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A NOTE ON DOMAINS OF DISCOURSE

Logical Know-How for Integrated Environmental Modelling

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Abstract

Building computer models means implementing a mathematical structure on a piece of hardware in such a way that insights about some other phenomenon can be gained, remembered and communicated. For meaningful computer modelling, the phenomenon to be modelled must be described in a logically coherent way. This can be quite difficult, especially when a combination of highly heterogeneous scientific disciplines is needed, as is often the case in environmental research. The paper shows how the notion of a domain of discourse as developed by logicians can be used to map out the cognitive landscape of integrated modelling. This landscape is not a fixed universe, but a multiverse resonating with an evolving pluralism of domains of discourse. Integrated modelling involves a never-ending activity of translation between such domains, an activity that often goes hand in hand with major efforts to overcome conceptual confusions within given domains. For these purposes, a careful use of mathematics, including tools of formal logic presented in the paper, can be helpful. The concept of vulnerability as currently used in global change research is discussed as an example of the challenges to be met in integrated environmental modelling.[1]

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1 *Introduction*

The present note assembles know-how concerning domains of discourse. This know-how is relevant, perhaps essential, for building computer models that combine knowledge from a variety of scientific disciplines. The relevant know-how stems mainly from that border region between philosophy and mathematics where the tradition of logic is cultivated. The note should be accessible to patient readers without much training in this field; equally patient readers trained in logic should be able to skim off the relevant arguments from paragraphs with a more introductory character than they would need.

The notation and concepts from logic that I use, in particular the concept of a domain of discourse around which the present note is built, are standard tools of logical inquiry. The way I use them, in particular the emphasis on exchanges of arguments and on bridges between domains of discourse, is more novel. It is oriented towards those kinds of scientific debates where scholars do not try to speak in chorus, but as participants in a conversation between professionals with different backgrounds. In this sense they are contributions to the business of managing interdisciplinary research.

I introduce domains of discourse as the contexts needed for reasonably coherent exchanges of logical arguments (2). Such arguments rely on an interplay between experience and tradition taking place in evolving domains of discourse (3). I emphasize the necessary pluralism of domains of discourse used by scientific communities and by society at large. The links between different domains are rooted in natural langauge and often rely on metaphor (4). In view of computer modelling, I am especially interested in linkages between mathematical and other domains of discourse; I I therefore give a closer look at the measurement processes that evolved with the rise of modern physics and mathematics (5). I then discuss linkages between mathematical and biological domains of discourse. Here, probability plays a key role (6). Across different domains of discourse, one can find the same rules of inference. However, rules of inference are heterogeneous, too. I argue that a pragmatic use of a variety of rules is appropriate (7). A pragmatic approach is particularly helpful whenever one must work without a consolidated domain of discourse. This is relevant for efforts to model social systems. I consider the example of vulnerability studies and highlight some related challenges (8). In particular, I look at the domains of discourse of economics in view of modelling vulnerability to global environmental change (9). Clarifying linkages between domains of discourse can help to overcome major conceptual confusions within a given domain as well. Mathematics is surprisingly useful, often simply indispensable to take care of existing linkages and to design new ones (10).

2 *Exchanging arguments*

If one is able to model some phenomenon or effect on a computer - e.g. the influence of greenhouse gas emissions on future sea level rise - , then this computer model can be used to argue about that topic. The model can be used to assess the pros and cons of various factual claims concerning the phenomenon as well as the pros and cons of various proposals for actions dealing with it.

Any exchange of arguments presupposes some shared context of objects, properties, relations, and inferences. This context is often called the domain of discourse (or the "universe of discourse", sometimes the "ontology"; for the history of the concept and various ways of using it see De Morgan 1847, Boole 1854, Peirce 1885, Mead 1934, Eco 1976, Brandom 1984, Goldblatt 1984, Gamut 1991, Heim and Kratzer 1998, Benjamins et al. 1999).

A given domain of discourse corresponds to a language game (Wittgenstein 1953; more on this in section 7). As Wittgenstein emphasised, however, not all language games are concerned with exchanges of arguments - language games of greeting, e.g., although vital for the human condition, are not. In the present note, I restrict the focus to language games built around exchanges of arguments. As I will discuss, these language games form a much more heterogeneous landscape than is sometimes assumed, and it is a landscape that is considerably larger than the territory claimed by scientific traditions.

In logic, terms referring to objects are either names of individual objects or variables ranging over all objects in a domain of discourse. Terms expressing properties and relations are called predicates. The arguments studied by logic deal with properties of and relations between objects. They build on inferences linking given statements to other ones.

With these and further concepts, logic makes pathways of thought explicit (Brandom 1984). It is interested in describing in general terms pathways of thought that appear repeatedly in wide ranges of arguments and are generally treated as valid. Logic is reflexive in the sense that logical concepts can be used to study the arguments of logic itself.

An important and well-reserached subject of logical inquiry are the natural numbers. They have names, e.g. "1", "2", "957" etc. Relevant properties are: "... is prime" or "... is even"; relevant relations: "... is the successor of ... " or "... $\geq$ ...". Relations may involve more than two individuals, as: "... is the greatest common divisor of ... and ... ", or: "... is the sum of ... and ...". The natural numbers, along with relations of succession, of being a sum, etc., form the domain of discourse of arithmetic.

One can refer to objects by using variables and predicates together with expressions saying that an object exists or that something holds for all objects. When dealing with natural numbers 1, 2, 3, etc., one can refer to the number 1 with the statement: "There exists an x such that for all $y : y \geq x$". In standard notation this reads: "$\exists x \forall y \ (y \geq x)$". The expressions "$\exists$..." (there exists ... such that) and "$\forall$..." (for all ...) are called quantifiers.

An elementary sentence states that some objects have certain properties or stand in certain relations to each other. Elementary sentences may be denied, as in: $\neg \exists x \forall y \ (x \geq y)$. Here "$\neg$" is used as negation sign. Elementary sentences may be connected into compound sentences, using sentential connectives like "and" (often written as $\wedge$), "or" ($\vee$), "implies" ($\Rightarrow$), and "if and only if" ($\Leftrightarrow$).

Logical argument is based on the fact of life that some sentences imply other ones. A simple example is: "All philosophers are wise. Some Greeks are philosophers. Therefore some Greek are wise"

To describe the pathway of thought involved in such an argument, logic uses variables for possible predicates along with variables referring to objects. This enables one to ask which pathways are sound for arbitrary predicates and objects - i.e. across different domains of discourse. Take "P", "Q", "T" to be three predicate variables and read "Px" as: "x has property P". Then the following sequence can be used to express the pattern of logical inference involved in our example:

$$\begin{array}{c} \forall x(Px \Rightarrow Qx) \\ \exists x(Tx \Rightarrow Px) \\ \hline \exists x(Tx \Rightarrow Qx) \end{array} \tag{1}$$

The sequence as such is not an argument, it is a form of possible arguments. It becomes an argument once the object variables are assigned to identifiable objects and the predicate variables are assigned to specific properties and relations. This happened in our initial example. The same holds for the inferences:

- "All gods are immortal. Some characters in the Ilyad are gods. Therefore some characters in the Ilyad are immortal."
- "All numbers divisble by three have a sum of the digits divisible by three. Some numbers divisible by seven are divisible by three. Therefore some numbers divisible by seven have a sum of the digits divisible by three"
- "All mammals breast-feed their babies. Some animals in the ocean are mammals. Therefore some animals in the ocean breast-feed their babies".

Arguments only work if the people involved in the discourse share an ability to identify relevant objects and to assess relevant properties and relations. This is a practical skill embedded in what Wittgenstein (1953) calls a form of life - a social pattern of human practice. Our first example presupposes a community whose members know how to distinguish philosophers from other beings, wise from non wise people, and Greek from non-Greeks. In a similar way, the ensuing arguments presuppose communities sharing specific domains of discourse - the ones of greek literature, arithmetic, and biology. In the words of G.H. Mead: "This universe of discourse is constituted by a group of individuals carrying on and participating in a common social process of experience and behavior, within which these gestures and symbols have the same or common meanings for all members of that group" (Mead 1933/1962, p.89).

A domain of discourse then involves a list of objects and a list of properties and relations. Both lists contain items that the members of some community are able to identify. Usually, the lists are open-ended; it is understood that further objects, properties and relations will emerge in the course of further arguments and investigations. However, not all lists will do: a domain of discourse also needs some kind of coherence. This coherence is based mainly on relations of inference. Sentences referring to a given domain of discourse hang together by patterns of inference, i.e. by possible arguments.

3 *Experience, tradition, and debate*

In patterns of inference, there is a fundamental difference between finite and infinite domains of discourse. To take a simple case, consider a domain of discourse consisting of Romeo and Julia. The statement that both Romeo and Juila are mortal implies that all individuals in this domain of discourse are mortal. Writing "M" for "is mortal", "R" for "is Romeo", and "J" for "is Julia" we get:

$$\exists x(Rx \land Mx) \land \exists x(Jx \land Mx)) \Rightarrow \forall x(Mx) \tag{2}$$

In a finite domain of discourse, a conjunction of a finite number of statements of existence can imply a universal sentence. In an infinite domain, this is not possible. To take a famous example, many even numbers are known to be the sum of two primes (as with $8 = 5 + 3$) and no even number is known for which this is not the case. But from these facts it does not follow that all even numbers are the sum of two primes. The latter statement - known as Goldbach's conjecture - still waits for proof.

This feature of infinite domains of discourse lies at the roots of Popper's (1935) influential view of how scientific inquiries work. As he noticed, for a statement to count as an observation it has to be a statement of existence. On the other hand, key insights of science are universal statements ranging over an infinity of possible cases. This is the typical form of what are called natural laws as well as of many other scientific findings. To say that salt is soluble in water is to make a universal statement about an infinite number of possible situations where salt and water might mix. In an infinite domain of discourse, arguments built on observations and nothing else would be utterly uninteresting: all they could do would be to rearrange these observations one way or other without ever leading to a universal sentence.

In real life, arguments are less boring than that because we can combine statements of existence with universal sentences stemming from a tradition of inquiry. Every child learning a language learns a whole lot of universal sentences with it. They may be challenged and displaced by other universal sentences, but the burden of proof lies with the challenger, and she will hardly be successful if she tries to talk without using universal sentences at all - language does not work that way. We have no doubt that all human beings are mortal although billions of them have not yet died. We have no doubt because it is part of a received wisdom cultivated through the criticism and experience of generations. Perhaps some day genetical engineers will advance a claim that they can produce immortal human beings. If so, a debate will be warranted on the credibility of that claim and what to do with it. For the time being, we will continue to live by our belief in the universal sentence that all human beings are mortal.

I leave open the question how far scientific inquiry really matches Popper's peculiar account of it and how scientific inquiry relates to other traditions of inquiry like democratic debate or religious search. What matters is that the coherence of a domain of discourse is due to a dynamic interplay between tradition, experience, and critical arguments. Tradition is not a preliminary source of knowledge to be supplanted by experience in the course of scientific progress: no observation is possible without concepts provided by some tradition. These concepts come with a fabric of sentences, including universal ones. Any particular sentence can be questioned, if there are reasons to do so. But to question a sentence already presupposes a whole fabric of sentences. Inquiries driven by reasonable doubt do not lead towards a world without tradition. What the progress of any particular inquiry does - be it in the realm of science or in other fields - is to cultivate traditions of inquiry by renewing them through fresh experiences and debates.

If a tradition of inquiry enables one to deal with an infinite domain of discourse, one can still cut out finite sub-domains for particular purposes. Logical arguments can be developed and exchanged in a domain of discourse involving the Earth and the sun only, as if no other bodies existed. Or one may study the permutations of just two or three numbers, disregarding all other numbers. Out of an infinite domain, other infinite sub-domains may be cut out as well. In arithmetic, e.g., for some purposes one may be interested in prime numbers only, leaving out the other ones. In physics, out of a class of possible oscillations of a pendulum for some purposes one may wish to consider only a certain - still infinite - sub-class.

Given a finite domain of discourse, one can always try to embed it in a larger one. Sometimes, it also turns out to be possible to aggregate two or more infinte domains into a more comprehensive one. As a domain of discourse is characterized by patterns of inference linking its various sentences, such aggregation only works if new patterns of inference running across the different domains considered can be found. In the development of science, successful aggregation of this kind is often seen as major progress. Such was the case with Maxwell's treatment of electricity and magnetism, and later on with Einstein's development of a single geometrical framework to deal with electromagnetism and gravity.

One may wonder whether integrated environmental modelling implies the possibility, perhaps even the necessity, to expand this kind of synthesis to the point where it would include biological, psychological, and socio-cultural phenomena as well. This would mean that the interplay of experience, tradition, and debate can, perhaps even must, lead towards one basic domain of discourse, providing a comprehensive frame in which all other domains are to be integrated.

4 *The case for pluralism*

Such a view of integrated environmental modelling gains some plausibility from the influential image of the world as ultimately consisting of physical matter. Often, such matter is visualized as an arrangement of small particles whose complex dynamic patterns lie behind the stars, stones, bodies, faces, melodies, thoughts, emotions and whatever else we experience. According to this image, there is a single ultimate reality that can be described by a single domain of discourse, the one of physics.

Recently, this image has been renewed thanks to advances in molecular biology and brain research. Our ability to relate important aspects of human behavior, perception, emotion, and thought to biochemical processes in the brain has been greatly enhanced. Moreover, several of these biochemical processes can be linked to genetic mechanisms that play out in biological evolution. Many see this as a confirmation of the view that there is an ultimate reality consisting of physical matter.

However, some arguments to the contrary deserve attention here. To discuss them, one needs a domain of discourse for elementary physical objects, call them "$a, a', a'', ...$", characterized by various physical predicates, call them "$P, P', P'', ...$". Starting with these objects and predicates, one can identify further physical objects, namely sets of elementary ones. To study these, one needs a predicate of membership, which I write in the usual fashion as ".. $\in$ ...". I treat sets in the standard way as collections of objects fulfilling some predicate P: $a' = \{x : Px\}$. In such a domain of discourse, logical argument about physical objects is possible in a rigorous fashion.

For example, one may form a term for a set of all red objects - with 'R' as the predicate for "... is red" one would simply write $a' = \{x : Rx\}$. If one can show that such a set exists and that it is unique - these conditions are less obvious than one may expect - to say that something is red amounts to the same thing as saying that it is an element of the set of red objects. With the membership predicate one can introduce not only sets of objects, but also sets of such sets, etc.

This leads to the venerable view of an atomistic universe. One can try to sketch such a universe by taking elementary particles as the logical atoms (the a, a', etc.), and imagining these as arranged in the atoms of modern physics, these in molecules, etc., etc. until one reaches the mountains and oceans, the planets and the stars. In these terms one can sketch a map of our galaxy and the world beyond.

4.1 *The necessity of non-physical objects*

In such a domain of discourse one can describe physical objects in mathematical terms as follows. Consider sets containing exactly one elementary object. Call the set of all these sets the number 1. More generally, call the set of all sets with n elementary objects as elements the number n. Now to say that some set of elementary objects has 7 members amounts to the same thing as saying that the set in question is an element of the 7-set. Along these lines, numbers can be treated as properties of elementary physical objects on a par with colors and the like.

Showing that logic, mathematics, and physics could be combined in such a way was one of the great achievements of work on the foundations of mathematics (Russell and Whitehead, 1910-13). Perhaps the greatest merit of that work, however, has been the gradual clarification of how incoherent a physicalist world-view is bound to be.

First, consider the set-theoretic proof that "there is no universe" (Halmos 1960, sec.2). Let a be an arbitrary set and a' the set of all elements of a that are not elements of themselves:

$$a' = \{x : x \in a \wedge \neg(x \in x)\} \tag{3}$$

Then $\neg(a' \in a)$. Suppose otherwise. By its definition, a' cannot be an element of itself (because it only contains elements that are not elements of themselves). But then if it were a member of a it would fulfill the definition of members of a' and so would bound to be an element of itself. This contradiction shows that a' cannot be an element of a. Therefore, a cannot contain everything, as it cannot contain a'. But a was a completely arbitrary set, and so there can be no set containing all sets.

What is this proof about? In a way, it is about the structure of language, more precisely: of the kind of language needed to talk about sets of objects.

If one wants to complement formal proof with intuitive associations, it may be helpful to think of the world for a while as consisting of quarks, photons, electrons, and other elementary particles. An atom then is a set of such particles and not an element of itself, while elementary particles are elements of themselves (as Quine, 1969, p.30ff, shows, this is a nice way of distinguishing elementary objects from non-elementary ones).

Next, I focus on the property called spin. Photons, electrons, hydrogen atoms all have spin (polarized sunglasses use it with photons). If one considers any of these entities in isolation under suitable experimental conditions, their spin is either up or down. If one considers a pair of such entities under analogous conditions, however, their spin can correspond not only to the four combinations of up and down, but also to an additional state that is sometimes called one of entanglement.

This kind of situation was ignored by classical physics. It is widespread in modern physics as in many other fields. One can describe it by distinguishing between a photon and a set consisting of a photon: taken in isolation, photons have spin up or down, as members of sets, they have spin up, down, or entangled. Of course one can use other words to describe this situation, but the key point is quite clear: being an element of larger structures changes the possible properties of elementary particles. The same holds when structures involving elementary particles are elements of further structures. When two hydrogen atoms, e.g., are combined in a hydrogen molecule, their two electrons behave as a two-electron cloud. In that cloud, there are no distinguishable trajectories of single electrons. To identify any property of a single electron, one must break it away from the molecule. But by studying an electron in isolation, one would never find out what electrons can do as components of molecules.

Knowing an elementary particle means knowing what it can do. One cannot learn this without considering the infinite set of possible structures they can be involved in. A world consisting of elementary particles consists of entangled pairs, n-particle clouds and suites of more complex structures as well, leading from quarks to jaguars and beyond (Gell-Man 1994). The "no-universe" argument then says that it would be logically incoherent to imagine some ultimate suite of structures out of which all the other ones could be explained. Whenever one can talk about some such suite of structures with the means of mathematical logic, one can find some even more comprehensive suite of suites.

The "no-universe" argument is not restricted to elementary particles. In general terms, it implies that one cannot effectively summarize all the possibilities involved in a situation where along with any given sets one also needs to consider sets of these sets.

Technical as they are, at the beginning of the 20^{th} century issues of this kind triggered one of the deepest crises in the study of the foundations of mathematics and of philosophy in general, because they suggested that scientific discourse is unable to refer to the world as a whole. The philosophy of Wittgenstein, one of the most influential philosophers of the 20^{th} century, can be read as a series of variations on this theme.

Second, consider the fact that any physical domain of discourse can be described with sentences expressed by finite strings of symbols. The symbols are taken from two kinds of lists. There is a finite list of symbols for quantifiers, connectives, and brackets, and there are infinite, but countable lists of symbols for objects, properties, and relations. Objects may be referred to by names, $a, a', a'', ...$ or variables, $x, x', x'', ...$ Properties and relations may be expressed by predicate symbols $P, P', P'', ...$ Expressions for names, object variables, and predicates can be iterated in order to represent arbitrarily long lists of objects and predicates. On the basis of these symbols, all strings containing just one symbol can be put in lexicographic order, then the same can be done for the strings containing two symbols, and so on for strings of arbitrary length. This yields a countable list, and the meaningful formulae are bound to be an infinite subset of this list. Therefore, the whole domain of discourse can be represented by a countable set of formulae. But this means that the objects that can be identified by these formulae will form a countable set, too. In other words, each identifiable object can be associated to a natural number.

Besides natural numbers, however, real numbers - like $\sqrt{2}$ and π - are basic for any meaningful description of physical realities: $\sqrt{2}$ matters wherever the diagonal of a square shows up, π matters wherever the diameter of a circle or the shape of a wave show up. Real numbers are remarkable among other things because they cannot be counted. Suppose otherwise. Then they could be put in a list. Imagine this list in tabular form with each row representing a real number between 0 and 1 in decimal notation. Take the first digit in the first row and add 1 modulo 9 to it: if the digit is $x \neq 9$, it becomes $x + 1$; if the digit is 9, it becomes 0. Following Cantor's (1891) diagonal argument, do the same for the second digit in the second row, the third in the third, etc. The result will be a real number that appears nowhere in the list. But the list was assumed to contain all real numbers. In other words, the assumption that the real numbers can be counted leads to a contradiction.

This means that while one can study $\mathbb{R}$, the set of real numbers, as an object within some domain of discourse, there will always be properties of $\mathbb{R}$ that cannot be specified in that domain. After all, containing some specific number as an element is a property of $\mathbb{R}$. Moreover, any formal theory of real numbers will be amenable to a rigorously defined re-interpretation that makes it a theory - although with very different meaning - over natural numbers. Any theory that can be formulated by a countable set of formulae allows for an interpretation over the set of natural numbers (Skolem 1941).

By the first argument, a physical domain of discourse does not enable one to talk about the world as a whole. By the second argument, such a domain of discourse needs objects - real numbers - and properties - those expressed with the help of real numbers - that cannot be fully specified in any given domain. A more sophisticated argument - due to Gödel (1931, with Hofstadter, 1979, providing an exposition accessible to the non-specialist) - shows that this already holds for any domain of discourse comprising the natural numbers. It also shows that, given a domain of discourse comprising the natural numbers, it is always possible to form a new domain specifying some additional property of these by a suitable formula.

The third argument to be considered shows that even a physical domain of discourse leads to non-physical objects. It runs as follows. Given an arbitrary set X and an arbitrary predicate P, one can always form the subset U of all elements v fulfilling that predicate:

$$\forall x \exists U \forall v \ ((v \in U) \Leftrightarrow (v \in X \wedge Pv)) \tag{4}$$

This is an example of a universal sentence corroborated by a tradition of critical inquiry: the various axiomatic and non-axiomatic approaches to set theory developed in the course of time all share it one way or another. Given a domain of physical objects, then, one can take an arbitrary set of physical objects (call it a'), form the predicate: "... is not an element of a' " and gets:

$$\exists U \forall v \ ((v \in U) \Leftrightarrow (v \in a' \wedge \neg(v \in a'))) \tag{5}$$

This set is empty, as nothing can simultaneously be a member of a' and not be a member of a'. So there is an empty set. Inconspicuous as it may look, it still is a non-physical object. And as a matter of fact, it is the ancestor of a whole domain of discourse of non-physical objects.

By received wisdom, two sets are equal if they have the same elements:

$$\forall u \forall x \forall y \ ((u \in X) \Leftrightarrow (u \in Y)) \Leftrightarrow (X = Y)) \tag{6}$$

As a result, there can be only one empty set, as there are no elements by which two empty sets could be distinguished from one another.

Another piece of received wisdom says that given an arbitrary set U with no more than one member, one can always form the set V containing that set.

$$\begin{aligned} &\forall U \forall x \forall y (((x \in U \wedge y \in U) \Leftrightarrow x = y) \Rightarrow \\ &\exists V \forall Z \ (U \in V \wedge (Z \in V \Leftrightarrow Z = U))) \end{aligned} \tag{7}$$

Therefore, there is a set containing only the empty set. This is a second non-physical object. It is different from the empty set precisely because it has a different element. By the same reasoning, one can construe an infninite sequence of sets, each one containing only its predecessor. But this sequence has all the properties known from natural numbers. Therefore, starting with the empty set, one can construe the natural numbers. From there, one can go on to construe the other kinds of numbers, functions between them, and the whole fabric of mathematical entitities.

But now we have two kinds of natural numbers: those introduced above as sets of elementary physical objects, and those constructed now out of the empty set. As if this were not enough, we can construe a third kind of natural numbers by repeating the first construction with sets of sets of objects - and this proliferation cannot be stopped: the attempt to describe the world in purely physicalistic terms leads to a world populated not only by elementary particles, but also by sets of such particles, by sets of such sets, ..., and by an uncountable multitude of natural number systems as well.

Quite likely, one may feel that the word "exists" is used in a different way when one says that a planet exists than when one says that a prime number exists. But this is precisely the point of distinguishing different domains of discourse. To say that something exists places it in some domain of discourse. To say that the word "exists" is used in different ways means to distinguish different domains of discourse.

4.2 *Mutually irreducible objects*

For the introduction of mathematical objects, it turns out to be quite irrelevant whether one starts with physical ones. The former are needed to talk about the latter, and they cannot be construed out of these. One could introduce mathematical objects starting with greek gods as well, and they cannot be construed out of those, either. Therefore, one may just as well start with mathematical objects for their own sake, as has become common practice.

If one cannot reduce mathematical domains of discourse to physical ones, does it work the other way round? This is the ancient idea that the ultimate reality is the one of mathematics. It was first developed by the Pythagoreans, later in a different version by Platonists. It is so much at odds with the fabric of contemporary science and technology that it is hard to propose as a serious view. Nevertheless, at least a short discussion is appropriate.

In the times of classical physics, typical objects of investigation were physical bodies, characterized by properties like position and momentum, and linked with each other by relations like attraction and impact. Meanwhile, physics itself has evolved so as to use other domains of discourse as well. Often, the relevant objects are regions in space-time with some given shape - cubes, spheres, whatever. These regions are then characterized by properties like the fact that in a specific region some given body will experience a specific acceleration, and relations like the fact that this very acceleration will change the analogous property of another region. Systems of space-time regions of arbitrary size, characterized by shared properties and relations, can be used to construe physical fields. By taking the limit of sequences of space-time regions converging to a single point, for example, one can construe a field for the density of some substance in arbitrary regions of space-time.

Before this background, one might argue that the three-dimensional space of real numbers, $\mathbb{R}^3$, "is" the physical space we humans live in, that time is an additional dimension of that space, with further dimensions characterizing the densities of mass, charge, and other variables in physical space.

There are many problems with such an argument. First, there is a myriad of situations that can be described in terms of three-dimensional spaces of real numbers. How are we to distinguish the description of the position of a body in physical space from, say, the description of three bodies characterized by their respective masses? We do it by relying on non-mathematical predicates like those used to talk about physical space and time as well as mass.

Second, since the days of Einstein it is clear that a description of space-time in terms of $\mathbb{R}^4$, the four dimensional space of real numbers, is quite tricky: what is simultaneous for one observer need not be so for another one. Additional subtleties come into play with quantum mechanics. Depending on circumstances, physicists use different mathematical objects to talk about time - and to say that they are actually talking about time means using a non-mathematical predicate.

"Theories pass. The frog remains", as the French biologist Jean Rostand said. "Frog" is a non-mathematical predicate, as is "star" or "gravitation". One can use mathematical objects in many ways to talk about non-mathematical ones, but one needs non-mathematical predicates to do so.

Recently, another way of looking at the physical world as rooted in mathematical reality has found increasing interest: as computers are built out of physical materials, one can try to describe the physical world as one huge computer (Zuse 1967, Wolfram 2002). This is the kind of image that can inspire wonderful insights as well as considerable nonsense. I will not try to explore the many questions that may be asked in such a setting, but focus on the technical point that matters here: can physical predicates be defined in terms of mathematical ones?

It is clear that a given mathematical structure - say, a specific computer program - can be implemented in two different pieces of hardware. Moreover, a given physical pattern can be looked at as representing two different mathematical structures. A striking example in the domain of computation is given by quine programs. These are programs that make a computer print exactly their own source code. They are named in honor of the logician W.v.O. Quine, whose work is relevant to our problem (Quine 1960, 1969). A quine program for the language Python looks as follows:

$$\mathtt{f} = \mathtt{lambda} : [`\mathtt{f} = \mathtt{lambda} : `, `; \mathtt{print}\ \mathtt{f}()[0], \mathtt{f}(), \mathtt{f}()[1]`]; \mathtt{print}\ \mathtt{f}()[0], \mathtt{f}(), \mathtt{f}()[1] \tag{8}$$

Read as a program, the first occurrence of the expression "f=lambda :" is part of a function definition; read as output the same expression is an element of the value returned by that very function. If we run the quine on a computer, it will be obvious from the circumstances which one is the case. If we are faced with (8) simply as a physical pattern of ink on paper, there is no way to tell.

This kind of ambiguity carries much further, as in the case - discussed in section 4.1 - of theories of real numbers that can be re-interpreted as theories defined over natural numbers. Because of such difficulties, a description of the physical world as a huge computer is surprisingly ambiguous. If the physical world can be described as a computer, then it can be described as other computers, too - and what makes these computers equivalent is the fact that we use them to describe the same physical world. There is no one-to-one correspondence between physical and mathematical objects. In the picture of the physical world as a computer as in the example of space-time discussed above, we use physical predicates to identify the non-mathematical objects we want to talk about.

With regard to the domain of discourse established with the emergence of modern physics one thing is quite clear: one does not need any mathematical objects or predicates to observe that Newton's apple falls down to earth, or that a stone and a brick dropped simultaneously from the oblique belltower known to Galilei hit the ground simultaneously. Physical predicates like mass, momentum, electrical charge, etc. do not depend on mathematical concepts. One certainly does not get very far in discussing physical objects without referring to mathematical ones, but the two inhabit different domains of discourse.

Besides the domains of discourse of physics and those of mathematics, a third kind of domains is unavoidable even for the most strenuous attempts to produce a physicalist account of the world: one needs a domain in which Newton, Einstein, as well as Dr. Jones from the lab nearby, can be identified by name. This is not just a nice add-on to provide due recognition to scholarly work, it is indispensable in order to discuss empirical observations. To do so, one must be able to refer to the person that made the observation. More generally, scientific inquiry proceeds by an on-going exchange of arguments where the participants need the ability to refer to each other. Trivial as this fact is, it implies that besides physical and mathematical domains of discourse, one also needs domains of discourse in which the relevant individuals are human beings. Of course we can describe ourselves as physical objects. But we can do so only after having identified a certain physical body as our own, or as the body of Einstein or of Dr. Jones. We build up the myriads of co-ordinate systems available to do physics starting with space-time regions that we can identify with our own bodies (Brandom, p.444f).

Once one has got acquainted to the notion of a plurality of irreducible domains of discourse, the variety of scientific disciplines and terminologies appears in a different light. The frog is a physical object no less than the stars, but there have been good reasons to develop domains of discourse to talk about living beings in different ways than about inanimate objects. As a matter of fact, even within physics there are rather different domains of discourse. The space-time regions of quantum mechanics, e.g., are different from the ones of relativity theory. And domains of discourse dealing with living organisms and their environments are even more heterogeneous.

Mathematical discourse has its pluralism, too. Even the natural numbers, innocuous as they may look, replicate themselves through the worlds of mathematics (see section 4.1). One way of dealing with this situation is to focus on categories of mathematical objects that form an interesting domain of mathematical discourse. Predicates like "...is a natural number system..." or "...is a vector space..." cannot be used to form sets. A set of all natural number systems, e.g., would be no less self-defeating than a set of all sets (the set of all natural number systems could be the first element of a still larger natural number system). But these predicates can be used to describe categories of mathematical objects that fit the same domain of discourse. "A category may be thought of in the first instance as a universe for a particular kind of mathematical discourse" (Goldblatt 1984, p.1).

In the course of time, such a focus on categories could lessen the weight currently put on set theory as the foundation of mathematics - for example, by helping to understand why the metaphor of a house is somewhat misleading here. The argument of section 4.1 could then be cast in a less set-theoretic format. When talking about physical objects, one might try to substitute the membership predicate with a predicate for "... is a part of ..."; when talking about mathematical objects, one might introduce the natural numbers directly via predicates of identity and succession. The literature on such attempts shows that they increase the range of "ways of worldmaking" (Goodman 1978) at our disposal, and thereby the pluralism of domains of discourse. The physicalist worldview would not be restored; rather, mathematical physics would use numbers and functions to talk about stars and stones, and these would complement each other as mutually irreducible objects.

Dr. Jones and other human beings enter the picture as further irreducible objects. I will not discuss domains of discourse dealing with human beings here, as I will do so in sections 8 and 9. The key point of the present section will be reinforced there: from a logical point of view, the landscape we live in is not one world, but many, not a fixed universe, but a multiverse resonating with an evolving pluralism of domains of discourse.

In ordinary language, of course, one can move across these domains and happily talk about the world as a whole. If asked what one means by that expression, one can draw on the resources of poetry rather than those of logic. One can build up metaphors, perhaps invoking images of waves and the ocean, of people linked by family resemblances, of the melodies in a fugue, asking one's interlocutor whether these metaphors are helpful to her, modifying them, introducing new ones until she feels comfortable to let the conversation flow further - or until one changes the subject. What one will not be able to do is to form a sensible expression for "the world as a whole" in the kind of language required to talk in a logically coherent way about the objects of physics. To the extent to which it makes sense to talk about the world as a whole, what is called "the universe" in physics (even if it includes all the parallel universes advocated by authors like Tegmark, 2003) is but one aspect of the world we live in.

This situation sheds new light on the relation between scientific knowledge and other forms of knowledge. Scientific knowledge as we know it usually tries to avoid moral or aesthetic judgement. But of course there are moral and aesthetic domains of discourse, and the scientific enterprise is by no means the only arena for exchanging arguments.

As soon as one starts talking about human beings as well as about quarks, one is quite likely to talk about human values as well as about the flavors of quarks. And once you recognize the plurality of domains of discourse, "you get rid of the notion that quarks and human rights differ in 'ontological status' " (Rorty 1998). The plurality of domains of discourse includes a wide array of scientific ones as rich and fascinating examples, but it is not at all restricted to the horizon of the scientific enterprise.

5 *Numbers, bricks, and coins*

Sentences about greek gods imply other sentences about greek gods and sentences about human beings imply other sentences about human beings. What is less clear, and perhaps more interesting, is the question whether, and if so, how, sentences about gods imply sentences about humans as well as the other way around. From a logical point of view, a similar relation holds between sentences about numbers and sentences about stars. The sentences of arithmetic hang together by logical inference, and so do the sentences of astronomy. It is clear that the two are related, but it is not so clear how.

Today, few people feel comfortable with seeing the world we live in as inhabited by greek gods. More feel comfortable with seeing it as inhabited by protons and electrons. Most, however, are quite comfortable with seeing our world inhabited by flowers and birds and to treat these as objects of discourse in their own right, without bothering about how they may relate to the world of quantum fields. To some extent, we have simply learnt to live in a plurality of domains of discourse. This raises the question of how we link these domains with each other in our thoughts and actions.

A particularly instructive example for studying this problem is given by the traditions of mathematics and physics. Clearly, over the last centuries, they have developed hand in hand. And the most elementary connection between mathematical and physical domains of discourse was and is provided by the practice of counting.

Counting things requires the ability to identify one of them, to identify a next one, and so on, without counting anything twice, and to identify a last one. When dealing with mathematical sets, one can try to establish a one-to-one correspondence between the members of a set and a finite sequence of natural numbers that starts with 1 and moves stepwise from one natural number to its successor. If one succeeds, the set has as many members as indicated by the last number in the sequence. To take a simple example, consider the set comprising the prime numbers smaller than 10, i.e. $\{1, 2, 3, 5, 7\}$ (counting 1 as prime). A one-to-one correspondence can be established with the sequence [1,2,3,4,5], so the set has five members. The diagonal argument considered above shows that no such correspondence can be established between the set of real numbers and any finite sequence of natural numbers (it even shows that no one-to-one correspondence can be established between the set of real numbers and any infinite sequence of natural numbers).

When dealing with non-mathematical objects, there are no clear rules as to when they are a set and when they are not, so things are slightly different. For an emotion, a cloud, a design for an airplane to become objects of logical reasoning, there must be some domain of discourse in which they appear. It is easy to produce such a domain involving a particular emotion, cloud, and design - e.g. by telling the story of how an engineer experiencing a particular emotion was inspired by a particular cloud to design a particular airplane. It is certainly instructive to try to produce a domain of discourse for emotions, clouds, and airplane designs in general. It is instructive especially because it is difficult, perhaps nonsensical to produce such a domain. So far, psychologists, meteorologists, and engineers live happily without any agreed domains of discourse for emotions, clouds, and airplane designs. Things do not get easier when the relevant objects are further delimited by some additional criterion and one tries to count them. A symphony, e.g., may trigger emotions, but it is not clear whether the emotions triggered by a symphony can be treated as a set on a par with mathematical sets, nor is it obvious how they could be put in a sequence. One may look at the sky and identify a particular cloud, but it is not so clear what it means to talk about the set of clouds in the sky, and perhaps even less what it would mean to count them. One may think about possible designs of airplanes for freight traffic - again it is not at all trivial to treat these as a set, let alone a countable one.

The paradoxes of set theory have taught logicians to be careful about how to introduce sets. By now there are rules for introducing mathematical sets in ways that are known to avoid logical paradoxes (although they leave more than enough other puzzles to think about). There are no similar rules for introducing non-mathematical sets. What one can say, however, is that many things - the fingers on our hands, the apples we collect from a tree, the people we meet at a party - can be put in finite sequences by a successor relationship, and then one can safely treat them as members of sets.

Statements like: "There are twelve Apostles", or: "This is the fifth mistake" can be rendered with a predicate involving a relation between two nodes: a natural number specified one way or another ("twelve", "fifth") and a group of objects specified by another predicate ("the Apostles", "the sequence of mistakes up to this one"). Writing a predicate for counting as "...C...", an arbitrary natural number as "n", and a predicate specifying the list of objects counted as "L...", one can express a count of things as:

$$\exists x(xCn \wedge Lx) \tag{9}$$

The ability to handle such predicates is trained woldwide in grammar school. It has become part of the global culture shared by humankind.

Counting is one way of associating things with numbers, namely with the natural numbers. One may say that by counting one measures the size of a countable set in terms of its members. What is usually called measurement provides a further link between things and numbers, this time the real numbers.

Consider a pile of bricks and a rod such that one brick is much shorter than the rod. One may be able to put several bricks in a line and count what is the maximum number of bricks yielding a line that is still not longer than the rod. Suppose the number is 3, with the rod still longer than three bricks. Then one may perform a second count with smaller bricks (or marks on the original bricks). Suppose one can use a second pile of bricks such that a line of ten short bricks is of equal length as one original brick, and that now the maximum number of bricks in a line not longer than the rod is 31, with still a difference left. One may go on with such counts, and find out that they can be expressed in terms of the original bricks by a number starting with the following digits: 3.14159...

Statements like: "The rod is 3.14159... bricks long", or: "The car drives at 70 kilometers per hour" can be rendered with a predicate involving a relation between three nodes: an object, a real number (including integers), and a unit of measurement whose dimension (length, weight, etc.) is specified by the predicate. Writing "Q" for the predicate rendering the measurement result, "r" for an arbitrary real number, "u" for the unit, and "P" for the predicate identifying the object to be measured, one gets:

$$\exists x(Px \wedge Q(x,u,r)) \tag{10}$$

Along with counting, the language of practical measurement has become a vital part of global culture - without it, we could not share the technologies that increasingly shape the world we live in. The language and practice of measurement are so pervasive in the contemporary world, and so important for purposes of integrated modelling, that they deserve careful analysis.

Practical measurement starts with the ability to make comparisons, e.g. to say that something is longer than someting else. Among the objects that are amenable to such comparisons, they provide a strict ordering relation, i.e. a relation that is both asymmetric and transitive:

$$\begin{aligned} &\forall x \forall y (x \succ y \Rightarrow \neg(y \succ x) \wedge \\ &\forall z (x \succ y \wedge y \succ z) \Rightarrow x \succ z) \end{aligned} \tag{11}$$

If an object is neither longer nor shorter than another one, they stand in a tolerance relation to each other (Chajda 1991, Ungváry 1983). A measure of length, however, is more than a tolerance relation.

Understanding tolerance relations is essential when looking at the relation between mathematical and non-mathematical domains of discourse. To talk about tolerance relations in general terms, one needs a suitable domain of discourse. Such a domain can be introduced by taking the relations one is interested in as objects and characterizing them in terms of further properties and relations. For example, one can say that a tolerance relation does not necessarily have the property of transitivitiy: One may not see any difference in length between a sequence of rods presented pairwise, but clearly perceive that the first rod is longer than the last one.

When engaging in practical measurement, we refine the judgements leading to tolerance relations by relying on a series of practical abilities, skills that have been developed over generations. They include the ability to produce measurable objects by combining other measurable objects - as when we put bricks in a line -, or by considering the difference between two such objects - as when we consider that part of a rod that extends beyond an adjacent brick. I write $\oplus$ for the combination of objects, and $\ominus$ for the forming of differences; I define ω as the empty difference, resulting when we compare an object with itself (as usual, I define a name by using the identity predicate preceded by a colon):

$$\begin{gathered} \forall x \forall y \exists u \exists v \\ (x \oplus y = y \oplus x = u \wedge \\ x \ominus y = v \ \wedge (x \oplus y) \ominus y = x \\ x \ominus x = y \ominus y) \\ \omega := x \ominus x \end{gathered} \tag{12}$$

Of course, the ability to manipulate measurable objects is useful only because it relates to the results of the measurement activities. The key link between the two is provided by the fact of life that if one combines a given object one after the other with two objects that differ in length, the difference shows up again in the resulting combinations:

$$\begin{gathered} \forall x \forall y \forall z (x \succ y \Rightarrow x \oplus z \succ y \oplus z \wedge \\ z \ominus y \succ z \ominus x \wedge x \ominus z \succ y \ominus z) \end{gathered} \tag{13}$$

In mathematical terms, the resulting structure can be described as a lattice ordered group (comparing objects provides the lattice order, combining objects into new ones provides the group structure). But keep in mind that so far I am not referring to mathematical, but rather to physical objects (this is also the reason why I am not focussing on mathematical measure theory). To discuss the link to the real numbers, I need a few more steps.

In practical measurement, we also rely on our ability to subdivide measurable objects into smaller ones - as when we put a mark on a brick to obtain a finer scale of measurement. One can easily describe this ability as follows:

$$\forall x \exists y (x = y \oplus y) \tag{14}$$

Moreover, we treat something as a single measurable property only if we can trust that the relevant objects are commensurable in the following sense: by taking an arbitrary measurable object and combining it with itself again and again, one can surpass the measure of any other given object after a finite number of steps. To express this so-called Archimedean property one needs a predicate "...*A*..." (for xAy read: x is a multiple of y) saying that with regard to the comparison under consideration an object x is equal to a a smaller object y combined with itself a finite numbers of times. As usual, I define a predicate using the "if and only if" connective preceded by a colon (I use three dots to indicate that the binary operation $\oplus$ is repeated a finite numbers of times):

$$xAy :\Leftrightarrow x = y \oplus y ... \oplus y \tag{15}$$

With this predicate one can then express the Archimedean property:

$$\forall x \forall y \exists z (z \succ x \wedge zAy) \tag{16}$$

Finally, in practical measurement we trust that at least in principle we could make our measurements as fine-grained as we wish. I express this principle with the help of an operation for best approximation: $x \star y$ (read: the best approximation of x by y) is the sequence of iterations of y that is the longest sequence still not longer than x:

$$\begin{aligned} z = x \star y &:\Leftrightarrow (zAy \wedge \forall u((uAy \wedge u \not\succ x) \Rightarrow u \not\succ z)) \\ &\forall u \forall v \exists x (v \succ u \ominus (u \star x)) \end{aligned} \tag{17}$$

Subdividing and combining things so as to get further commensurable things, securing the Archimedian property, being able to make finer and finer comparisons - it is this whole combination of practical skills and orienting principles that enables us to make the step from tolerance relations between physical objects to measurements associating these objects with real numbers expressing their length.

To do so, we pick a unit of length - perhaps an iron rod placed in a building in Paris, or a more subtle device described in widely accepted documents - and express the length of objects in terms of this unit. If we choose another unit, we can then convert the measurements performed with the first one. All we need to do ist to divide these measurements by the length of the second unit as measured in terms of the first one. This simple transformation is due to the rich structure described above.

I have discussed practical measurement taking the example of length. A similar account can be made for the measurement of time, although here one needs a particular concept of repetition: putting two rods in a line is not the same as repeating a periodic process like the swing of a pendulum. Other measurements deal with mass, electrical charge, and many more physical properties. Sometimes the idea of combining units is straightforward, as with mass, sometimes it is more subtle, as with mass per volume or with temperature. I give a brief look at these cases because - due to their subtleties - they are useful to assess the role of practical measurement in other domains of discourse, in particular with regard to the role of measurement units.

As an example for comparing masss per volume, consider the case where a piece of solid matter - say the crown whose gold content Archimedes had to assess - is put in a recipient containing some liquid in which that piece of matter sinks to the ground. If this operation is repeated with a piece of some other substance, with the same mass as the original one, one can easily assess whether it displaces a larger quantity of liquid or not, thereby comparing mass per volume. (By taking a piece of pure gold of equal weight as the crown, Archimedes could prove that the crown was by no means made of pure gold.) To produce a practical measurement, one would need a seqence of comparisons where at each step the mass per volume used for reference is split in smaller components. Of course, this cannot be achieved by splitting an object such as a piece of gold, as one might consider in order to measure mass. Weight per volume is not an extensive variable. One way of splitting it would be to look for a piece of solid matter such that it would be equal in mass to the original object, while its volume would be twice as large.

Temperature is similar to mass per volume in so far as it can be expressed as energy per particle - where both "energy" and "particle" may need to be further specified depending on the problem at hand. To split the temperature of a gas contained in a given volume, one can compress the gas to half the volume, cooling it down until its pressure is equal to the one in the original volume. To split the temperature of solids or liquids, one can use a volume of gas with the same temperature as the solid or liquid.

Temperature is particularly instructive because for historical reasons it is often expressed in physically weird units - weird because they do not allow for the simple transformation we know from meters and yards. The reason, however, is not some odd property of temperature, but a peculiar system of introducing the measurement units. One would get the same situation with length if a distance of zero meters were defined as equal to three yards and a distance of one meter as equal to seven yards. If on the other hand one sticks to the absolute zero of physical temperature as the reference point, the same conversion as with units of length results.

I have looked at counting and practical measurement as two ways of associating physical objects to numbers - as special cases of measurement, if you like. They are not the only ones, although historically they have been the most influential. A third such link stems from the notion of probability. It relies on a double sequence of counts, as practical measuring does. But in this case, at each iteration one counts an event that may happen or not happen - as when we ask whether tossing a coin will yield heads up or not. The event count, therefore, will never exceed the iteration count. If one labels the event count as v and the iteration count as n, then $\frac{v}{n}$ is the relative frequency of the event under consideration. When this relative frequency converges to an identifiable limit, this limit is usually called the probability of the event.

Based on probabilities, additional measures can be defined, like the one for entropy in statistical mechanics or the similar one for information in communication theory. Unfortunately, much confusion has been generated by the fact that the word probability is also used in very different, but equally fruitful ways. They run through heterogeneous domains of discourse, ranging from mathematics to decision theory and other disciplines, and often they have little to do with relative frequencies. But as far as bridges between mathematics and physics are concerned, the frequentist account is the one that matters.

So far, I have discussed different kinds of measures - based on counting, practical measuring, and probability estimates - as linking mathematics and physics. They provide bridges between mathematics and chemistry as well, mainly on the basis of distinguishing different chemical substances and then measuring their physical properties.

In close relation with chemistry (Primas 1981), it would be interesting to analyse measurement processes dealing with quantum mechanical systems. There, counting acquires new relevance, because many quantities come in discrete steps only. Moreover, probability plays a particular role. When considering, say, an electron, it is possible to give a probability distribution for its possible positions, and another such distribution for its possible momenta. These distributions, however, cannot be simply combined to generate a joint probability distribution for position and momentum, as this would contradict the uncertainty relation between these two variables. And by now it is widely understood that in quantum mechanics measurement is by no means the kind of detached observation known from astronomy, but actually modifies the system under consideration. Discussing these matters, however, would require a far-reaching study adressing a whole range of unresolved issues. A few related remarks in section 7 will have to suffice.

6 *Numbers in the Garden*

The domains of discourse used in biology rely on practical measurements of time, distance, mass, and other physical and chemical properties. Specifically biological predicates, however, involve counting and probability rather than practical measurement. Examples are the number of cheetahs living today and the probability of survival for specific genes under specific environmental conditions. Without going too far into the world of biology, it is instructive to see how a biological domain of discourse can be linked to the mathematical notion of probability. First, consider a classification of kinds of behavior as agressive vs. non-agressive. One may also establish some comparisons between more and less agressive behavior. This will result in tolerance relations of the kind discussed in section 5.

With regard to biological predicates, however, the combination of skills that enabled us to proceed from tolerance relations to practical measurement in physics don't seem to be of much additional help. It is not clear how one would combine the objects to be measured into suitable new objects. Bricks can be lined up to yield longer objects of the same kind and split so as to yield shorter ones. With agressive behavior, it is not clear what it would mean to line-up or split items of such behavior so as to yield suitable new items. Suitable here means: the new items must be amenable to coherent comparisons of agressivity along similar lines to those I used for refining tolerance relations of "longer than" until I reached a function expressing length as a real number (I will look at related difficulties when discussing human vulnerability in sections 8 and 9).

What is often quite fruitful is to study the probability - in the frequentist sense introduced in section 5 - that some animals will display agressive behavior under given circumstances. It may then be useful to develop a different way of comparison by saying that one animal is more agressive than another one if it has a higher probability of agressive behavior under given circumstances. Along these lines one can then describe at least some behavioral traits as continuous variables.

As a second example for the relevance of probability measures with regard to biological domains of discourse, consider the frequencies of different genetic sequences in a population of organisms at a given moment in time. In complex ways, different genetic sequences govern different traits of living organisms. The frequencies of sequences as well as those of traits develop through an interplay between the three fundamental principles of Darwinian evolution: reproduction, mutation, and selection (Page and Nowak 2002).

The frequency of a given genetic sequence in the population changes with reproduction. Via mutation, a given sequence can be transformed into another one; via selection, the relative frequency of certain sequences may increase at the cost of other ones. The selection process depends both on environmental conditions and on the relative frequencies of different sequences with the organismic traits they govern. Along these lines, evolutionary domains of discourse involving relative frequencies of genetic sequences and organismic traits can be linked to mathematical ones involving probabilities. Evolutionary game theory (Maynard-Smith 1982), Lotka-Volterra type population dynamics (Lotka 1920, Volterra 1926, May 2001) and further approaches to Darwinian dynamics (Michod 2000) take advantage of this possibility.

For such a link to be possible, the evolutionary domain must have certain non-trivial characteristics. For example, there must be distinct genetic sequences governing different traits of organisms, these traits must have different - possibly interdependent - reproductive potentials, changes in the environment must be slow in relation to the reproduction of organisms, etc. "Nature" then influences the population of organisms living in a given environment much like a gardener breeding flowers with specific characteristics might do: certain characteristics are fostered, other ones are gradually weakened by environmental pressure. With suitable specifications, the operation of such a gardener can be modelled in terms of dynamic processes based on mathematical probabilities.

This kind of link between biological and mathematical domains of discourse is relevant for efforts to model vulnerability in biodiversity research. Consider an environment in which favorable and unfavorable conditions for the reproduction of specific genetic sequences alternate stochastically. Then one can define a variable for the evolutionary vulnerability of a genetic sequence as the probability of its extinction over a given time-span. Of course, this is not identical to the probability of unfavorable conditions, as these do not necessarily lead to extinction, but it is related to it. Evolutionary vulnerability depends on the probability of unfavorable environmental conditions, but also on the mechanisms by which organisms can react to unfavorable conditions, and on the interaction between individual organisms.

The bridges between biological and mathematical domains of discourse are robust enough to allow for a rigorous analysis of evolutionary vulnerability. Such analyses can also be performed for geographically well-delimited territories and for well-defined changes of environmental conditions. The latter possibility is particularly relevant in view of current threats to biodiversity from human activities. However, biodiversity analysis quickly leads to two hard questions. How can one compare the prospect of extinction of corals supporting whole reefs with the prospect of extinction of some kind of cockroaches with peculiarly colored backs? And how can one assess possibilities of extinction by unprecedented environmental changes for which no reliable statistical evidence is available? In section 9, I will introduce human preferences, allowing to address the first question, and subjective probabilities, allowing to address the second one.

Having looked, albeit briefly, at the relation between organisms and numbers, what about the relation between organisms and molecules? How are physical and biological domains of discourse linked to each other? To some extent, there is a rather straightforward overlap. Organisms occupy reasonably well-defined space-time domains, and these can be described with the predicates of physics and chemistry. Along these lines one then can measure the weight, speed, etc. of a given organism. However, the same organism can be described with biological predicates like the ones specifying species, sex, behavior, habitat, etc. Investigating inferences linking the two kinds of predicates is one of the exciting challenges of current research. The development of the biosphere has drastically changed the chemical composition of the atmosphere as well as many physical and chemical properties of the surface of the Earth. On the other hand, the development of life clearly depends on physical and chemical processes.

Since the days of Aristotle, from time to time bold thinkers have wondered whether the language needed to talk about organisms - a language of birth and death, of growth and form, of chance and resilience - was not appropriate to talk about mountains and clouds as well. Still, few people think that physical predicates like mass and momentum could be explained in terms of biological ones. On the other hand, attempts to explain biological predicates - including the one for life itself - in terms of physical and chemical ones have been remarkably fruitful in some domains.

Whether the fruits of such efforts will be a reduction of biological predicates to physical ones is another question. A plurality of domains of discourse enables one to perceive, understand, and shape a much richer environment than one would recognize if one were bound to a single domain. If one were able to translate every word and expression from one domain into another one, there would be much less to be gained from their co-existence. Since the origins of modern science, researchers have made amazing progress in unifying widely differing topics into comprehensive domains of discourse. By now, we may have reached a point where similar progress is warranted in taking care of the linkages - based on overlap, metaphor, and other means - between heterogeneous domains.

7 *Changing logics*

So far, I have discussed heterogeneous domains of discourse as differing in the objects, predicates, and inferences involved, leaving the rules of inference in the background. Some logicians (e.g. Quine 1960) like to think that a unique pattern of inference forms the backbone of human knowledge, and that this pattern is made explicit by classical logic. Others (e.g. Specker 1960, Putnam 1968) are less sure. As an illustration of the problems involved, consider the pattern of inference implicit in the following reasoning.

"The Greeks and the Trojans hope to win tomorrow's battle,
and
the Greeks will win or the Trojans will win."
Therefore:
"The Greeks and the Trojans hope to win tomorrow's battle and the Greeks will win,
or
the Greeks and the Trojans hope to win tomorrow's battle and the Trojans will win."

The relevant pattern of inference is represented by the transition from the first to the second formula in the following scheme (S, S', S'' stand for arbitrary sentences):

$$\frac{S \wedge (S' \vee S'')}{(S \wedge S') \vee (S \wedge S'')} \tag{18}$$

The scheme is quite similar to the one implied by the equation $3 \cdot (2+6) = (3 \cdot 2) + (3 \cdot 6)$. Akin to the arithmetic operators for multiplication and addition, the logical operators for conjunction ($\wedge$) and alternation ($\vee$) are assumed to follow a distributive pattern. In fact, such is the use of these operators in standard logic. In the domain of discourse of quantum mechanics, however, it is quite standard to use rules of inference that do not allow this distributive inference without additional justification. Otherwise, false predictions for the behavior of photons, electrons, etc. follow.

As Putnam (1968) emphasizes, this is not an arcane problem whose relevance would be restricted to the study of sub-atomic particles. The example of the battle used above is instructive here. It is a fact of life that even if one has no doubt that there will be a battle tomorrow and that the battle can only end with either the Greeks or the Trojans winning, the outcome of the battle can still be open. In particular, it can depend on a myriad of human decisions not yet taken, decisions which will in turn create the context for new decisions.

In terms of logic, the sentence: "The Greek will win or the Trojans will win" can be true in one of the following ways: the Greek will win, the Trojans wll win, or both will win (to keep things simple, I exclude the third possibility). If one knows the sentence "The Greek will win or the Trojans will win" to be true, one needs not and cannot imply that one of its components - say: "the Greek will win" - is true. If by a meaningful sentence one means a sentence that is either true or false, then "the Greek will win" is not a meaningful sentence in this context. Somebody saying such a thing claims a kind of knowledge that is not available. It is not available because it presupposes a fully predetermined future, and this presupposition is not warranted (Feyerabend 1999, suggests ways to relate the - clearly distinct - universes of discourse of Homer and those of quantum mechanics).

Of course, many people claim that the future is fully predetermined even if we cannot know it in advance. In their lives, this claim may be quite helpful, it may enable them to keep their temper where others might lose it and to achieve things they would never have achieved otherwise. Still, it is a claim that is hard to justify once it has been challenged within some reasonably comprehensive tradition of inquiry. Clearly, there are religious traditions in which the claim of a predermined future plays an important role. More surprisingly, unrestricted use of standard logic implies the same claim. However, this claim is at odds with democratic culture - democratic institutions become meaningless if there is no freedom to take decisions about an open future. It is also at odds with the culture of scientific research - scientific research would be a sequence of superstititious rituals if experiments, as well as other research settings, would not depend on deliberate human choices (see Habermas 2002 for a thoughtful reflection on the interplay between religious, political, and scientific traditions of inquiry).

Of course, one can choose to simply ignore this problem. Conceptual inconsistency does not always hinder scientific progress. But sooner or later much greater progress will become possible by addressing the problem than by ignoring it.

One way to proceed might be to use variants of quantum logic - i.e. a logic without the generalized distributive law - in domains of discourse dealing with human beings. Quantum logic as such does not imply that the future is not predetermined, but it differs from standard logic by not implying the contrary either. Therefore, it may be interesting to investigate relations between quantum logic and another kind of non-standard logic, namely temporal logic (von Weizsäcker 1985). There, sentences are treated differently depending on how they are embedded in temporal sequences. The meaning of the phrase "temporal sequences" is determined in part by common sense, in part by the logical patterns brought into play. An important domain of discourse for temporal logic are computer programs, as these organize temporal sequences of connected events.

There are other possibilities for non-standard logics, and there are other reasons to look at standard logic in new ways. I will not explore these in depth here (see Agazzi 1981 for a survey), it suffices to mention a few examples.

Just as in quantum logic one relies on non-standard ways of combining conjunction and alternation, one can also use negation in non-standard ways. In computer science, e.g., there is often an important difference between two ways in which a program may fail to compute a certain result. On one hand, the program may compute some other result, on the other, it may engage in an endless flow of operations. One way of expressing this difference is by distinguishing between strong and weak negation. This can then be further elaborated by distinguishing between two ways in which a sentence can fail to be true, leading to logical patterns where sentences can be not only true and false, but also indeterminate.

The last option can be used to expand on the problem of the distributive law. The sentence "The Greeks will win tomorrows battle" can be treated as meaningful, but indeterminate, and the compound sentence "$S \vee S'$ " can be treated as true in case its components are both indeterminate. Another reason for studying non-standard logics is the importance of intentions and norms for human life - both are hard to deal with in standard logic.

In general, non-standard logics have been developed by exploring both non-standard uses for standard logical symbols - including the quantifiers - and by introducing non-standard symbols, like symbols to say: "it is possible that", "for all actual objects", "x believes that", etc. The fate of Euclidean geometry can help one thinking about these innovations in logic. Today, Euclidean geometry is still considered a milestone in the history of human thought, it still invites careful study and provides fascinating research problems, but it is one variant among several. Very few mathematicians would treat it as the one and only appropriate way to describe the physical spaces human beings live in. In the case of logic, over the past decades many versions of logical reasoning have been designed and investigated. A single domain of discourse may be investigated with different logical frameworks, and none of them is likely to be the single best choice in all cases.

Arguing about logic is a subtle affair, however. What rules of inference shall one use when arguing about rules of inference? Once more, this question points to the relation between tradition and critical debate. After all, doubts about rules of inference can only arise in a given historical situation, where some rules of inference are already in use. This clearly holds when some irritating result casts doubt on seemingly well-established rules of inference.

As an example, consider Russel's famous paradox (see the "no-universe" proof in section 4.1 for further illustration). Cantor had developed set theory on the basis of a rule of inference according to which one could always form a set of those objects satisfying a given predicate. If "$P(...)$" is an arbitrary predicate, then, in the following scheme one can infer the second formula from the first:

$$\frac{\exists x(P(x))}{\exists y \forall x\ (x \in y \Leftrightarrow P(x))} \tag{19}$$

Russell showed that this leads into a contradiction when $P(x)$ is taken to be $\neg(x \in x)$. In a nutshell:

$$\begin{gathered}\exists y \forall x\ (x \in y \Leftrightarrow \neg(x \in x)) \\ y := \{x : \neg(x \in x)\} \\ y \in y \vee \neg(y \in y) \\ y \in y \Rightarrow \neg(y \in y) \\ \neg(y \in y) \Rightarrow y \in y\end{gathered} \tag{20}$$

The last two lines, both of which follow from the definition of y, are clearly contradictions. The problem has been solved by distinguishing predicates that correspond to sets from predicates that do not. For arbitrary predicates, then, Cantor's rule of inference has been modified so as to gear it to some predicate conferring sethood (P' in the scheme below):

$$\frac{\exists z \forall x\ (x \in z \Leftrightarrow P'(x)) \exists x(P(x))}{\exists y \forall x\ (x \in y \Leftrightarrow (P'(x) \wedge P(x))} \tag{21}$$

Another kind of doubts about rules of inference arises when one is faced with different groups of people subscribing to different rules of inference. As an example, consider the relation between standard and fuzzy logic. In standard logic, sentences are treated as being either true or false. In fuzzy logic, sentences are assigned degrees of plausibility represented by real numbers between zero and one. A proponent of fuzzy logic might argue that standard logic misses a key feature of human thought, namely the fact that our concepts fit reality to varying degrees under different circumstances, and that one needs ways to deal with the resulting differences in plausibility of whole sentences. She might add that standard logic is a special case of fuzzy logic, namely the one where all relevant sentences are either totally plausible or totally implausible. And she might highlight the practical use of fuzzy logic for a whole range of engineering purposes. To this a proponent of standard logic might reply that the vagueness of everyday concepts is a problematic state of affairs and that rather than reinforcing it one should try hard to reach the conceptual clarity of standard logic. As for practical use, he could mention specific technologies relying on standard logic as a key design tool, and he might argue that standard logic greatly fostered scientific progress.

When trying to make up one's mind about such debates, one interesting question is whether rules of inference proposed for some domain of discourse can also be used for domains of discourse consisting of sentences about the former one. Are the rules of inference proposed for the study of electric circuits also appropriate for the study of sentences describing electric circuits? It is one of the great strengths of standard logic that it meets this test.

To take an extremely simple example, consider the classical rule deducing the sentence "B" from "If A, then B" and "A". This rule of inference is equally appropriate when discussing electric currents as when discussing sentences about electric currents. "If the switch is on, the current flows; the switch is on; therefore, the current flows" is an example referring to currents. "If the sentence 'the switch controls the current' is true, then its negation is false; the sentence is true; therefore, its negation is false" is an example referring to sentences about currents.

Now take the case of fuzzy logic applied to the same electric currents. Suppose that the sentence "The voltage along this circuit is 7V" is assigned a plausibility of 0.7. Then according to fuzzy logic the sentence "The voltage along this circuit is not 7V" is assigned a plausibility of $1 - 0.7 = 0.3$. Next, turn to the domain of discourse consisting of sentences about electric currents. For fuzzy logic to work in the domain of electric currents, one cannot assign plausibility 0.7 - or any other plausibility except 1 or 0 - to the sentence expressing the arithmetic of plausibilities used before. The same holds for all other sentences expressing rules of fuzzy logic. In this respect, therefore, one must revert to standard logic. This argument is no objection against the use of fuzzy logic when studying electric currents, but it is an argument against the claim that fuzzy logic is a viable alternative to standard logic when studying sentences.

Debates about rules of inference, then, are necessary and possible. Over two millennia, they have led to the mathematical formulation of standard logic in the 19$^{\text{th}}$ century. Subsequent debates have produced a situation where standard logic, while still treated as the main point of reference, is complemented by other varieties in a somewhat pragmatic fashion. This pluralism blurs the distinction between general rules of inference and rules for the use of specific predicates. For example, the history of the membership predicate shows how debates about rules of inference can be tied to specific predicates, as opposed to an emphasis on patterns of arguments that are independent from any particular predicate.

Debates about rules of inference do not follow fixed rules of inference, but explore tentative modifications of rules given by some tradition or other. To do so, one must be able to describe the relevant traditions as dynamic entitities, projecting them into the future, showing how the proposed modifications take care of important strengths of given rules while amending critical weaknesses. This kind of debates cannot be framed in terms of a given domain of discourse, and therefore it cannot be carried out in a formalized language.

If one could rely on a single set of formalized rules of inference, all one would want to say about these rules might be said in the same formal language used to express the rules themselves. To some extent, such an elegant form of self-reference is possible with standard logic. This is the modern version of the classical image - shaped by Plotinus, Spinoza, and others - of the world as expressing a single logical structure (Carnap, 1928).

As far as this logical machinery works, one can then describe domains of discourse simply as sets of objects, taking the rules of inference for granted, identifying names with individual objects, and predicates with subsets of a universal set that can be taken for granted as well. Therefore, the universal set cannot contain all the sets of modern mathematics', because, as discussed in section 4, these don't fit any universal set. Moreover, the names and predicates to be used must be given in a clear-cut way. The ambiguity and openness of ordinary language must somehow have evaporated before logical argument begins.

As a matter of fact, this ambiguity and openness is put aside whenever one can rely on what Brandom (1994, p.440ff) calls canonical designators. These are names used by some community to refer to objects whose existence and identity can be taken for granted. "The sun" and "the Earth" are such names in astronomy, "hydrogen" and "oxygen" in chemistry, "the U.S." and "China" in political science, "1" and "2" in arithmetics, "Plato" and "Aristotle" in philosophy, etc. Without canonical designators, there would be no domains of discourse amenable to logical argument.

Canonical designators, however, come with whole lists of predicates and inferences attached. Talking about domains of discourse as given sets of objects makes sense for particular purposes, but it already presupposes the availability of canonical designators. This is the reason why I talk about domains of discourse as linked not just to some range of objects, but to a language game and the life form that comes with it. The ability to refer to objects by canonical designators is a key resource shared by those exchanging arguments within a given domain of discourse. This resource already draws on a range of rules of inference, and these rules are neither cast in stone nor completely uniform across domains. From time to time, canonical designators may change, and so may the rules of inference used in some domains.

It takes the resources of ordinary language, in particular its ability to operate as its own meta-language, and its potential to generate new meaning through metaphor, to develop rules of inference in a reasoned debate. In this sense, logical argument is embedded in broader forms of reasoning, where as participants in a debate we "make up the rules as we go along" (Wittgenstein 1953, para. 83) - not in an arbitrary way, but in a creative one. There is no way of establishing a universal domain of discourse in which all exchanges of arguments could be embedded. But there are ways of story-telling that work as "ways of world-making" (Goodman 1978), of creating logical spaces for exchanges of arguments - and for their own future transformation.

8 *Modelling social systems*

Rules of inference can be amended, and so can the properties and relations used to identify objects of discourse. For purposes of computer modelling, this is often essential. Without a reasonably well specified domain of discourse, computer modelling is pointless. Often, modelling begins with some vaguely circumscribed problem and without a suitable domain of discourse already at hand. As long as the relevant predicates are not clarified and specified so as to yield such a domain of discourse, computer models are bound to be about as meaningful and about as problematic as older forms of superstition.

This problem is particularly relevant for the task of modelling social systems. There are two different challenges to be met here. First, the words and concepts used to talk about social systems are much closer to ordinary language than those used to talk about the systems studied by astronomers, engineers, etc. - basically because language is itself a key ingredient of social systems. Second, the words and concepts used to talk about social systems in contemporary culture are fraught with conceptual confusions of a kind that seems particularly hard to deal with. To take but one major example, after centuries of scholarly debate, story-telling by poets and playwrights, and explorations and suffering by ordinary people, the relation between passion and rationality still provides an incredibly difficult beast to wrap one's head around (Frank 1989, provides an entry point into the literature on this topic). Before this background, I will now discuss main linkages between social and mathematical domains of discourse.

There is a huge variety of predicates expressing all sorts of social properties and relations. They range from concepts of gender and kinship relations (like daughter, uncle, wedding) to concepts of broad collectives and their organization (like nation, power, parliament), from personal characteristics (like introverted, team-player, leader) to historical categories (like traditional, urban, post-modern). All these predicates can be used to identify countable objects of social research. And as I have discussed with regard to physical domains of discourse, frequency counts open the doors to probabilistic analysis. The possibility of counting then provides valuable bridges between social domains of discourse and mathematical ones. These bridges can be greatly enlarged because objects - individuals, actions, institutions, events, etc. - identified by means of social predicates can be further described with biological and physical predicates. Distinctions between male and female as well as differences of age come to mind. Nearly as important are physical distances: population density, e.g., is a variable of huge importance for the dynamics of social systems.

With regards to modelling social systems, then, a wide variety of predicates can be used to build domains of discourse suitable for modelling in a more or less *ad hoc* way. Often these domains will combine specifically social predicates affording possibilities of counting with physical predicates affording possibilities of measuring. Countless mathematical models of specific phenomena have been built along these lines, in particular for purposes of statistical analysis. Broader social theories, however, are usually not amenable to formal modelling because they do not provide sufficiently well-defined domains of discourse.

As an example, consider Max Weber's (1920) theory of the protestant ethic. A key claim of this theory is that the protestant ethic was a necessary condition for the development of capitalism. What are the objects constituting the relevant domain of discourse? They may be certain - all? - human actions, physical persons, institutions, systems of beliefs, and more. What are key predicates? Perhaps a predicate stating that somebody believes wealth to be a sign of divine grace, or a predicate stating that a certain sequence of actions is motivated by the belief that every human being has a calling to fulfill in her or his life, or a predicate describing the prevalent denomination in specific economic regions. What are rules of inference? Perhaps one can infer from the fact that somebody believes an action to be appropriate under specific circumstances that she or he will take that action under these circumstancen - and perhaps not.

What these instances of vagueness and ambiguity show is not that the theory is false, or meaningless, or uninteresting. What they show is that the theory can only be understood and used by somebody who is willing and able to settle all sorts of indeterminacy by exerting reasonable and sometimes creative judgement on a case to case basis, informed by a wealth of background knowledge that is even harder to express in terms of a domain of discourse. Developing that kind of judgement is essential in all scientific disciplines, but particularly so in disciplines like the social sciences, where mechanistic procedures rarely provide reliable guidance when dealing with the problems that these disciplines tackle.

Building a mathematical model in line with a theory like the one of the protestant ethics is not impossible, but it requires one of two things. The first possibility is to drastically simplify the theory to the point of preserving only some rudimental features. For example, one may use a domain of discourse where the relevant objects are nations between the years 1500 and 1900, and where these objects are described by two predicates. First, nations may be protestant or not protestant, and second, they may experience a transition from non-capitalist ways of life to a persistent process of capital accumulation. The theory might then be expressed by two inferences: If the first predicate is false, then the probability for the second predicate to be true is close to zero; if the first predicate is true, then the probability for the second predicate to be true is larger than half. Interesting as such a statement may be, it is clear that it misses most of the claims and insights conveyed by the theory under consideration.

The second possibility is to elaborate the theory further to the point where a domain of discourse that can be linked to a mathematical one becomes available. This may be quite rewarding, but it certainly is very demanding. It may well require substantial advances in both the social science theory under consideration and the mathematics to be used for modelling purposes. Before this background, it is understandable that up to now attempts to formalize broad social theories have been blessed with rather limited success.

As a result, in the social sciences one often finds a mixture of theories that do not come with a domain of discourse amenable to mathematical modelling, and models - usually statistical in character - that can be used to test specific empirical hypotheses, but that do not convey the main thrust of the theories on which they rely. I call the resulting style of analysis "semi-formal theorizing" - without implying that this is only a preliminary stage on some supposed route towards the holy grail of "fully formal" theorizing.

Still, there are two instances of highly elaborated and extremely influential formal theories dealing with social domains of discourse: the theory of formal languages on the one hand, economics on the other. I consider them in that order.

The theory of formal language has been remarkably successful in linking social and mathematical domains of discourse by refining linguistic predicates like "word", "sentence", "verb", "noun", etc. Building on Chomsky's (1957) pioneering idea of a generative grammar, researchers in linguistics, computer science, and mathematics have engaged in this endeavour with great enthusiasm. Surprisingly, perhaps, their main impact has been on software engineering. To what extent this theory is appropriate for the study of natural languages is more controversial (Baker and Hacker 1984), but it certainly has inspired a remarkable body of linguistic literature. At least as important for our present purpose is the fact that the various versions of formal logic currently available can be studied as formal languages. In principle, this opens the door to models of human knowledge. Such models are currently explored in the scientific communities dealing with Artificial Intelligence and with Multi-Agent-Simulations.

A formal language is defined as a subset of all possible strings written in a finite alphabet. To see the relevance of this definition for natural languages, compare the strings "acquainted", "vertraut", "acvarsti". How does somebody who is acquainted with English and German recognize the first word as English, the second as German, and the third as neither English nor German? Clearly, different languages somehow separate admissible from inadmissible strings in different ways even when they can be expressed by similar or identical alphabets.

In view of the theory of formal languages, it is a remarkable feature of finite alphabets that any string written in any finite alphabet can be unambiguously translated into an alphabet with a single sign. As an example, I translate the string "she" - based on a standard alphabet with 26 characters plus a blank sign - into an alphabet with a single sign plus a blank. First, I associate to each letter of the original alphabet the number of its position in this alphabet. Then I associate the sequence of prime numbers (2,3,5,...) to the first, second, third, etc. position in the string under consideration. Now I set each prime number to the power of the number corresponding to the sign at that position. The product of these numbers can then be used to encode the text in a single number that can be unambiguously decoded again. (Numbers unambiguously representing arbitrary finite texts written in a given finite alphabet are called Gödel Numbers - after the mathematician who used them to crack some hard problems in mathematical logic.) Finally I write the resulting number as a simple string of as many marks of the one-sign alphabet as correspond to that number. An example is given below.

Text:	"she"
Encoding:	"s" $\mapsto$ 19, "h" $\mapsto$ 8, "e" $\mapsto$ 5
Signs to numbers:	19, 8, 5
Positions to prime-numbers:	2, 3, 5
Product of prime-numbers to the powers of numbers:	$2^{19} \cdot 3^8 \cdot 5^5$
Gödel number of text:	10'749'542'400'000
Representation by one-sign alphabet:	IIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIII... (10'749'542'400'000 marks)

Table 1: Encoding words with numbers

By this procedure, a formal language - defined as a subset of possible strings from a finite alphabet - can be analysed as a subset in the set of natural numbers. When studying natural languages, finite alphabets come into play with written texts. In spoken language, an analogous role is sometimes played by phonemes, i.e. specific sounds that can carry meaning in the language under consideration.

Once strings of signs have been associated with natural numbers, two closely related questions arise. First, how can strings belonging to a given language be distinguished from other strings, and second, how can they be generated?

As for the first question, it is possible to define generative grammars, i.e. functions that transform a set of acceptable strings (i.e. Gödel numbers) into a larger such set (i.e. further Gödel numbers). These functions are computable in a strict sense. To highlight the importance of this feature, consider the seemingly simple function that maps any rational number into 1, any irrational number into 0. There is no way to implement this function on a computer, because there are plenty of numbers for which it is not known whether they are rational or irrational, and there is no computational procedure to settle that question. For the functions defining generative grammars, on the other hand, it is possible to indicate procedures that compute their value for all acceptable inputs.

Starting with a finite set of initial strings, one can then produce an infinity of acceptable strings by repeatedly applying such a function to its own result. With sufficiently complex grammars, each string so generated can be characterized by some identifiable syntactic and phonetic structure. Acceptable strings can include propositions like: "Yesterday I went to the beach" and non-propositional utterances like: "Ladies and Gentlemen, all the way from Michigan: Slim Shady!"

As to the second question, for a given generative grammar it is possible to define functions with the following property: they take a given string as argument and either return 0 (or any predefined number) if the string is not acceptable according to that grammar, or a list of numbers that label the syntactic and phonetic structure of the given string. Until now, only small patches of natural languages have been successfully described in such terms. On the other hand, the design and analysis of formal languages has made tremendous progress over the last decades. Many of them are sufficiently similar to natural languages to be extremely useful for structuring the interface between the mathematical structures implemented in computers and the human beings designing and using these computers.

The world of computers is particularly instructive when it comes to heterogeneity of domains of discourse. Computer scientists operate on a wide spectrum of issues, some of which are closer to the operations of the machine, some to the activities of the user, some in between. In principle, it would be possible to deal with issues across the whole spectrum in terms of a unique domain of discourse. This is so because the correspondence between formal languages and arithmetics works both ways: not only is it possible to transform any formal language into a computable function, it is also possible to describe computable functions by formal languages. One way to do so is to take standard logic and use the natural numbers as the domain of discourse. The predicates and relations needed to design and analyze computable functions can then be introduced in this setting.

In practice, however, professionals working on different aspects of computing use quite heterogeneous domains of discourse. The tasks that some user must or wants to tackle by means of computers are described in very different languages than the steps performed by a computer shuffling single bits around in its memory. In computing as elsewhere, heterogeneous domains of discourse are an essential feature of the social division of labor.

After having looked at the theory of formal languages, I will turn to the second major formal theory of social systems, the one of economics. I take that term in a broad sense, so as to encompass not only the study of mechanisms closely tied to money, but also the impressive theoretical toolkit provided by decision and game theory. To prepare that discussion, I give a closer look at the concept of vulnerability as currently used in research on sustainable development (Adger and Kelly 1999, McCarthy et al. 2001, Smith and Klein 2003). It seems fair to say that this research is performed largely in the style that earlier in this section I have characterized as "semi-formal theorizing".

Consider anthropogenic climate change. Given the prospect of sea-level rise induced by future climate change, there are reasons to say that The Netherlands are more vulnerable to such change than Switzerland. And given differences in well-being and in population density, it is plausible that Bangladesh is more vulnerable than The Netherlands (Nicholls et al. 1999). Here, the word "vulnerability" is not used in the sense of evolutionary vulnerability discussed in section 6, but rather as a concept of human vulnerability.

As Wittgenstein (1953) argued, the meaning of concepts is usually defined by examples. Only from time to time is it possible and sensible to define concepts in terms of other concepts. Most concepts are best explained by showing how to use them under specific practical circumstances. Their further use is then shaped by these examples, but so as to leave room for creative interpretation later on. Borderline cases may arise where the people using the concept will have to decide how to apply it to these cases - a decision that is usually not taken by explicit deliberation, but by the shared development of a social practice.

In this sense, I introduce the concept of human vulnerability by examples. The examples show that vulnerability is used as a comparative concept, that can be expressed with a relation "x is more vulnerable than y" (I will also write it as: "$x \vdash y$"). I take this relation to be an ordering relation (see (11) in section 5). One can then define a relation for being neither more nor less vulnerable:

$$x \sim y \ :\Leftrightarrow \neg(x \vdash y \vee y \vdash x) \tag{22}$$

This is a tolerance relation like those discussed in section 5. As usual with tolerance relations, $x \sim y$ need not be transitive: Perhaps there are no good reasons to say that Saudi Arabia is more or less vulnerable to climate change than either South Africa or Spain, but there may still be strong reasons to say that South Africa is more vulnerable than Spain.

Given the importance of physical measurements in contemporary culture, one is tempted to imagine that wehrever there is a comparative predicate - like: "... is more vulnerable than ..." - there is bound to be some property that can be expressed by real numbers. Tolerance relations are then understood to work as follows. The relevant objects have some quantitative property $Q(x)$ that can be expressed by real numbers and there is some number Δ that defines a tolerance. And a tolerance relation T is based on that quantitative property:

$$xTy :\Leftrightarrow Q(x) - Q(y) \leq \Delta \tag{23}$$

But tolerance relations can be well-defined without any real numbers being involved. It may be perfectly reasonable to say that I am as grateful to Peter as to Paul while it would at best be a joke to say that my gratitude to them is 3.7 with a tolerance of 0.5.

In section 5, I have discussed tolerance relations - those of physical measurement - that can be refined up to the point where they refer to quantitative properties expressed by real numbers. In section 9, I will discuss under what conditions a tolerance relation for human vulnerability can be refined in a similar, but not identical way. The relevant conditons, however, cannot be taken for granted in general, they only apply under rather limited circumstances. Therefore, tolerance relations are indispensable for vulnerability analysis.

Fortunately, in the case of human vulnerability the relevant tolerance relation has some additional structure. Small island states in the pacific are clearly more vulnerable to climate change than most other countries in the world, even if they might survive as states one way or another. And states like India or Brazil, while being less vulnerable than small island states, may still be more vulnerable than, say, Norway or New Zealand. The key features are the fact that we are talking about a finite universe of discourse (of states or regions) ordered by a tolerance relation and that no entity is incomparable with all other ones. We can describe the relevant structure as follows:

$$\forall x \exists y (x \vdash y \vee y \vdash x) \tag{24}$$

On this basis, we can identify countries (or whatever entities we are dealing with) with maximal vulnerability: they are more vulnerable than at least some other one without being less vulnerable than any other one. In the same vein, we can identify countries with minimal vulnerability, and treat the remaining ones as countries with intermediate vulnerability. If some country were completely incomparable with any other country in terms of vulnerability, that country would be simultaneously maximal and minimal, making that classification pretty meaningless - but this possibility is excluded by the structure just described.

Therefore, a rather robust kind of vulnerability analysis consists of defining three broad categories of vulnerability to climate change: maximal, intermediate, and minimal. Three, rather than two, four, or twelve categories because we are dealing with a tolerance relation with the structure just introduced. No metric is required to identify maximal, intermediate, and minimal elements in a set ordered by such a relation. In practice, such identification may well be possible in a way that is not very controversial amongst researchers and stakeholders interested in the issue. It will require a structured dialogue sorting out the relevant arguments as far as possible and accepting that a few ambiguous cases be resolved in an arbitrary way. Along these lines a predicate of human vulnerability can be introduced in the spirit of semi-formalized theorizing discussed above.

Just as we have looked at countries, we can look at regions, at social groups and eventually at single individuals. Paying special attention to individuals is characteristic for economic theory. It is particularly appropriate here, because it can help to investigate issues of social solidarity that are intimately related to human vulnerability.

9 *Human Vulnerability and Global Change*

Consider an individual - call her Bianca - operating in a reasonably stable environment with some stochastic features. By luck, reasoning, imitation or whatever means, she has found a way of action she cannot improve upon with the means at her disposal. Now suppose something changes in Bianca's environment. To stick to the climate change example, two severe floods may happen in her surroundings within a surprisingly short timespan.

How shall we - researchers interested in human vulnerability - assess Bianca's vulnerability after these events? As she is a human being just like ourselves, and perhaps knows more about her life than we do, we could start by asking her. She might answer that she feels scared because people have died in these floods. We could observe her, and we might notice that she starts following the daily weather reports much more carefully than she did before. We could ask ourselves whether we consider these reactions of Bianca's to be appropriate. Given our understanding of what it means to die in a flood as well as our knowledge about those two floods, we might conclude that her reactions are quite appropriate. On the basis of these facts, we may well say that Bianca is more vulnerable than she was before.

So far, the description of Bianca's vulnerability still looks quite similar to the tolerance relation for human vulnerability discussed in section 8. However, now I have explicitly introduced the "subjective" view of a human individual, and balanced it with the "objective" view of an external observer. This provides the ground for assessing under what conditions and how far one can move from a tolerance relation for vulnerability towards a measurement of vulnerability in terms of real numbers (Downing et al. 2001). In global change research, vulnerability is usually seen as a function of impact and adaptive capacity, with impact in turn a function of exposure and sensitivity (McCarthy et al. 2001, Schröter et al 2003). Accordingly, I will proceed in four steps, addressing exposure, sensitivity, adaptive capacity, and vulnerability.

9.1 *Exposure*

When studying human vulnerability, exposure to the risk of dying certainly matters. But of course we are all bound to die some day: what really matters for vulnerability analysis is untimely, cruel, unjust death. Now these are all predicates that different people will use in very different ways - one may say that they express a strong subjective component. I will come back to this point, for the moment I simply take it for granted that Bianca would prefer not to die in a flood and that most people have no difficulty in understanding this preference of her.

Untimely death is by no means the only event to be considered when analysing human vulnerability, but it certainly is an important one. And once such a critical event has been identified, human vulnerability can be analysed in terms of the probability of that event (Parry et al. 2001).

This enables one to move beyond the tolerance relation introduced in the last section, and also beyond the distinction between maximal, minimal, and intermediate vulnerability. To some extent, one can now start working with a quantitative concept of vulnerability, as the probability of a critical event can cover the whole range between zero and one. Writing "c" for the critical event and "$\bar{\pi}(c)$" for the probability of that event, one gets a simple definition for a preliminary concept of exposure that can be expressed in terms of real numbers between 0 and 1, namely $E := \bar{\pi}(c)$.

As in the study of other living beings, probabilities provide a link between a domain of discourse dealing with human beings and a mathematical domain. In the mathematical domain of probability theory, probability is a function from a set of sets into the interval

[0,1] of real numbers. The function must satisfy several requirements, in particular: the function of the union of two sets must be equal to the sum of the function of these two sets. The set of sets must satisfy a few requirements, too, in particular: it must contain the union of two sets whenever it contains these two sets.

In domains of discourse dealing with human beings, I will use the word risk to refer to situations where a human being is faced with more than one possibility, at least one of which has a consequence that will make that person suffer (in decision theory, such situations are often labelled "lotteries"). Notice that to talk about risks in this sense probabilities are not needed, nor does one need to quantify damages; however, some kind of preference between the different possible consequences is required.

As for probabilities, in domains dealing with human beings they come in two kinds: one relies on the relative frequencies discussed in section 5 (and to be further discussed in the present section), the other relies on the subjectivity that distinguishes humans from other beings. Subjective probabilities are not an easy topic, but an extremely important one. I approach it by considering the difficulty of measuring exposure to various stresses in terms of probabilities of critical events.

Non-probabilistic data can be vital for describing actual suffering, but they are of little use for describing potential suffering. Vulnerability does not refer to actual wounds - taking wounds as metaphors for suffering - but to potential ones. Finding the data required for a probabilistic description is no trivial task. Only this kind of data, however, provides a reasonable basis for moving from a tolerance relation for vulnerability towards a variable expressing vulnerability in terms of real numbers.

Some steps in that direction are possible even where the probability of the critical event is not well known. Imagine that Bianca must choose one of two roads for some journey by car. There is an important difference between the two roads, however: one of them leads through a deep canyon prone to flash flooding, while the other one does not. She can die on both roads, be it by car accident or by drowning in the floods, and she knows that the relative frequency of people dying on the first road is much higher than on the second. The idea of a relative frequency that is somehow characteristic of a sequence is rather elementary for human beings - we all distinguish between fast and slow rythms, even when they are quite irregular. And even if Bianca cannot tell a number for the relative frequency of deaths along the two routes, she may still choose one road rather than the other. The argument can be refined with more complex cases, as when the same road is travelled more than once, and when various combinations of the two roads are feasible. This is needed to assess subjective probabilities in detail, but I will not discuss this here.

In order to describe Bianca's behavior under these circumstances I introduce the following terminology. If in a sequence of events one can identify the relative frequency of some event well enough for practical purposes, I call that frequency a characteristic frequency. This is not a mathematical concept, it sits in a domain of discourse dealing with human beings, not with the functions from sets to real numbers that are the stuff mathematical probability is made of. In domains of discourse dealing with humans, characteristic frequencies express a frequentist view of probability.

Next, I consider different risks with the same possible outcomes and look at the preferences of some decision-maker - e.g. Bianca - when choosing among them. If these preferences lead to a ranking of the risks, I call the resulting ranks the preferability of the different risks. As the different risks under consideration all have the same possible outcomes, risk preferability is closely tied to the decision-maker's knowledge of or guesses about characteristic frequencies of different risks. The nature of this tie, however, is far from simplel. Sometimes, the decision-maker knows a characteristic frequency and this knowledge shapes her preferences between risks. Sometimes the decision-maker has very strong preferences between risks - perhaps because of strong emotional experiences in the past - and these shape the characteristic frequencies the decision-maker takes into consideration. Sometimes, there is some mixture between these two extremes. In any case, risk preferability expresses a subjectivist view of probability. This is not to say that there is no objectivity here: risk preferability captures a relation between a decision-maker - in our case Bianca - and a range of risks. If Bianca prefers the risk of the first road to that of the second, then saying this is as objective as saying that she is more fluent in English than in French.

Mathematical probabilities can be used to model the interplay between characteristic frequencies and risk preferability. Characteristic frequencies can be represented by limiting values of relative frequencies, and these limits may be called first order probabilities. Risk preferability is more subtle, because here one must deal with the whole range of possible characteristic frequencies. It is often useful to represent risk preferability by using mathematical probability distributions defined over all possible first order probabilities, i.e. over the interval $[0, 1]$. The resulting probabilities may be called second order probabilities.

The distinction between first and second order probabilities is closely related to the influential distinction between risk and uncertainty introduced by Knight (1921). According to that distinction, risk is characterized by well-defined probabilities of outcomes, uncertainty by unknown probabilities and only partial knowledge of outcomes. In this paper, I use the word risk to characterize situations with different possible outcomes. Risks can then be further characterized by first and second order probabilities. Knightian risk is the extreme case where second order probabilities select a single first order probability as the one that matters. The case where second order probabilities are uniform is as far as one can get in the direction of Knightian uncertainty in the present framework. Fascinating unresolved questions of decision theory and economics arise in this area. In particular, in some cases it can be quite misleading to assume that an agent acts on the basis of a single well-defined function of second order probabilities (Jaeger 1988). This is not always a big issue, however, as sometimes only rough features of second order probability distributions really matter

Second order probabilities do not represent relative frequencies, they represent risk preferability, and thereby propensities for action - different distributions of second order probabilities mean that Bianca will make different decisions when she has to select the roads she will travel. Preferences and probabilities are closely related: it is an important piece of received wisdom that under the circumstances Bianca is faced with in our example, any reasonable person will and should take the second road. This holds even if the choice is about a single action that will never be repeated, and where therefore it is hard to show

how and why relative frequencies matter at all (Putnam 1968 calls this "Peirce's puzzle" - after the founder of American pragmatism, who thought about these issues long ago).

By writing $\tilde{\pi}(\bar{\pi}(c))$ for: "the density of second order probabilities of first order probabilities of a critical event" - remember: it's Bianca drowning in a flood! -, one can combine first and second order probabilities and define exposure to a critical event as follows:

$$\mathbf{E}_c \ := \ \int_0^1 \bar{\pi}(c) \cdot \tilde{\pi}(\bar{\pi}(c)) \ d\bar{\pi}(c) \tag{25}$$

The resulting number is the expected value for a first order probability given a distribution of second order probabilities. Exposure to different events can then be described by a list of such expected values (I call it $\bar{\mathbf{E}}$). In order to aggregate them to a single magnitude, one needs to take into account human sensitivity to risk exposure. Before doing so, however, I will look into a key advantage of combining first and second order probabilities (for more background, see Kreps 1988, p.145ff).

This combination enables one to compute a meaningful measure of exposure in cases where little or nothing is known about relative frequencies. When nothing is known, $\tilde{\pi}(\bar{\pi}(c)) = 1$ everywhere, because the decision-maker has no reason to put greater weight on any particular first order probability. The integral indicated above then yields: $\mathbf{E}_c = 1/2$. Incidentally, this suggests that with unknown first order probabilities, the probability of one out of two events happening twice in a row should be assessed not as 1/4 but as 1/3, because $\int_0^1 \bar{\pi}(c)^2 d\bar{\pi}(c) = 1/3$ - a point with some relevance for the assessment and management of environmental risks.

With this way of studying exposure, one can also take advantage of a mathematical result known as de Finetti's theorem. For the present purposes, the main insight can be illustrated as follows. Consider sequences of events such that at each step one out of two possibilities is realized. At each step, one can count the relative frequency of each of these possibilities. Suppose a sequence looks like the following one:

$$0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0,\ ... \tag{26}$$

The relative frequencies of "1"s then look like this:

$$0,\ \frac{1}{2},\ \frac{\mathbf{2}}{\mathbf{3}},\ \frac{1}{2},\ \frac{2}{5},\ \frac{2}{6},\ \frac{2}{7},\ \frac{\mathbf{1}}{\mathbf{4}},\ \frac{1}{3},\ \frac{2}{5},\ \frac{5}{11},\ \frac{1}{2},\ \frac{7}{13},\ \frac{4}{7},\ \frac{5}{3},\ \frac{5}{8},\ \frac{11}{17},\ \frac{\mathbf{2}}{\mathbf{3}},\ \frac{12}{19},\ ... \tag{27}$$

Relative frequencies may oscillate without ever converging to some limit (in the present sequence, characters in boldface mark turning points of an oscillation). Under such circumstances, the frequentist concept of probabilities is hard to apply. Still, risk preferability works with non-converging sequences of relative frequencies, too: Bianca, like anybody else, is free to prefer one risk to another one, much as she is free to prefer apples to oranges. Both for converging and non-converging sequences, her preferences can be expressed by means of suitably distributed second order probabilities.

De Finetti's theorem turns out to be useful whenever Bianca is faced with a sequence that does have a limiting relative frequency. The theorem shows that she can systematically improve her second order probabilities in these cases by applying Bayes rule (see Jaynes 2003 for a Bayesian vision of logic, probability, and science in general). If Bianca improves her second order probabilities each time she learns about a new event in the sequence, these probabilities will get more and more focussed on the single first order probability towards which the relative frequencies are bound to converge under the given assumption.

To take but one example, suppose that the critical event to be considered is an unusually high flash flood and that the last flash flood has been unusually high. Then second order probabilities can be improved by the following formula. (I write $\bar{\pi}'(c)$ for an arbitrary first order probability of the critical event, $\tilde{\pi}_t(\bar{\pi}'(c))$ for the second order probability assigned to $\bar{\pi}'(c)$ at time t, and $\tilde{\pi}_{t+1}(\bar{\pi}'(c))$ for the second order probability to be assigned one tide later.)

$$\tilde{\pi}_{t+1}(\bar{\pi}'(c)) = \tilde{\pi}_t(\bar{\pi}'(c)) \cdot \frac{\bar{\pi}'(c)}{\int_0^1 (\tilde{\pi}_t(\bar{\pi}(c)) \cdot \bar{\pi}(c)) d\bar{\pi}(c)} \tag{28}$$

Therefore, researchers can help Bianca even if their data are insufficient to justify an estimate of a first order probability. To take an extreme case: nobody would want to estimate a first order probability on the basis of a single event. If however, Bianca entertains second order probabilities of a kind that give very small weights to large first order probabilities for large floods, a single large flood allows her to improve her second order probabilities by increasing those weights by a small, but well-defined amount.

For the analysis of exposure, this means that one first needs data on second order probabilities for the relevant agent. By observing her behavior and asking her directly, one can usually get a rough idea of how the agent assesses different possible characteristic frequencies. Even if the data will hardly be sufficient to fully specify a distribution of second order probabilities, they can provide a sufficient basis for showing how additional data, now relating to first order probabilities, can be used to get a more focussed distribution of second order probabilities. This interplay of data on second and first order probabilities makes a probabilistic description of exposure feasible in the rather frequent cases where a purely frequentist approach would be hopeless.

So far, I have argued that probabilistic data of exposure can be used to make a step beyond a simple tolerance relation for vulnerability by assessing the probability of critical events. That step becomes easier when one combines first and second order probabilities. When studying exposure in these terms, however, one must check whether the people under consideration can really be characterized by single distributions of second order probabilities. If not, it may be appropriate to stick to tolerance relations for vulnerability as described in section 8.

Of course, different critical events engender different probabilities. Exposure then is a list of probabilities. It is easy to establish a tolerance relation between such lists by using the Pareto-criterion: exposure is greater in situation A than in situation B if and only if both situations involve the same critical events and all probabilities in situation A are greater than the corresponding probabilities in situation B. The attempt to aggregate the description of exposure by a list of probabilities into a single number leads to a next link between socio-economic and mathematical domains, and thereby to the notion of sensitivity.

9.2 *Sensitivity*

Nearly always, people find the risk of death worse than the risk of loosing their property. Of course, such a comparison is based on the assumption that the relevant probabilities are the same. When running over a street to stop a thief from running away with her bag, Bianca prefers a small risk of dying in a car accident to a high risk of loosing her bag to the thief. There is some sort of balance between her preferences between the two critical events and her assessment of the uncertainties she is faced with.

An additional point should be made in this context, however. There is a fundamental difference between death and the possibility of death; if somebody is willing to trade a small risk of dying for a big opportunity to get her bag back from a thief, that does not mean that she would be willing to accept death for any amounts of bags. The suffering from actual death is a different matter than the suffering from the possibility of death.

Now consider the sentence: "Bianca prefers the risk of c_2 happening with probability $\bar{\pi}(c_2)$ to the risk of c_1 happening with probability $\bar{\pi}(c_1)$". I write it as:

$$[c_2, \bar{\pi}(c_2)] \ \rhd_b \ [c_1, \bar{\pi}(c_1)] \tag{29}$$

It is often sensible to treat "... $\rhd_b$ ..." as an ordering relation. Writing x, y, z for different risks with probabilities attached, one gets:

$$\forall x \forall y \forall z (((x \rhd_b y \wedge y \rhd_b z) \Rightarrow x \rhd_b z) \wedge (x \rhd_b y \Rightarrow \neg y \rhd_b x)) \tag{30}$$

As discussed in sections 5 and 8, this kind of comparisons leads to a tolerance relation. But when dealing with critical events, one can do more. Suppose Bianca is faced with a choice between a road with car accidents, but no flash floods, and one with flash floods but no car accidents, and suppose that her second order probabilities are the same for both roads. She may still prefer the road with car accidents. One may then say that she finds dying in a flood even worse than dying in a car accident (for the sake of the argument, I assume that in both cases the risk is one of death, and only one of death). With heterogeneous critical events, risk preferability may not only lead to subjective probabilities, but also to a further ranking, this time according to the agent's sensitivity to the danger of different critical events.

Consider probabilities for which Bianca is indifferent between the risk of death and the risk of loosing her bag (indifference will show by her neither running after the thief without hesitation despite the traffic, nor clearly desisting from trying to catch the thief because of the traffic). Pairs of such probabilities may well be characterized by the same proportion (I come back to this point below). If so, one can use that proportion to describe a ranking of Bianca's suffering from these two risks. Call two such probabilities $\bar{\pi}(c_1)^*$ and $\bar{\pi}(c_2)^*$, take the first risk as reference point by setting $s(c_1) = 1$ and define:

$$s(c_2) := \frac{\bar{\pi}(c_1)^*}{\bar{\pi}(c_2)^*} \tag{31}$$

If one can identify a fixed proportion between probabilities that make Bianca indifferent between two kinds of risks, one can use that proportion to express her sensitivity to each one of these risks by a single number, Bianca's sensitivity to the critical event in question. Subjective probabilities rank risks across different possible characteristic frequencies, sensitivity values rank them across different critical events. Bianca's sensitivity values are real numbers because they are related to characteristic frequencies: they indicate the proportion between characteristic frequencies that would make Bianca indifferent between risks involving different critical events.

Sensitivities can be identified for the risk of loosing one's life as well as for the risk of loosing one's harvest, and in the latter case they can be identified for the risks of loosing various fractions of that harvest as well. It is plausible that loosing two thirds of the harvest is worse than loosing one third; learning at what probabilities Bianca will be indifferent between these two risks can enable one to find a number expressing her sensitivitiy to different losses. One can then introduce sensitivity as a function **S** defined over a set C of mutually exclusive critical events and yielding a real number $S(c)$ for each critical event $c \in C$:

$$\mathbf{S} : C \Rightarrow I\!R, c \mapsto S(c) \tag{32}$$

A sensitivity function represents the way a decision maker assesses the suffering that would be experienced when a risk materializes, but also the anxiety experienced by a decision-maker living with a frightening risk. Therefore, an impact assessment by means of a sensitivity function takes the important, if often neglected, problem of anxiety into account.

The use of a sensitivity function does not imply a claim that suffering in general could be quantified, only that a specific human decision-maker will prefer some risks to other ones depending on the characteristic frequencies involved. In principle, a sensitivity function can have positive as well as negative values, where the latter may be read as expressions of happiness. This is the usual way of framing utility functions, where people can be more or less happy, but happiness as such is unattainable. Vulnerability analysis pays attention to suffering, and when doing so one may wish to pay attention to happiness as well. One way of doing so would be to identify happiness with the absence of suffering. This would imply that suffering can be so transformed as to lead not only to further pursuit of elusive happiness, but to actual happiness (a claim sometimes associated with Buddhist traditions). In modelling terms, this would result in a function $S(c)$ having positive values bounded below by an absolute zero, like temperature and other physical measures. In his seminal paper on optimal growth theory, Ramsey (1928) advocated such an approach long ago, arguing that economic growth might enable humankind to reach a state of bliss, as he put it. Vulnerability analysis based on sensitivity functions may or may not follow this route.

Human sensitivity to risk is different from the susceptibility of a piece of iron to rust under the influence of humidity. Human beings care about meaning, and this enables them to do wonderful as well as terrible things. We are animals suspended in webs of meaning that we ourselves have spun, to take up Geertz's (1973, p.5) beautiful phrase. Sensitivity is related to humans acting in the face of an uncertain future, it reflects the priorities human beings set in the face of various risks. These priorities are of course shaped by biological properties of the human body, but they are equally shaped by the webs of meaning that human beings keep creating and transforming.

For this reason, caution is required when working with sensitivity functions. The reality they are meant to represent is usually highly contingent on social context. As a rule, one should expect human decision-makers to draw on a whole set of sensitivity functions, where context as well as personal autonomy can determine which one of them will shape a given decision. Moreover, one should always be prepared to the situation where suffering can be described by a tolerance relation, but this relation cannot be translated into a meaningful sensitivitiy function.

9.3 *Impact*

If and only if a sensitivity function can be identified, one can quantify the impact experienced by Bianca or any other agent by combining exposure and sensitivity to different risks. The risks to be considered can then be represented by a subset of relevant critical events out of the total set of conceivable critical events. I write $C^* \subset C$ for the subset of relevant events, and "$... \otimes ...$" for the operation of combining exposure $\bar{\mathbf{E}}$ and sensitivity $\mathbf{S}$ into a function indicating the impact $\mathbf{I}$ of a possible combination of risks as a real number:

$$\mathbf{I} \;:=\; \bar{\mathbf{E}} \otimes \mathbf{S} = \sum_{c \in C^*} \left(\int_0^1 \bar{\pi}(c) \cdot \tilde{\pi}(\bar{\pi}(c)) \cdot S(c) \, d\bar{\pi}(c) \right) \tag{33}$$

When discussing global change, environmental research is required to identify the relevant subset of critical events, e.g., by assessing whether there is a risk of a one-meter sea-level rise from climate change in the next fifty years. Assessing the probability of that risk requires the combination of first order and second order probabilities discussed above. Assessing the sensitivity of some agent to that risk requires data on the preferences of the agent in question when faced with choices between risks of sea-level rise and other risks.

An impact function of this type is a special case of the broader class of cardinal utility functions for decision-making under uncertainty, as developed in different variants by von Neumann and Morgenstern (1947), Savage (1954) and many others. The numerical value for the impact of different risks still has a clear meaning. It is rooted in the possibility that risks can be mixed by concatenating them. Faced with a choice between two roads, Bianca may roll a dice and take the first road if she gets a 1 or 2, the second otherwise (risks can be concatenated by destiny as well, without Bianca having to roll a dice). Clearly, risk mixtures modify the probabilities she has to deal with. Sensitivity values indicate what characteristic frequencies would make Bianca indifferent between different risks. Impact values indicate what mixture ratios would make Bianca indifferent between mixtures of different risks.

The caveats expressed above first for exposure, then for sensitivity, mean that any impact function must be handled with considerable care. In impact assessments, monetary damages are often taken as a proxy for sensitivity. That may or may not be a useful thing to do. The key point is to ask whose suffering is being assessed - it will then often be obvious whether monetary damage is a good proxy or not. When used in the wrong circumstances or in inappropriate ways, impact functions may well do more harm than good. In these cases, it is much better to work with the tolerance relation for vulnerability discussed in section 8. Sometimes, it may also be possible to work with tolerance relations for exposure and for sensitivity. But these can be combined into a tolerance relation for impact only in the most obvious cases. Suppose that we wish to compare Bianca's vulnerability in two situations characterised by two critical events, namely floods and droughts. Let the probability for both kinds of extreme events be higher in situation A than in situation B. Now suppose that Bianca's sensitivity for the relevant risks can be characterized by a tolerance relation only. If the relation is such that she always prefers lower probabilities to higher ones, then the impact in situation A is clearly higher than in situation B. This may well be useful statement to make, but it is a long shot from a quantifiable impact. On the other hand, in a given social context given human decision-makers often do show the sort of preferences over risks implied by a specific impact function, and making such a function explicit can help to predict, analyse and sometimes improve their decisions.

9.4 *Adaptive Capacity*

A key issue in vulnerability analysis is the role of adaptive capacity (Burton et al. 2002, Mendelsohn and Neumann 1999). To assess the vulnerability induced by some form of global - or other - change, therefore, one must consider how Bianca - or any other agent - can shift from a pattern of behavior that was appropriate under earlier circumstances to one that is more appropriate under new ones. For this purpose, one needs some idea of Bianca's opportunities for action, her adaptive capacity. It will depend on various kinds of resources, ranging from money to knowledge, from friends to social security. Of course, the critical events under consideration may well impair these resources, and therefore there may be considerable difference in adaptive capacity before or after some such event. Hence the importance of environmental research providing - as far as possible - reasonable forecasts of critical events in the future.

To a considerable extent, Bianca's resources will depend on the actions of other people. Whether she has a job, how much she earns, how reliable her friends are, how much attention the state pays to people like her when they are in difficulties, etc. - the answer to all these questions depends critically on actions by others. When she is well endowed with such resources, a flood will be a very different event than when these resources are lacking. This raises the issue of social interaction and social solidarity, to which I will come back in section 9.5.

Adaptive capacity will rarely, if ever, have a meaningful representation as a single number. In general, it is best represented as a function selecting feasible actions from a set of conceivable actions depending on a list of available resources. I call the list of these resources r, the set of all possible lists of resources R, conceivable actions x_i, and the set of all conceivable sets of actions P. Then one can represent Bianca's adaptive capacity by the following function:

$$\mathbf{A} : R \rightarrow P = \{\{x_1\}, \{x_1, x_2\}, ...\}, \quad r \mapsto A(r) \tag{34}$$

9.5 *Vulnerability*

Until now, I have looked at the consequences for Bianca of a given course of action in the face of various risks. If environmental conditions - including the social environment - change, Bianca may need to reassess her past choice of action. To analyze this choice, one must specify the probabilities for critical events as conditional probabilities. They are conditional on Bianca's actions, but also on those of other agents. Remember that the critical event is not a hurricane, it is Bianca drowning in the ensuing flood, and the probability of her drowning depends on her actions, too. With these conditional probabilities, one can then combine impacts and adaptive capacity to get Bianca's vulnerability.

I write x for Bianca's actions, $\bar{\pi}(c|x))$ for the first order conditional probabilities of critical events, and $ArgMin$ for the minimal value of the impact function. The minimum is taken over those actions that are feasible with Bianca's resources, $x \in A(r)$. I write "... $\oslash$ ..." for this combination of the impact function with the function for adaptive capacity and get:

$$\mathbf{V} := \mathbf{I} \oslash \mathbf{A} \ = \ ArgMin_{[x \in A(r)]} \sum_{c \in C^*} \left(\int_0^1 \bar{\pi}(c|x) \cdot \tilde{\pi}(\bar{\pi}(c|x)) \cdot s(c) \ d\bar{\pi}(c|x) \right) \tag{35}$$

Bianca's vulnerability is the minimal value of the impact function over the set of her feasible actions. Whether her vulnerability increases as a result of some environmental change depends on the comparison between this value before and after the change.

Clearly, the feasibility of actions in the contemporary world often depends on the availability of money - therefore, money is a key resource for adaptive capacity, too. Money provides a remarkable link between socio-economic and mathematical domains of discourse. When we measure the length of an object like a table or a stone, the object is there and we ask how long it is. When we assess the relative frequency of heads in a sequence of throws of a coin, the sequence is there and we ask whether it can be characterized by a probability measure. Money as we know it, however, consists in numbers stored in peculiar ways. Money is not based on measuring some given thing one way or other, but on using numbers in a way defined by institutions like credit, accounting rules, and the like.

Of course, there are coins, and there were the times of the gold standard, when nuggets found in the mountains could be as good as coins of money. But today we live in a world where numbers stored in computers can bring governments down, without any relation with gold reserves. And numbers stored on computers determine to a considerable extent what options people all over the world do and do not have when dealing with their vulnerabilities. Whether these numbers could function even without the circulation of coins and bank notes remains to be seen.

Economic theory relies on two major links betweens socio-economic and mathematical domains of discourse. On one hand, there are utility functions, more or less similar to the index of suffering that we have used to describe human sensitivity to global change, and sometimes geared to subjective probabilities. On the other hand, there are production functions. They are used to represent physical production structures, like the amount of steel needed to produce a car. As far as they rely on physical properties of the goods under consideration, they can use the broad and solid bridges between physical and mathematical domains of discourse. Where they are formulated in more generic terms, as when capital is treated as a single magnitude, things are less clear. One way or other, monetary magnitudes must enter the picture, and a whole array of difficult questions is usually swept under the carpet in order to use those magnitudes without further delay.

This is a pity, as those questions provide opportunities for fruitful and important research. A good example is provided by the long traditions of studies showing that human preferences sometimes systematically violate the conditions required for a neat index of human suffering - or more generally, a utility function - to exist under conditions of uncertainty (milestones of this research are Allais 1953, Ellsberg 1961, Kahneman and Tversky 1979).

The problem is not that one cannot use a utility function, the problem is that one needs a plurality of such functions even to describe a single agent. In particular, this will be the case when indifference between two risks involving the same critical events results with more than one proportion between probabilities. And much as one needs more than one utility function to describe an agent, one may need more than one distribution of subjective probabilities. Which one will be activated then depends on the social context that agent finds herself in. The well-known research on anchoring, framing, and similar effects shows that it is possible to identify rules linking social contexts to different probability and utility functions.

An important example of that plurality of utility functions is given by lexical preferences. Many people feel that it is never appropriate to consider a trade-off between a human life and a standard consumer product: No amount of chewing gum is sufficient to justify a decision to let a human being die. The same people may feel that it is necessary and appropriate for a hospital to decide which patients to keep alive with great effort and which ones to let die as peacefully as possible.

Ethical reasoning usually has a lexical structure: ethical arguments often imply that certain trade-offs are appropriate only as long as they don't affect specific ethical norms. At the same time, ethical choices often concern trade-offs between compliance with different ethical norms. With regard to vulnerability to global environmental change, lexical preferences are particularly relevant for the difficult, but necessary debate about equity (Kasperson et al. 2001).

When possible actions can be represented by real numbers, no single utlitity function can represent such lexical preferences. Such preferences can be easily represented, however, by a series of utility functions combined according to suitable rules. A similar argument holds for subjective probability: many difficulties with designing an empirically sound distribution of subjecitve probabilities are due to an unnecessary attempt to work with a single such function. A whole set of functions together with context-dependent selection rules seems more appropriate.

So far I have discussed vulnerability in view of a single individual, in our case named Bianca. But clearly Bianca's opportunities for action are linked to those of other individuals in many ways. In the face of suffering, human beings may or may not establish and maintain bonds of solidarity. Such bonds can be very effective when they are based on face-to-face relations between people who know each other personally, but they can also be highly effective in other ways when they are based on institutional mechanisms like insurance schemes, provision of public goods by political action, etc. Bonds of solidarity enable people to think and act as a team, to develop shared and interlocking preferences as well as shared and interlocking resources. Ultimately, any vulnerability analysis of human communities like nations, regions, etc., must carefully check whether large parts of these communities are being instrumentalized for the purposes of other individuals and groups, or whether those communities really have developed some common preferences and a capability for joint action in the light of these.

A critical thread in the literature on vulnerability deals with social solidarity as a cornerstone of adaptive capacity (Sen and Dréze 1999, Ribot et al. 1996). However, it is remarkably difficult, often strictly impossible, to aggregate given individual preferences into social preferences in a meaningful way (Arrow 1951, Sen 1970, Kirman 1992). Such aggregation is often thought to be performed automatically by the market. But of course, the market aggregates preferences by weighting them with individual wealth - and the distribution of wealth in today's world is not exactly a hallmark of social solidarity.

Aggregating individual preferences into social ones is not always desirable. Where it is, it requires changes of preferences induced by exchanges of argument - something one might claim lies at the core of human freedom. Debates about preferences and solidarity are impossible, however, as long as one maintains a simple dichotomy between scientific facts - that are supposed to be amenable to rational argument - and human values - that are supposed to be either self-evident or arbitrary. One reason to pay attention to the reality of heterogeneous domains of discourse is the fact that it helps overcoming that dichotomy without falling back into a Medieval quest for a single compelling truth (Putnam A.R. 1985, Putnam H. 2001, Jaeger et al. 2001).

Debates about human vulnerability and solidarity in the face of global change may well lead to new kinds of institutions. In the lifetime of our grandchildren, they may support forms of global solidarity that humankind has not known in the past. Before rushing into the playground of institution design, however, it may be appropriate to acknowledge the need for a patient effort to enlarge our understanding of human vulnerability in the face of current global change. This will hardly be possible without major advances in our capability for integrated environmental modelling; integrated, that is, across highly heterogeneous domains of discourse.

10 *Modelling in the Multiverse*

The word "New York" refers to a city in America, the word "and" does not refer at all, it fulfills another role in human conversations. Within a given domain of discourse, we can refer to its objects, without such domains, we could not talk about anything. We can refer to objects by names, but also by identifying them with suitable predicates - as when we talk about the largest city on the East coast. Domains of discourse often have some regions of overlap. In these cases, we may talk about the same object within different domains. In one domain, the horizon does not move as the sun rises in all its glory, in another one the earth rotates so that changing parts of its surface are exposed to the rays of the sun (see Goodman 1978 and Putnam 1992 for further discussions of the rather interesting philosophical issues involved in this example).

Sometimes, we use an object to represent another one, as when one uses an icecube in a glass of water to show how the tip of an iceberg is much smaller than its invisible part. I use the words "refer" and "represent" to highlight the difference between on the one hand the relation linking a name or an expression to the object it refers to and on the other hand the relation between two objects where the first is used to represent the second. Representing one object of discourse with another is useful in order to learn, remember, communicate all sorts of things about the object represented. Usually this is based on the fact that the representing object is either more familiar or easier to handle than the represented one.

The word "model" is used in many ways. When we talk about a computer model, we usually refer to a piece of software running on some hardware equipment so as to represent a specific object. The hardware is a physical object made of sylicon, metals, plastics, etc., and put in a sequence of states according to the software. The software is an algorithm, a mathematical object. The relation between hardware and software raises some deep issues, sometimes triggering speculations about the relation between body and mind. I will not explore these here. What matters for the present purpose is the fact that in computer modelling we use mathematical objects to represent some other object - an organization, a conflict, a car, a stock market bubble, a prime number, an ocean, etc.

Computer modelling is a sophisticated thread in the fabric of human culture. The physical equipment is linked to the represented object via mathematical objects, namely algorithms, but usually the link between these and the represented object is established by further mathematical objects. When modelling an insurance market, e.g., one may use a fixed point of some continuous mapping to represent market equilibrium and then an algorithm operating over a finite range of natural numbers to represent that fixed point. The key interface between the represented object and the computer model is not the hardware, not even the software, it is the mathematical object captured by that software. And the key question in computer modelling is how to represent the intended object by such a mathematical object. This requires a suitable domain of discourse for the object to be represented. On the one hand, it must enable one to talk about interesting features of the object one wants to represent. On the other hand, it must correspond to a logical structure that fits the relevant domains of mathematics. By and large, such domains of discourse are available for physical objects - but things tend to become murkier when one is dealing with living beings. In the social sciences of our times such domains of discourse are not to be found ready made.

In the past decades, several powerful ways of using mathematical objects to represent socio-economic ones have been developed, in particular in the domains of statistics, linguistics, and economics. But serious issues remain unresolved. They concern the arbitrary character of many predicates used when construing "measures" of social phenomena, the questionable use of formal languages as models of natural languages, the restrictive assumptions needed to represent preference structures by utility functions, and more. Such problems cannot be resolved by simply construing more models, but only by a patient clarification of socio-economic domains of discourse and their relationships with other domains. Such clarification, in turn, can be greatly facilitated by the thoughtful design and implementation of computer models.

Clarifying relations between different domains of discourse, especially when they are so heterogeneous as to involve different logical patterns of inference, is rewarding way beyond the realm of computer modelling. Coming to terms with the pluralism of domains of discourse may well be one of the most important challenges humankind is faced with in the present global culture. This holds both for the plurality of scientific and non-scientific domains and for the plurality of cultural traditions flourishing in different parts of the world. And addressing some of the most pressing issues of our times - like the ones of global environmental change and the related ones of overcoming poverty at a planetary scale - clearly require the ability to combine knowledge rooted in highly heterogeneous domains of discourse.

The business of establishing, restoring, improving mutual understanding between researchers specializing in widely differing domains of discourse can take advantage of some of the most highly specialized domains of discourse developed so far: those of mathematics. The standards of reliability cultivated over centuries by mathematicians have led to a wealth of techniques and insights that can be used to improve the concepts of other domains of discourse. This implies an interative movement between a mathematical and a non-mathematical domain, patiently asking questions about both as well as about their relation, and boldly exploring new inventions that hold promise of fresh insights. There is no general recipe here, only the pleasure of cracking one nut after the other. The remarks on the concept of vulnerability in the present section offer a basket of nuts to be cracked in this way.

That said, let us keep in mind that links between different domains of discourse are ultimately forged in natural languages. The great dream of a unified language for scientific discourse, a language that would then provide the scaffolding for human knowledge in general, has a long and impressive history. In a variety of forms, it has been shared by Frege, Poincaré, Whitehead, Wittgenstein, and many others. The publication of Kuhn's essay on 'The Structure of Scientific Revolutions' - ironically in the first volume of the Intemational Encyclopedia of Unified Science (1962) - marks the awakening from that dream.

The linkages between different domains of discourse are due to the fact that these domains are embedded in natural languages, not to the illusion that somewhen in the future they might all be integrated with the tools of logic, or mathematics, or whatever. One cannot model the multiverse on a computer, one can only model things in the multiverse by using other things, in particular computers. If as modellers we want to talk about the world as a whole, it is probably a good idea to pause for a moment first, perhaps listen to non-scientists like various stakeholders next (Kasemir et al 2003), and then talk in natural language. And there is not one natural language lying at the root of all others (although of course languages like Latin or English play a pivotal role in specific historical periods): the unity of the human mind resides in a plurality of languages that offers opportunities for translation.

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