{(0)}) = \rho^{(0)} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(0)} = -\partial_z (\rho^{(0)} w^{(0)}) \tag{125}$$

Subtracting ∂_t of the above equation from the $\boldsymbol{x}$-scale divergence of the horizontal momentum equation (H^0) yields a two-dimensional Poisson problem for the second order pressure $p^{(2)}$:

$$\begin{aligned}\Delta_{\boldsymbol{x}} p^{(2)} &= \partial_t \partial_z (\rho^{(0)} w^{(0)}) \\ &- \rho^{(0)} \nabla_{\boldsymbol{x}} \cdot \Big(\boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u}^{(0)} + w^{(0)} \partial_z \boldsymbol{u}^{(0)} \Big) \\ &- \frac{1}{Ro} \nabla_{\boldsymbol{x}} \cdot \Big((\rho \boldsymbol{u}^{\perp} \mathcal{S})^{(0)} + \boldsymbol{i} (\rho w \mathcal{C})^{(0)} \Big) \\ &+ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{D}^{(0)}_{\rho \boldsymbol{u}}\end{aligned} \tag{126}$$

Given a right hand side and suitable boundary conditions for $p^{(2)}$ equation (126) provides, for each z, $\boldsymbol{\xi}$ and t, the second order pressure. The time derivative on the right hand side of (126) points back to the assumption $w^{(0)}$ being known.

To find the vertical velocity $w^{(0)}$ consider the entropy equation (E^0). Together with (115) this equation yields

$$w^{(0)} \partial_z s^{(0)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) D^{(0)}_{\rho e} \tag{127}$$

and one has to distinguish between the two cases $\partial_z s^{(0)} \neq 0$ and $\partial_z s^{(0)} = 0$.

Leading order stratification $\partial_{\mathbf{z}} \mathbf{s}^{(0)} \neq \mathbf{0}$. Equation (127) yields the vertical velocity $w^{(0)}$ in terms of the *differential heating* i.e. of the fluctuations of microscopical energy transport and radiative heating (remember that, because of (120) and (124), $w^{(0)}$ and $D^{(0)}_{\rho e}$ are fluctuations):

$$\boxed{w^{(0)} = \frac{{\rho^{(0)}}^{-\gamma} (\gamma - 1) D^{(0)}_{\rho e}}{\partial_z s^{(0)}}} \tag{128}$$

If these fluctuations vanish (or, equivalently, if $\nabla_{\boldsymbol{x}} D^{(0)}_{\rho e} = 0$) then $w^{(0)} = 0$. In this case the divergence constraint (125) becomes

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(0)} = 0 \tag{129}$$

Just as in section 4.2.2 equation (128) is consistent with the stationary limit of small scales deep convection (69) of density fluctuations and is the non-dimensional form of equation (2.2).

First order stratification $\partial_{\mathbf{z}} \mathbf{s}^{(0)} = \mathbf{0}$, $\partial_{\mathbf{z}} \mathbf{s}^{(1)} \neq \mathbf{0}$. In this case equation (127) yields

$$D^{(0)}_{\rho e} = 0 \tag{130}$$

i.e. the leading order differential heating must be zero. Consider the first order entropy equation (E^1). Because of (115), (116), (130) and since $\partial_z s^{(0)} = 0$ this equation becomes

$$\partial_t s^{(1)} + w^{(0)} \partial_z s^{(1)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) D_{\rho e}^{(1)} \tag{131}$$

Averaging over $\mathcal{D}_{\boldsymbol{x}}$ and taking into account that $\rho^{(0)}$, $s^{(1)}$ do not depend on the $\boldsymbol{x}$-coordinate and that $\overline{w}^{\boldsymbol{x}} = 0$ one has

$$\partial_t s^{(1)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) \overline{D_{\rho e}^{(1)}}^{\boldsymbol{x}} \tag{132}$$

Subtracting equation (132) from (131) yields

$$w^{(0)} \partial_z s^{(1)} = {\rho^{(0)}}^{-\gamma} (\gamma - 1) \widetilde{D_{\rho e}^{(1)}}^{\boldsymbol{x}} \tag{133}$$

The last equation shows that if $\partial_z s^{(1)} \neq 0$ the leading order vertical velocity $w^{(0)}$ is defined in terms of the first order perturbation of the differential heating. Just as in the case of leading order stratification if $D_{\rho e}^{(1)}$ has no fluctuation in $\boldsymbol{x}$ (i.e. $\nabla_{\boldsymbol{x}} D_{\rho e}^{(1)} = 0$) $w^{(0)}$ is identically zero and the divergence constraint (125) reduces to (129).

Higher order stratification $\partial_{\mathbf{z}} \mathbf{s}^{(0)} = \partial_{\mathbf{z}} \mathbf{s}^{(1)} = \mathbf{0}$. As in the single scale analysis, the case of very weak stratification leads to a qualitatively different regime in which large vertical displacements of mass elements are possible without the action of entropy changing sources. On the other hand, any entropy changes greater than $O(M)$ that are imposed will immediately lead to deep convection, since in the bulk of the atmosphere there is no balancing layer in which the buoyancy of the affected mass element would be neutralized. The mathematical structure of the resulting limit equations will be discussed in a separate report.

5.2.3 The anelastic and Boussinesq approximations.

See section 4.2.3.

5.2.4 First order perturbation.

Averaging (C^1), (H^0), (V^{-1}) and (E^1) over $\mathcal{D}_{\boldsymbol{x}}$ and taking into account the results of the leading order analysis yields

$$
\begin{aligned}
\partial_t\rho^{(1)} + \rho^{(0)}\nabla_{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}} + \partial_z(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}) &= 0\\
\rho^{(0)}\partial_t\overline{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}} + \nabla_{\boldsymbol{\xi}}p^{(1)} &= -\frac{1}{Ro}\rho^{(0)}\left(\overline{\boldsymbol{u}^{(0)\perp}\mathcal{S}}^{\boldsymbol{x}} + \boldsymbol{i}\overline{w^{(0)}\mathcal{C}}^{\boldsymbol{x}}\right)\\
&\quad -\partial_z(\rho^{(0)}\overline{\boldsymbol{u}^{(0)}w^{(0)}}^{\boldsymbol{x}}) + \overline{\boldsymbol{D}^{(0)}_{\rho\boldsymbol{u}}}^{\boldsymbol{x}}\\
\partial_z p^{(1)} &= -\rho^{(1)}g\\
\partial_t s^{(1)} + \overline{w^{(1)}}^{\boldsymbol{x}}\partial_z s^{(0)} &= {\rho^{(0)}}^{-\gamma}(\gamma-1)\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}}
\end{aligned}
\tag{134}
$$

We start by deriving an equation for $p^{(1)}$ by eliminating $\nabla_{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}}$ from the above system. Partial differentiation of (S^1) with respect to time yields, together with $(134)_1$, $(134)_4$:

$$
\begin{aligned}
\partial_t p^{(1)} + {c^{(0)}}^2\rho^{(0)}\nabla_{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}} &= -{c^{(0)}}^2\partial_z(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}) - {\rho^{(0)}}^{\gamma}\overline{w^{(1)}}^{\boldsymbol{x}}\partial_z s^{(0)}\\
&\quad + (\gamma-1)\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}}
\end{aligned}
\tag{135}
$$

Subtracting the large scale divergence of ${c^{(0)}}^2$ times $(134)_2$ from the time derivative of the above equation yields

$$
\begin{aligned}
\partial_{tt}p^{(1)} - {c^{(0)}}^2\Delta_{\boldsymbol{\xi}}p^{(1)} =& \frac{1}{Ro}{c^{(0)}}^2\rho^{(0)}\nabla_{\boldsymbol{\xi}}\cdot\left(\overline{\boldsymbol{u}^{(0)\perp}\mathcal{S}}^{\boldsymbol{x}} + \boldsymbol{i}\overline{w^{(0)}\mathcal{C}}^{\boldsymbol{x}}\right)\\
&-{c^{(0)}}^2\partial_t\partial_z(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}) - {\rho^{(0)}}^{\gamma}\partial_z s^{(0)}\partial_t\overline{w^{(1)}}^{\boldsymbol{x}}\\
&+{c^{(0)}}^2\partial_z(\rho^{(0)}\overline{\boldsymbol{u}^{(0)}w^{(0)}}^{\boldsymbol{x}})\\
&+(\gamma-1)\partial_t\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}} - {c^{(0)}}^2\nabla_{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{D}^{(0)}_{\rho\boldsymbol{u}}}^{\boldsymbol{x}}
\end{aligned}
\tag{136}
$$

Equation (136) shows that the first order perturbation pressure evolves according to a wave-type equation. In low Mach number non-rotating flow without gravity forces we know $p^{(1)}$ to be an *acoustic* pressure, see [8],[1]. Here the situation is more intriguing.

The first term on the right hand side represents the effects of rotation. In the next section we will show that rotation couples the time evolution of $p^{(1)}$ with the vorticity dynamics on the large (meso and synoptic) scales.

The sum of the second and of the third terms represents the coupling between average vertical motion and $p^{(1)}$. Recalling (S^1) and $(134)_3$ these terms can be written as

$$-c^{(0)^2}\partial_t\partial_z(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}) - \rho^{(0)^\gamma}\partial_z s^{(0)}\partial_t\overline{w^{(1)}}^{\boldsymbol{x}} = \rho^{(0)}\partial_t\left(g\overline{w^{(1)}}^{\boldsymbol{x}} - c^{(0)^2}\partial_z\overline{w^{(1)}}^{\boldsymbol{x}}\right) \tag{137}$$

In the classical theory of perturbations of the rest state for non-rotating, compressible, stratified flows this is the only term on the right hand side of the evolution equation for the perturbation pressure, see equation (32).

The fourth term of (136) represents the effects of correlations between fluctuations of $\boldsymbol{u}^{(0)}$ and of $w^{(0)}$ on the $\boldsymbol{x}$-scale. In fact

$$\overline{\boldsymbol{u}^{(0)}w^{(0)}}^{\boldsymbol{x}} = \overline{\boldsymbol{u}^{(0)}\widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}} = \overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}}\widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}} \tag{138}$$

The first equality holds because of (120). The second equality follows from the fact that averages are defined on the whole domain of dependence and therefore averages of fluctuations vanish, equation (31).

Finally the last two terms of the $p^{(1)}$ equation represent the effects of microscopical transport and radiative heating.

5.2.5 First order pressure and large scale vorticity.

Consider the Coriolis term on the right hand side of (136). Recalling (120) and substituting $\overline{\mathcal{S}}^{\boldsymbol{x}} + \widetilde{\mathcal{S}}^{\boldsymbol{x}}$ for $\mathcal{S}$ and $\overline{\mathcal{C}}^{\boldsymbol{x}} + \widetilde{\mathcal{C}}^{\boldsymbol{x}}$ for $\mathcal{C}$ yields

$$\nabla_{\boldsymbol{\xi}}\cdot\left(\overline{\boldsymbol{u}^{(0)\perp}\mathcal{S}}^{\boldsymbol{x}} + \boldsymbol{i}\overline{w^{(0)}\mathcal{C}}^{\boldsymbol{x}}\right) = \overline{\mathcal{S}}^{\boldsymbol{x}}\nabla_{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}} + \nabla_{\boldsymbol{\xi}}\cdot\left(\overline{\widetilde{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}}\widetilde{\mathcal{S}}^{\boldsymbol{x}}}^{\boldsymbol{x}} + \boldsymbol{i}\overline{\widetilde{w^{(0)}}^{\boldsymbol{x}}\widetilde{\mathcal{C}}^{\boldsymbol{x}}}^{\boldsymbol{x}}\right) \tag{139}$$

In writing the first term on the right hand side of (139) we used the fact that $\mathcal{S}$ is a function of $\boldsymbol{x}$ only and and $\overline{\mathcal{S}}^{\boldsymbol{x}}$ is constant. Notice that, according to the definition of $\boldsymbol{u}^{\perp}$ (equation (20)), the following identities hold

$$\nabla_{\shortparallel}\cdot\boldsymbol{u}^{\perp} = -\,\boldsymbol{k}\cdot\nabla_{\shortparallel}\times\boldsymbol{u} \tag{140}$$

$$\boldsymbol{k}\cdot\nabla_{\shortparallel}\times\boldsymbol{u}^{\perp} = \nabla_{\shortparallel}\cdot\boldsymbol{u} \tag{141}$$

Introducing the synoptic density-weighted vorticity of the average leading order horizontal velocity

$$\zeta^{(0)} := \rho^{(0)}\boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\overline{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}} = -\rho^{(0)}\nabla_{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}} \tag{142}$$

equation (136) can be re-written as

$$\partial_{tt}p^{(1)} - c^{(0)^2}\Delta_{\boldsymbol{\xi}}p^{(1)} = -c^{(0)^2}\overline{f}^{\boldsymbol{x}}\zeta^{(0)} + P \tag{143}$$

In the above equation $\overline{f}^{\boldsymbol{x}} := \overline{\mathcal{S}}^{\boldsymbol{x}}/Ro$ is the mean Coriolis parameter. P consists of the second term on the right hand side of (139) (i.e. the Coriolis effects due to $\boldsymbol{x}$-scale velocity fluctuations) and all but the first term of the right hand side of (136).

$$
\begin{aligned}
P := & -c^{(0)^2}\partial_t\partial_z(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}) - \rho^{(0)^\gamma}\partial_z s^{(0)}\partial_t\overline{w^{(1)}}^{\boldsymbol{x}} \\
& +\frac{1}{Ro}c^{(0)^2}\rho^{(0)}\nabla_{\boldsymbol{\xi}}\cdot\left(\overline{\widetilde{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}}\widetilde{\mathcal{S}}^{\boldsymbol{x}}}^{\boldsymbol{x}} + \boldsymbol{i}\overline{\widetilde{w^{(0)}}^{\boldsymbol{x}}\widetilde{\mathcal{C}}^{\boldsymbol{x}}}^{\boldsymbol{x}}\right) \\
& +c^{(0)^2}\partial_z(\rho^{(0)}\overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}}\widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}}) + (\gamma-1)\partial_t\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}} - c^{(0)^2}\nabla_{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{D}^{(0)}_{\rho\boldsymbol{u}}}^{\boldsymbol{x}}
\end{aligned}
\tag{144}
$$

Taking the $\boldsymbol{k}$-component of the large scale curl of $(134)_2$ yields

$$
\begin{aligned}
\partial_t\zeta^{(0)} = & -\frac{\overline{\mathcal{S}}^{\boldsymbol{x}}}{Ro}\rho^{(0)}\nabla_{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}} \\
& -\frac{1}{Ro}\rho^{(0)}\boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\left(\overline{\widetilde{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}}\widetilde{\mathcal{S}}^{\boldsymbol{x}}}^{\boldsymbol{x}} + \boldsymbol{i}\overline{\widetilde{w^{(0)}}^{\boldsymbol{x}}\widetilde{\mathcal{C}}^{\boldsymbol{x}}}^{\boldsymbol{x}}\right) \\
& -\boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\partial_z\left(\rho^{(0)}\overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}}\widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}}\right) + \boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\overline{\boldsymbol{D}^{(0)}_{\rho\boldsymbol{u}}}^{\boldsymbol{x}}
\end{aligned}
\tag{145}
$$

Using the continuity equation $(134)_1$ to rewrite the large scale divergence of $\overline{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}}$ in terms of the time derivative of $\rho^{(1)}$ and of vertical gradients of $\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}$ and the state equation (S^1) to express $\partial_t\rho^{(1)}$ in terms of $\partial_t p^{(1)}$ and $\partial_t s^{(1)}$ the above equation becomes

$$
\partial_t\zeta^{(0)} = \frac{\overline{f}^{\boldsymbol{x}}}{c^{(0)^2}}\partial_t p^{(1)} + Q \tag{146}
$$

The symbol Q represents the effects of vertical motion, correlations of fluctuations, microscopical transports and radiative heating:

$$
\begin{aligned}
Q := & \overline{f}^{\boldsymbol{x}}\partial_z(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}) - \frac{1}{Ro}\rho^{(0)}\boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\left(\overline{\widetilde{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}}\widetilde{\mathcal{S}}^{\boldsymbol{x}}}^{\boldsymbol{x}} + \boldsymbol{i}\overline{\widetilde{w^{(0)}}^{\boldsymbol{x}}\widetilde{\mathcal{C}}^{\boldsymbol{x}}}^{\boldsymbol{x}}\right) \\
& -\frac{\overline{f}^{\boldsymbol{x}}}{c^{(0)^2}}(\gamma-1)\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}} - \boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\partial_z\left(\rho^{(0)}\overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}}\widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}}\right) + \boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\overline{\boldsymbol{D}^{(0)}_{\rho\boldsymbol{u}}}^{\boldsymbol{x}}
\end{aligned}
\tag{147}
$$

Thus the time evolution of $p^{(1)}$, $\zeta^{(0)}$ is described by the following coupled system

$$
\boxed{
\begin{aligned}
\partial_{tt}p^{(1)} - c^{(0)^2}\Delta_{\boldsymbol{\xi}}p^{(1)} &= -c^{(0)^2}\overline{f}^{\boldsymbol{x}}\zeta^{(0)} + P \\
\partial_t\zeta^{(0)} &= \frac{\overline{f}^{\boldsymbol{x}}}{c^{(0)^2}}\partial_t p^{(1)} + Q
\end{aligned}
}
\tag{148}
$$

Notice that (148) must be closed with some equation for the average vertical velocity $\overline{w^{(1)}}^{x}$ which appears both in P and in Q. In the case $P = Q = 0$ the system (148) supports traveling waves of the form

$$(p^{(1)}, \zeta^{(0)}) = (p_0^{(1)}, \zeta_0^{(0)}) e^{i(kx+ly-\omega t)} \tag{149}$$

provided that $\kappa^2 := k^2 + l^2$ satisfies the dispersion relation

$$\omega^2 = {c^{(0)}}^2 \kappa^2 + \overline{f}^{x\,2} \tag{150}$$

This is the dispersion relation for perturbations of the *shallow water* approximation linearized about a *geostrophic* basic state, see [4] page 15.

5.2.6 First order closure

The closure of system (148) for the first order perturbation pressure $p^{(1)}$ and for the large scale vorticity $\zeta^{(0)}$ requires an equation for the vertical velocity $\overline{w^{(1)}}^{x}$.

In the classical theory of perturbations of the rest state this equation is obtained by eliminating the perturbation density from the linearized continuity and vertical momentum equations, see [5] pages 170-171.

In our low Mach number asymptotics the vertical velocity $\overline{w^{(1)}}^{x}$ is defined, in much the same way as the leading order velocity $w^{(0)}$, in terms of differential heating through the first order entropy evolution equation $(134)_4$. As for the leading order velocity one has to distinguish between the cases of leading order, first order and higher order stratification.

Leading order stratification $\partial_{\mathbf{z}} \mathbf{s}^{(0)} \neq \mathbf{0}$. Differentiating the state equation (S^0) and using the vertical momentum equation (V^{-2}) one has

$$\partial_z s^{(0)} = -{\rho^{(0)}}^{-\gamma} \frac{\gamma p^{(0)}}{g} g \left(\frac{1}{\rho^{(0)}} \partial_z \rho^{(0)} + g \frac{\rho^{(0)}}{\gamma p^{(0)}} \right) \tag{151}$$

Taking the time derivative of the first order state equation (S^1) yields

$$\partial_t p^{(1)} = {\rho^{(0)}}^{\gamma} \partial_t s^{(1)} + {c^{(0)}}^2 \partial_t \rho^{(1)} \tag{152}$$

Using the vertical momentum equation $(134)_3$ the time derivative of $s^{(1)}$ can be expressed in terms of the leading order solution and of vertical and time derivatives of the first order perturbation pressure $p^{(1)}$:

$$\partial_t s^{(1)} = \frac{{\rho^{(0)}}^{-\gamma}}{g} \partial_t \left(g p^{(1)} + {c^{(0)}}^2 \partial_z p^{(1)} \right) \tag{153}$$

Substituting (151), (153) into the left hand side of $(134)_4$ yields

$$\boxed{\begin{aligned} -g\left(\frac{1}{\rho^{(0)}}\partial_z\rho^{(0)} + g\frac{\rho^{(0)}}{\gamma p^{(0)}}\right) {c^{(0)}}^2\overline{w^{(1)}}^{\boldsymbol{x}} = & -\frac{1}{\rho^{(0)}}\partial_t\left(gp^{(1)} + {c^{(0)}}^2\partial_z p^{(1)}\right) \\ & + \frac{g}{\rho^{(0)}}(\gamma-1)\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}} \end{aligned}} \tag{154}$$

For a stably stratified atmosphere $(\partial_z s^{(0)} > 0)$ the factor multiplying $\overline{w^{(1)}}^{\boldsymbol{x}}$ is positive and usually written in terms of a buoyancy frequency

$$ {N^{(0)}}^2 := -g\left(\frac{\partial_z\rho^{(0)}}{\rho^{(0)}} + g\frac{\rho^{(0)}}{\gamma p^{(0)}}\right) \tag{155}$$

In this case the system (148),(154) can be written as follows

$$\boxed{\begin{aligned} \partial_{tt}p^{(1)} - {c^{(0)}}^2\Delta_{\boldsymbol{\xi}}p^{(1)} = & -{c^{(0)}}^2\overline{f}^{\boldsymbol{x}}\zeta^{(0)} + \rho^{(0)}\partial_t\left(g\overline{w^{(1)}}^{\boldsymbol{x}} - {c^{(0)}}^2\partial_z\overline{w^{(1)}}^{\boldsymbol{x}}\right) \\ & + \frac{1}{Ro}{c^{(0)}}^2\rho^{(0)}\nabla_{\boldsymbol{\xi}}\cdot\left(\overline{\widetilde{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}}\widetilde{\mathcal{S}}^{\boldsymbol{x}}}^{\boldsymbol{x}} + \boldsymbol{i}\overline{\widetilde{w^{(0)}}^{\boldsymbol{x}}\widetilde{\mathcal{C}}^{\boldsymbol{x}}}^{\boldsymbol{x}}\right) \\ & + {c^{(0)}}^2\partial_z(\rho^{(0)}\overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}}\widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}}) \\ & + (\gamma-1)\partial_t\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}} - {c^{(0)}}^2\nabla_{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{D}^{(0)}_{\rho\boldsymbol{u}}}^{\boldsymbol{x}} \\ \partial_t\zeta^{(0)} = & +\frac{\overline{f}^{\boldsymbol{x}}}{{c^{(0)}}^2}\partial_t p^{(1)} + \overline{f}^{\boldsymbol{x}}\partial_z(\rho^{(0)}\overline{w^{(1)}}^{\boldsymbol{x}}) \\ & - \frac{1}{Ro}\rho^{(0)}\boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\left(\overline{\widetilde{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}}\widetilde{\mathcal{S}}^{\boldsymbol{x}}}^{\boldsymbol{x}} + \boldsymbol{i}\overline{\widetilde{w^{(0)}}^{\boldsymbol{x}}\widetilde{\mathcal{C}}^{\boldsymbol{x}}}^{\boldsymbol{x}}\right) \\ & - \boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\partial_z\left(\rho^{(0)}\overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}}\widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}}\right) \\ & - \frac{\overline{f}^{\boldsymbol{x}}}{{c^{(0)}}^2}(\gamma-1)\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}} + \boldsymbol{k}\cdot\nabla_{\boldsymbol{\xi}}\times\overline{\boldsymbol{D}^{(0)}_{\rho\boldsymbol{u}}}^{\boldsymbol{x}} \\ {N^{(0)}}^2{c^{(0)}}^2\overline{w^{(1)}}^{\boldsymbol{x}} = & -\frac{1}{\rho^{(0)}}\partial_t\left(gp^{(1)} + {c^{(0)}}^2\partial_z p^{(1)}\right) \\ & + \frac{g}{\rho^{(0)}}(\gamma-1)\overline{D^{(1)}_{\rho e}}^{\boldsymbol{x}} \end{aligned}} \tag{156}$$

System (156) supports internal gravity waves, acoustic waves and the Lamb wave and represents the link between our asymptotic framework and the classical theory of small perturbations of the state of rest for compressible stratified fluids, see section 2.

There are two major differences between the classical theory and the present case. The first one is in the equation for the vertical velocity $\overline{w^{(1)}}^{\boldsymbol{x}}$. In the classical theory such equation is obtained through combination of vertical momentum and continuity and contains a second order term $\partial_{tt}\overline{w^{(1)}}^{\boldsymbol{x}}$ on the left hand side.

The second differences lie in the right hand side of (156) where $\boldsymbol{x}$-scale correlations of fluctuations of the leading order solution, microscopical transport and radiative heating appear as forcing terms in the equations. In the classical theory these terms are absent due to the particular choice of the basic state (of rest) and to linearization.

Notice that the stationary limit of (156) is a geostrophic motion on the large horizontal scales. In this motion space variations of the Coriolis parameter, $\boldsymbol{x}$-scale correlations of fluctuations of the leading order solution and turbulent stresses perturb the geostrophic balance. This balance, however, does not depend explicitely on the vertical velocity. In much the same way as $w^{(0)}$, the vertical velocity $\overline{w^{(1)}}^{\boldsymbol{x}}$ is defined in terms of average first order heating and zero order stratification.

First order stratification $\partial_{\mathbf{z}}\mathbf{s}^{(\mathbf{0})} = \mathbf{0}$, $\partial_{\mathbf{z}}\mathbf{s}^{(\mathbf{1})} \neq \mathbf{0}$. Given a model for microscopical energy transport and radiative heating equation $(134)_4$ can be integrated to yield the first order perturbation entropy $s^{(1)}$. In turn $s^{(1)}$ can be plugged into the state equation (S^1) to express the right hand side of the vertical momentum equation $(134)_3$ in terms of the unknown pressure $p^{(1)}$ (and of $s^{(1)}$). Thus $(134)_3$ can be integrated in z to yield the $p^{(1)}$-decomposition

$$p^{(1)}(\boldsymbol{\xi}, z, t) = p_b^{(1)}(\boldsymbol{\xi}, t) + \pi^{(1)}(\boldsymbol{\xi}, z, t) \tag{157}$$

In this equation $\pi^{(1)}$ is the solution of $(134)_3$ with initial data $\pi^{(1)}(\boldsymbol{\xi}, z_b(\boldsymbol{\xi}), t) = 0$ on some arbitrary large scale dependent vertical level $z = z_b(\boldsymbol{\xi})$.

Notice that, since $\partial_z s^{(0)} = 0$, the third term of (144) vanishes and the right hand side of the system (148) only depends on the z (but not on $\boldsymbol{\xi}$ and time) derivatives of $\overline{w^{(1)}}^{\boldsymbol{x}}$.

Expressing the pressure $p^{(1)}$ in (148 through (157) and averaging over

$\mathcal{D}_z$ yield the following system for $p_b^{(1)}$, $\overline{\zeta^{(0)}}^{z}$:

$$
\boxed{
\begin{aligned}
\overline{c^{(0)\,-2}}^{z}\,\partial_{tt} p_b^{(1)} - \Delta_{\boldsymbol{\xi}} p_b^{(1)} = & - \overline{f}^{\boldsymbol{x}}\,\overline{\zeta^{(0)}}^{z} - \partial_t \left(\rho^{(0)} \overline{w^{(1)}}^{\boldsymbol{x}} \Big|_t - \rho^{(0)} \overline{w^{(1)}}^{\boldsymbol{x}} \Big|_b \right) \\
& + \frac{1}{Ro} \nabla_{\boldsymbol{\xi}} \cdot \left(\overline{\rho^{(0)} \widetilde{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}} \widetilde{\mathcal{S}}^{\boldsymbol{x}}}^{\boldsymbol{x},z} + \boldsymbol{i}\, \overline{\rho^{(0)} \widetilde{w^{(0)}}^{\boldsymbol{x}} \widetilde{\mathcal{C}}^{\boldsymbol{x}}}^{\boldsymbol{x},z} \right) \\
& + \rho^{(0)} \overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}} \widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}} \Bigg|_t - \rho^{(0)} \overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}} \widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}} \Bigg|_b \\
& + (\gamma - 1)\, \overline{c^{(0)\,-2} \partial_t D^{(1)}_{\rho e}}^{\boldsymbol{x},z} - \nabla_{\boldsymbol{\xi}} \cdot \overline{\boldsymbol{D}^{(0)}_{\rho \boldsymbol{u}}}^{\boldsymbol{x},z} \\
& - \overline{c^{(0)\,-2} \partial_{tt} \pi^{(1)}}^{z} + \overline{\Delta_{\boldsymbol{\xi}} \pi^{(1)}}^{z} \\
\partial_t \overline{\zeta^{(0)}}^{z} = & + \overline{c^{(0)\,-2}}^{z}\, \overline{f}^{\boldsymbol{x}} \partial_t p_b^{(1)} + \overline{f}^{\boldsymbol{x}} \left(\rho^{(0)} \overline{w^{(1)}}^{\boldsymbol{x}} \Big|_t - \rho^{(0)} \overline{w^{(1)}}^{\boldsymbol{x}} \Big|_b \right) \\
& - \frac{1}{Ro} \boldsymbol{k} \cdot \nabla_{\boldsymbol{\xi}} \times \left(\overline{\rho^{(0)} \widetilde{\boldsymbol{u}^{(0)\perp}}^{\boldsymbol{x}} \widetilde{\mathcal{S}}^{\boldsymbol{x}}}^{\boldsymbol{x},z} + \boldsymbol{i}\, \overline{\rho^{(0)} \widetilde{w^{(0)}}^{\boldsymbol{x}} \widetilde{\mathcal{C}}^{\boldsymbol{x}}}^{\boldsymbol{x},z} \right) \\
& - \boldsymbol{k} \cdot \nabla_{\boldsymbol{\xi}} \times \left(\rho^{(0)} \overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}} \widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}} \Bigg|_t - \rho^{(0)} \overline{\widetilde{\boldsymbol{u}^{(0)}}^{\boldsymbol{x}} \widetilde{w^{(0)}}^{\boldsymbol{x}}}^{\boldsymbol{x}} \Bigg|_b \right) \\
& - \overline{f}^{\boldsymbol{x}} (\gamma - 1)\, \overline{c^{(0)\,-2} \partial_t D^{(1)}_{\rho e}}^{\boldsymbol{x},z} + \boldsymbol{k} \cdot \nabla_{\boldsymbol{\xi}} \times \overline{\boldsymbol{D}^{(0)}_{\rho \boldsymbol{u}}}^{\boldsymbol{x},z} \\
& + \overline{f}^{\boldsymbol{x}}\, \overline{c^{(0)\,-2} \partial_t \pi^{(1)}}^{z}
\end{aligned}
}
\tag{158}
$$

6 Conclusions, open questions and future work

The analysis presented in sections 3,4 and 5 has interesting implications for the issues of modeling and computing atmospheric flows. So far, we can draw the following conclusions:

1. The analysis shows that, in the limit of vanishing Mach and Froude numbers, pressure perturbations of order $O(M^2)$ affect the velocity field at leading order. Therefore, single variable representations of the pressure field do not allow a meaningful computation of pressure gradients. This is a well known problem in low Mach number flows, see e.g. [15], [10], [8], [9], [14], [6], [1].

2. The analysis on meso and synoptic scales shows that, in the limit $M \to 0$ and for a stable or moderately stable stratification, the vertical velocity w must satisfy a diagnostic constraint. This constraint

takes the form of a simple relationship between vertical velocity, stratification and heating (see, for instance, equation (90)) and is *not* a form of the continuity equation. This result has three implications:

(a) Parameterizations of sub-scale physical processes involved in the energy budget (e.g. turbulent heat transfer) are not completely free and must satisfy an integral constraint as well.

(b) The second order perturbation pressure can be computed by solving, at each $z =$ const. level, a two dimensional elliptic problem. This problem involves a plain Laplace operator and can be solved very efficiently with standard methods.

(c) There is no truly three-dimensional motion at meso and synoptic scales. Thus, the question is whether numerical methods based on the unconstraint integration of the full three-dimensional equation of motion can predict vertical winds which are consistent with the asymptotic behavior of the true solution. If this is not the case these methods are not suitable for computing slow atmospheric motion on such scales and under strong or moderate stratification.

Notice that the last question *can* be answered by carefully defining simple numerical experiments. In these experiments initial conditions, boundary conditions and the parameterizations are prescribed in such a way that the exact solution, albeit unknown, evolves in the low Mach number regime. One can then compare the vertical velocities (and the divergence of the horizontal winds) obtained from the numerical solution with the results of the asymptotic analysis.

Let us stress the following point: this kind of computations can be run at negligible costs, do not require the acquisition, storage, management and analysis of huge amounts of observational data, do not raise the problem of evaluating the meaning, accuracy and consistency of such data and provide a simple and clearcut scheme for validating existing numerical methods and new computational approaches. The vertical velocity issue is a paramount example of how results of the asymptotic analysis can be used in the validation of numerical methods. In the context of slow atmospheric motion the vertical velocity constraint plays the same role as the divergence-free constraint in the incompressible Navier-Stokes equations. Other kinds of constraints might be relevant as well. No matter what these constraint are, we believe that such kind of experiments provide the scientific basis for a quantitative assessment of the efficiency and of the accuracy of numerical methods and should play a crucial role in the development, validation and evaluation of these methods.

3. The analysis of deep convection on the micro scales (section 3) reveals a vertical velocity-density perturbation dynamics which is perfectly consistent with the above mentioned diagnostic constraint for w: in the quasi-steady limit equation (68) simply reduces to equation (90).

4. On meso and synoptic scales pressure tendencies are $O(M)$. These can be computed by integrating a system of second order partial differential equations. This system supports internal and barotropic gravity waves, acoustic waves and Lamb waves and describes the dynamics of first order perturbation pressure, first order average vertical velocity and large scale vorticity (of the average horizontal velocity). It is a generalization of the set of equations obtained in the classical theory of perturbations from the state of rest for compressible stratified flows.

In the next step we want to apply the results obtained through asymptotic analysis in the construction of efficient and robust methods for the numerical prediction of atmospheric flows. The approach is similar to that in [13] where we have shown that it can be successfully used to extend Godunov-type methods for hyperbolic systems of conservation laws to cope with incompressible variable density flows. We want to follow the same strategy for low Mach number - low Froude number motion in the atmosphere. The asymptotic analysis shows that, at vertical scales comparable with the scale height, gravity strongly affects the structure of solutions of the governing equations in the limit of vanishing Mach and Froude numbers. Some of the results obtained in [8] (and systematically used in [13]) still hold: their reformulation in numerical methods, however, is not straight-forward. We also intend to derive guidelines and/or restrictions for a consistent parameterization of unresolved physical processes.

To achieve these goals we need a twofold strategy. On the theoretical side we must

1. Better understand the limit of neutral stratification i.e. of $\partial s/\partial z \to 0$. In particular we need to understand the behavior of the vertical velocity in this limit.

2. Better understand the long time cumulative effects of advection processes.

3. Better understand the implications of deviations from the derived constraints on the leading order solution, e.g. in initial data that are inferred from (under-resolved) measurements.

At the same time we need to define a small set of test problems on which the "dynamics" kernel of standard methods for the numerical computation of atmospheric motion can be run at different resolutions. These test problems should be as simple as possible (dry atmosphere, no bottom topography, rectangular computational domain, periodic boundary conditions) and at the same time retain the essentials of the dynamics of atmospheric motions: three space dimensions, various levels of stratification, rotation. The test problems are to be used both to analyze numerical solutions obtained via standard methods w.r.t. consistency with the results of the asymptotic analysis and to test the computational strategies.

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