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Climate, vegetation, and global carbon cycle: the simplest zero-dimensional model

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Abstract

The mechanisms of interaction between climate and biosphere are studied for some hypothetical zero-dimensional (point) planet, where all parameters are globally averaged over the two-dimensional surface of the planet, which is without ocean. These mechanisms are formed by two causal loops: vegetation → albedo → temperature → vegetation and vegetation ↔ atmospheric carbon → temperature → vegetation with a strong non-linear interaction. Using the conservation law for the total amount of carbon in the system and taking into account the assumption about quasi-stationary evolution of the system under anthropogenic CO₂ emission, we reduce the dimension of the basic system of differential equations to two. The reduced system is then studied by qualitative methods. The system can have up to five equilibria, three of them can be stable. Here there are two bifurcation parameters: total amount of carbon (A) and product of maximal plant productivity and residence time of carbon in the biota. Considering the system evolution under increase of A, we can observe the change of the planet 'status' from 'cold desert' to 'green cold planet' (first bifurcation), then a 'tropical planet' arises (second bifurcation), and, as a result of further increase of carbon in the system, the planet transforms to a 'hot desert'. In conclusion the model was calculated for 'quasi-Earth' values of parameters. © 1997 Elsevier Science B.V.

Keywords: Minimal model; Biosphere; Carbon cycle

1. Introduction

In our first publication (Svirezhev and von Bloh, 1996) we formulated the following questions:

- How does the 'Biosphere machine' operate?
- Is our Earth Biosphere unique or could, maybe, other, virtual biospheres exist?

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In this publication we continue to answer these questions, considering the system ‘biosphere + climate’ as a system with very strong non-linearities and multiple equilibria. Note that we remain within a framework of the simplest zero-dimensional models. The results obtained by the analysis do not depend on the explicit form of the chosen functions and can be generalized to a system obeying the same topological structure.

It is obvious that vegetation dynamics depend on temperature, precipitation, and the concentration of carbon in the atmosphere. On the other hand, temperature dynamics depend on the concentrations of carbon and water vapour in the atmosphere, and on the albedo of planetary surface. For instance, the albedo of ‘white sands’ desert is equal to 0.4; for coniferous forest it is about 0.1 (Robock, 1980). Our model includes the following simple submodels: global carbon cycle, vegetation and the equation for annual global temperature.

A few words about the history of this problem. Kostitzin (Kostitzin, 1935) realized Vernadsky’s (Vernadsky, 1926) idea about an interdependence between vegetation and climate in the form of a mathematical model for the coevolution of the atmosphere (climate) and the biota. It is interesting that this was the first mathematical model of a global carbon cycle. His ‘epoques glaciers’ act as self-oscillations of this system. Further attempts at the modelling of climate-vegetation interactions were made by (Watson and Lovelock, 1983) (Daisyworld, a model of a hypothetical planet). In 1994 the so-called ‘virtual biospheres’ concept was formulated (Svirezhev, 1994). According to this, the contemporary Earth Biosphere is one of many possible (virtual) biospheres, corresponding to multiple equilibria of some strongly nonlinear dynamic system ‘climate + biosphere’.

In the course of our planet’s history and own evolution this system passed through several bifurcation points, when random factors (small perturbations) determined which branch of the solution the system would take. A moving force of this evolution could be the evolution of the ‘Earth green cover’, which has, in turn, several bifurcation points, e.g. the appearance of terrestrial vegetation and the change from coniferous to deciduous forest.

Note that this concept contradicts Vernadsky’s ‘ergodicity axiom’, according to which the contemporary Earth Biosphere is unique, beyond dependence on its initial and previous states.

2. The model: formulation and simplification

The proposed model is called the simplest model because it contains the minimal set of state variables to realize a global vegetation model incorporating the global carbon cycle. The climate of our hypothetical planet is described by one variable, namely, the annual average temperature of its surface. The planetary atmosphere is an isotropic one, containing one ‘greenhouse’ gas-CO₂-with carbon concentration $C(t)$. The equation for temperature will be (Goody, 1964, Petoukhov, 1995):

$$k\dot{T} = S(1 - \alpha) - \sigma\varphi(C)T^4 \quad (2.1)$$

where k is the surface heat capacity, S is the solar radiation, α is the surface albedo, and σ is the Stephan-Boltzmann constant. The ‘greenhouse’ effect is described by the function $\varphi(C)$, which is a monotonous decreasing function with saturation for $C \rightarrow \infty$: $\varphi(C) \rightarrow \varphi_\infty$, $\varphi(0) = 1$. A good approximation is a hyperbola.

We consider the ‘point’ planet to be without ocean, covered by vegetation with the density ρ (in carbon units per unit surface). If Σ is the total area of the planet, then $N = \rho\Sigma$ is the total amount of carbon contained in the vegetation. (Without loss of generality we can put $\Sigma = 1$).

As in the previous model we assume that the albedo α depends on the density of vegetation N , so that α is a monotonous decreasing function of N (see Fig. 1).

We include in the model the simplest two-compartmental submodel of the global carbon cycle, when the total carbon is allocated to the compartments: atmosphere and biota (vegetation), with corresponding concentrations C and N .

The equation for biota is:

$$\frac{dN}{dt} = P(C, N, T) - mN \tag{2.2}$$

where P is the annual net-production of vegetation, m is the value which is inverse to the residence time of carbon in the biota, τ_N .

The equation for atmospheric carbon is

$$\frac{dC}{dt} = -P(C, N, T) + mN + e(t) \tag{2.3}$$

where $e(t)$ is the annual anthropogenic emission. The total amount of carbon $A(t) = C(t) + N(t)$ has the following evolution in time:

$$\frac{dA}{dt} = \frac{dC}{dt} + \frac{dN}{dt} = e(t) \tag{2.4}$$

and integrating the equation one obtains

$$A(t) = C(t) + N(t) = A_0 + \int_{t_0}^t e(\tau) d\tau \tag{2.5}$$

where $A_0 = C(t_0) + N(t_0)$. Using the equality to exclude the variable $C(t)$ from the Eq. (2.2) and Eq. (2.3), we get

$$\begin{aligned} \frac{dT}{dt} &= \frac{1}{k} [\Psi(N) - \sigma\varphi(A(t) - N)T^4] \\ \frac{dN}{dt} &= P(A(t) - N, N, T) - mN \end{aligned} \tag{2.6}$$

where

$$\psi(N) = S(1 - \alpha(N)) \tag{2.7}$$

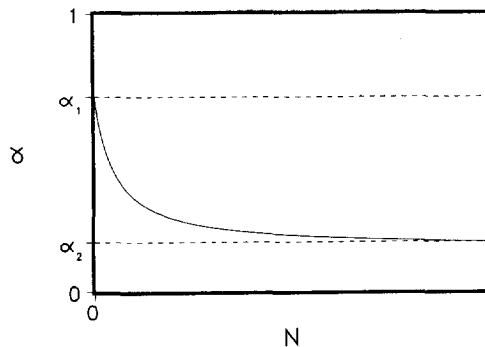


Fig. 1. Albedo α as a function of the carbon mass in the vegetation N .

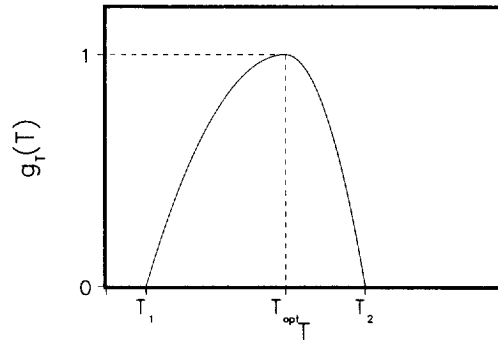


Fig. 2. Function $g_T(T)$ describing the dependence of productivity on temperature T .

The system is the simplest model of the biosphere of our hypothetical planet. Due to the explicit time dependence of $A(t)$ it is a non-autonomous dynamical system. If we suppose that $A(t)$ changes quasi-stationarily with t , i.e. is a slow process in relation to T and N then

$$\frac{dA(t)}{dt} = e(t) \ll A(t) \quad (2.8)$$

Therefore we analyse the stationary solutions of the system with $A(t) \equiv A$.

2.1. About the productivity function $P(C, N, T)$

The productivity of the vegetation is a function of C , N , and T alone. As in Keeling (1973) we assume that the productivity function $P(C, N, T)$ can be presented in the multiplicative form (accordingly with Liebig's principle):

$$P = P_m g_T(T) g_C(C) g_N(N) \quad (2.9)$$

where P_m is the maximum productivity of the vegetation and g_C , g_N are functions: $R^{\geq 0} \rightarrow [0, 1]$. Here $g_T(T)$ is a unimodal function: $R \rightarrow [0, 1]$ (see Fig. 2), according to (Lüdeke et al., 1995); the function $g_C(C)$ is a monotonous increasing function with saturation (see Fig. 3).

The function $g_N(N)$ is a monotonous increasing function, tending to one when $N \rightarrow \infty$ (as in Fig. 3). Since $C = A - N$ then the product $g_C \cdot g_N$ can be presented in the form $G_N(N)$ for fixed $A(t)$ (see Fig. 4).

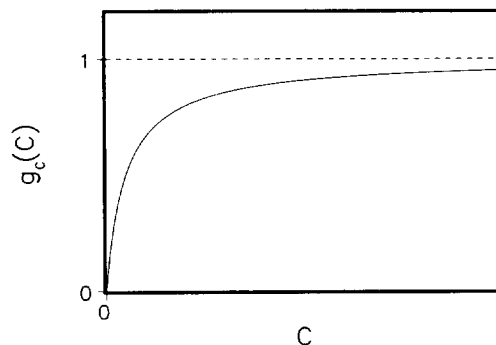


Fig. 3. Function $g_C(C)$ describing the dependence of productivity on carbon content C in the atmosphere.

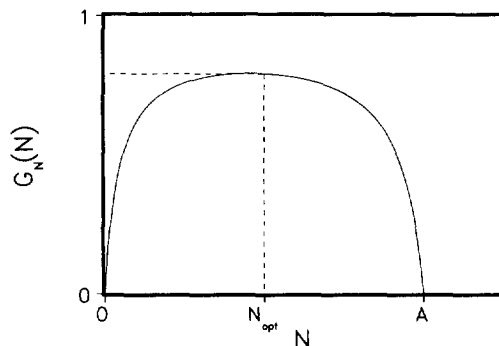


Fig. 4. The function $G_N(N) = g_C(A - N) \cdot g_N(N)$ as a function of N .

The function G_N is a unimodal function where $G_N(0) = G_N(A) = 0$. It is defined in the interval $N \in [0, A]$.

2.2. About the function $\Psi(N)$ and $\varphi(A - N)$

The function $\varphi_C = \varphi$, as a function of N , will have the following form (see Fig. 5). Analogously to the Daisyworld model by Watson and Lovelock, the vegetation cover changes the albedo α of the planet. We suppose that the surface albedo decreases with an increase of vegetation. In Fig. 6 the functional form of $\Psi(N) = S(1 - \alpha(N))$ is plotted.

In order to detect the equilibria we shall need the function $\Phi(N) = \Psi(N)/\varphi(A - N)$. It is obvious that $\Phi(0) = s_1/\varphi_A$, where $s_1 = S(1 - \alpha_1)$, and $\Phi(A) = \Psi(A)/\varphi(0) = \Psi(A)$. If $(\ln \Psi)'_N = (\ln \varphi)'_N$ at some point $N_{cr} \in [0, A]$, where the expression f'_N is defined as

$$f'_N := \frac{\partial}{\partial N} f \tag{2.10}$$

then $\Phi(N)$ has a maximum at N_{cr} (see Fig. 7).

If $N_{cr} > A$ then $\Phi(N)$ is a monotonous increasing function, and if $N_{cr} < 0$, $\Phi(N)$ is a monotonous decreasing one.

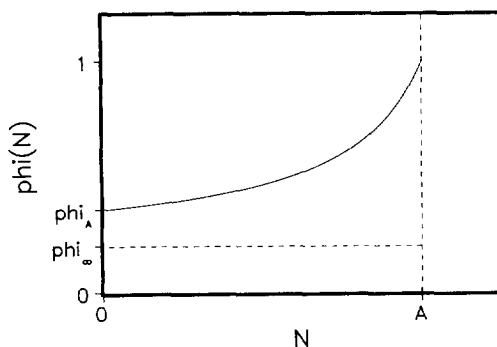


Fig. 5. The function $\varphi(C) = \varphi(A - N)$ (considering the greenhouse effect in the model) as a function of N . The value $\varphi(A) = \varphi_A$, corresponding to zero value of N , is determined by the value A . Since we shall consider this function as a function of $N: \varphi = \varphi(N)$, then $\varphi_A = \varphi(0)$.

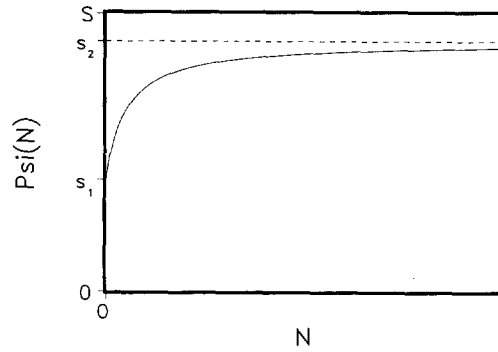


Fig. 6. The function $\Psi(N) = S(1 - \alpha(N))$ considering the solar irradiance: $s_1 = S(1 - \alpha_1)$, $s_2 = S(1 - \alpha_2)$.

2.3. Equilibria

The equilibrium points for the system Eq. (2.6) and Eq. (2.7) are determined by

$$T^* = \left\{ \frac{\Psi(N^*)}{\sigma\phi(N^*)} \right\}^{1/4} = \left\{ \frac{\Phi(N^*)}{\sigma} \right\}^{1/4} \tag{2.11}$$

and by solutions of the equation:

$$g_T(T^*)G_N(N^*) = \frac{m}{P_m} N^* \tag{2.12}$$

Since $G_N(0) = 0$ we present G_N in the form: $G_N = f(N)N$, where $\lim_{N \rightarrow 0} f(N) = f(0) < \infty$.

Then

$$N^* = 0, \quad T^* = T_0 = \left\{ \frac{\Psi(0)}{\sigma\phi(0)} \right\}^{1/4} \tag{2.13}$$

is one equilibrium. Since $N^* = 0$ this equilibrium corresponds to the ‘naked’ planet, i.e. desert, when no vegetation exists and all carbon is in the atmosphere.

The other equilibria are the solutions of the system of Eq. (2.11) and the equation:

$$g_T(T^*) = \frac{m}{P_m} \frac{1}{f(N^*)} \tag{2.14}$$

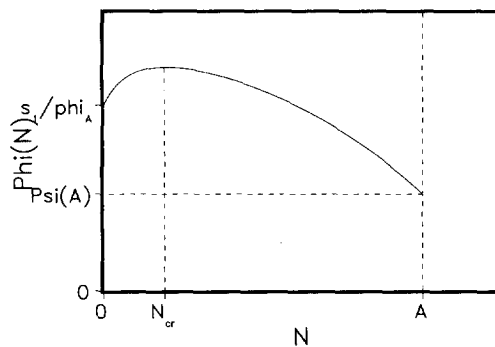


Fig. 7. The function $\Psi(N) = \Psi(N)/\phi(N)$; at the point N_{cr} one gets $\Psi' = 0$, or $(\ln \Psi)'_N = (\ln \phi)'_N$.

It is obvious that $T^* \in (T_1, T_2)$.

2.4. The cycles

Let us prove, using the Dulac criterion (see (Hale and Kocak, 1991)), that there are no cycles among the solutions of the system Eq. (2.6) and Eq. (2.7). Note that $f'_N(N) < 0$.

Since the expression

$$\begin{aligned}
 D &= \frac{\partial}{\partial T} \left\{ \frac{1}{N} \frac{1}{k} [\Psi(N) - \sigma \varphi(N) T^4] \right\} + \frac{\partial}{\partial N} \left\{ \frac{1}{N} [P_m g_T(T) f(N) N - mN] \right\} \\
 &= -4\sigma T^3 \frac{\varphi(N)}{kN} + P_m g_T(T) f'_N(N)
 \end{aligned}
 \tag{2.15}$$

is negative for any T and N which belong to the interior of the positive quadrant, then the system Eq. (2.6) and Eq. (2.7) does not have any closed trajectory.

2.5. Stability analysis

After linearization of the system Eq. (2.6) and Eq. (2.7) in the vicinity of the equilibrium points, we get the Jacobi matrix:

$$J = \begin{pmatrix} \frac{-4\Psi(N^*)}{kT^*} & \left[\ln \frac{\Psi(N^*)}{\varphi(N^*)} \right]'_N \frac{\Psi(N^*)}{k} \\ P_m N^* f(N^*) [g_T(T^*)]'_T & P_m g_T(T^*) \{ f(N^*) + N^* f'_T(N^*) \} - m \end{pmatrix}
 \tag{2.16}$$

Let us consider the point $\{N^* = 0; T^*_0\}$. Then

$$J_0 = \begin{pmatrix} \frac{-4\Psi(0)}{kT^*_0} & \left[\ln \frac{\Psi(0)}{\varphi(0)} \right]'_N \frac{\Psi(0)}{k} \\ 0 & P_m g_T(T^*_0) f(0) - m \end{pmatrix}
 \tag{2.17}$$

and the corresponding eigenvalues are equal to:

$$\lambda_1 = \frac{-4\Psi(0)}{kT^*_0} > 0; \lambda_2 = P_m g_T(T^*_0) f(0) - m
 \tag{2.18}$$

i.e. this equilibrium is a stable node, if

$$g_T(T^*_0) < \frac{m}{P_m f(0)}
 \tag{2.19}$$

and it is an unstable saddle, if

$$g_T(T^*_0) > \frac{m}{P_m f(0)}
 \tag{2.20}$$

Since $g_T(T) \equiv 0$ for any $T \notin (T_1, T_2)$, then $\lambda_2 = -m$, and any equilibrium $\{N^* = 0, T_0^* \notin (T_1, T_2)\}$ is stable. If $N^* \neq 0$ then

$$J = \begin{pmatrix} \frac{-4\Psi(N^*)}{kT^*} & \frac{\Psi}{k} \left\{ \ln \frac{\Psi(N^*)}{\varphi(N^*)} \right\}'_N \\ mN^* \{ \ln g_T(T^*) \}'_T & mN^* \{ \ln f(N^*) \}'_N \end{pmatrix} \quad (2.21)$$

and the corresponding eigenvalues are equal to

$$\lambda_{1,2} = \frac{1}{2} \left\{ - \left(\frac{4\Psi}{kT} - mN(\ln f)'_N \right) \pm \sqrt{\left[\frac{4\Psi}{kT} + mN(\ln f)'_N \right]^2 + \frac{4\Psi mN}{k} (\ln g_T)'_T \left(\ln \frac{\Psi}{\varphi} \right)'_N} \right\} \quad (2.22)$$

Since $(\ln f)'_N < 0$ for any $N \leq A$, then $4\Psi/kT - mN(\ln f)'_N > 0$ and the equilibrium with $N^* \neq 0$ cannot be an unstable node.

This equilibrium is a saddle point if

$$F(N^*, T^*) = \left(\ln \frac{\Psi}{\varphi} \right)'_{N^*} (\ln g_T)'_{T^*} + 4(\ln f)'_{N^*} (\ln T)'_{T^*} > 0 \quad (2.23)$$

If $F(N^*, T^*) < 0$, we have a stable node or stable focus, moreover for the latest

$$\left[\frac{4\Psi}{kT} + mN(\ln f)'_N \right]_{T^*, N^*}^2 + \left[\frac{4\Psi mN}{k} (\ln g_T)'_T \left(\ln \frac{\Psi}{\varphi} \right)'_N \right]_{T^*, N^*} < 0 \quad (2.24)$$

3. Parametrization

In order to make our investigation simpler and more visual, we use parametric presentations for functions $\Psi(N)$, $\varphi(N)$, and $f(N)$.

Let

$$\varphi(C) = 1 - \frac{(1 - \varphi_\infty)C}{k_c + C} \quad (3.1)$$

and substituting C by $C = A - N$, we get (see Fig. 5)

$$\varphi(N) = \frac{k_c + \varphi_\infty(A - N)}{k_c + A - N} \quad (3.2)$$

A good approximation for $\Psi(N)$ (see Fig. 6) is given by the function:

$$\Psi(N) = s_1 + \frac{(s_2 - s_1)N}{k_x + N} = \frac{s_1 k_x + s_2 N}{k_x + N} \quad (3.3)$$

Then

$$\Phi(N) = \frac{\Psi(N)}{\varphi(N)} = \frac{(s_1 k_x + s_2 N)(k_c + A - N)}{(k_x + N)(k_c + \varphi_\infty A - \varphi_\infty N)} \quad (3.4)$$

or

$$\Phi(N) = \frac{s_2}{\varphi_\infty} \frac{(a_1 + N)(k_c + A - N)}{(k_x + N) \left(\frac{k_c}{\varphi_\infty} + A - N \right)} \tag{3.5}$$

Let

$$g_T(T) = \begin{cases} \frac{4}{\Delta T^2} (T - T_1)(T_2 - T), & \text{if } T \in [T_1, T_2] \\ 0, & \text{if } T \notin [T_1, T_2] \end{cases} \tag{3.6}$$

where $[T_1, T_2]$ is the tolerance interval for vegetation, $\Delta T = T_2 - T_1$, i.e. the length of this interval. It is obvious that $T_{opt} = (T_1 + T_2)/2$ and $g_T(T_{opt}) = 1$.

For parametrization of $f(N)$ we must remember that $f(N) = G_N/N$. Because $G_N(N)$ can be described by parabola

$$G_N = \frac{4}{A^2} N(A - N)$$

and, respectively,

$$f(N) = \frac{4(A - N)}{A^2} \tag{3.7}$$

Using this parametrization we can investigate the evolution of the phase plane with an increase of the bifurcation parameter A .

3.1. The behaviour of $\Phi(N, A)$ as a function of A

Since $N \leq A$ then for $A \rightarrow 0$ we have

$$\Phi(A = 0) = s_1 \tag{3.8}$$

Let us detect the point N_{cr} , where $\Phi'_N = 0$. Since

$$\Phi'_N = \Phi \left\{ \left(\frac{1}{s_1 k_x / s_2 + N} - \frac{1}{k_c + A - N} \right) - \left(\frac{1}{k_x + N} - \frac{1}{k_c / \varphi_\infty + A - N} \right) \right\} \tag{3.9}$$

then N_{cr} is determined from the equation:

$$\frac{1}{s_1 k_x / s_2 + N} + \frac{1}{k_c / \varphi_\infty + A - N} = \frac{1}{k_x + N} + \frac{1}{k_c + A - N} \tag{3.10}$$

It is easy to show that for a large A this equation has the solution $N_{cr} \in (0, A)$. The derivative

$$\frac{\partial \Phi}{\partial A} = \frac{s_2}{\varphi_\infty} \frac{s_1 k_x / s_2 + N}{k_x + N} \frac{k_c(1 - \varphi_\infty)}{\varphi_\infty((k_c / \varphi_\infty + A)A + k_c / \varphi_\infty - N)^2} = \Phi \frac{k_c(1 - \varphi_\infty)}{\varphi_\infty} \frac{1}{(k_c + A - N)(k_c / \varphi_\infty + A - N)} > 0$$

and the second derivative $\partial^2 \Phi / \partial A^2 < 0$. Since

$$T^* = \left\{ \frac{1}{\sigma} \Phi(N^*) \right\}^{1/4}$$

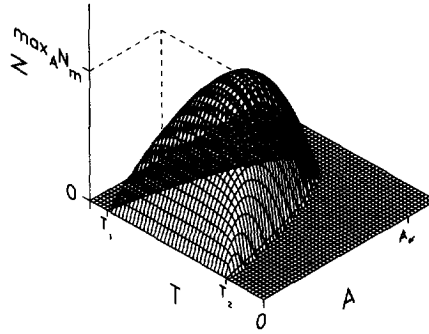


Fig. 8. The carbon mass in vegetation in equilibrium N as a function of equilibrium temperature T and the total amount of carbon A in the system. For $A > A_{cr} = 1/\beta$ equilibrium with $N = 0$ exist alone.

then $T^*(A)$ is a growing function of A for any N^* .

3.2. Equilibrium $N^* > 0$ as a function of g_T and A

The equilibrium $N^* > 0$ of the system is determined by the solution of Eq. (2.14). For $T \in [T_1, T_2]$ we have

$$g_T = \frac{m}{P_m} \frac{A^2}{4(A - N)}$$

or

$$N = A \left\{ 1 - \frac{mA}{4P_m g_T(T)} \right\} \quad (3.11)$$

where $g_T = (4/\Delta T^2)(T - T_1)(T_2 - T)$. It is obvious that $N \leq A$. Since $N > 0$ then

$$g_T > \frac{m}{4P_m} A \quad (3.12)$$

i.e. $N^* > 0$ only within the interval (T'_1, T'_2) , where

$$g_T(T'_1) = g_T(T'_2) = \frac{m}{4P_m} A$$

It is interesting that the interval is reduced to a point, i.e. $T'_1 = T'_2$, if $\beta A_{cr} = 1$, where β is defined as

$$\beta := \frac{m}{4P_m} \quad (3.13)$$

It is obvious that, if $A > A_{cr} = 1/\beta$, $N^* > 0$ does not exist, and there are the solutions $N^* = 0$ and T^* alone. In Fig. 8 the different $N(T)$, corresponding to different values of A , are shown. The maximum of N in respect to T is $N_m(A) = \max_T N = A(1 - \beta A)$. It is obvious that $\max_A N_m = 1/4\beta$ and it reaches that point at $A^* = 1/2\beta = A_{cr}/2$.

Let us remember the stability condition for the equilibrium $\{N^* = 0; T_0^*\}$ (see Eq. (2.19) and Eq. (2.20)). Since $f(0) = 4/A$ then from Eq. (2.19) it follows:

$$g_T < \frac{m}{4P_m} A \quad (3.14)$$

i.e. this equilibrium is stable, and if

$$g_T > \frac{m}{4P_m} A \tag{3.15}$$

it is unstable.

Let us compare these expressions to Eq. (3.12). Hence, this equilibrium is always unstable, if $T_0^* \in [T_1, T_2]$, and it is stable if $T_0^* \notin (T_1, T_2)$. It is obvious that for $A > A_{cr}$ the unique equilibrium $\{N^* = 0, T_0^*\}$ exists and is stable.

3.3. The ‘naked’ planet

This equilibrium is the equilibrium of the ‘naked’ planet. Let us consider it in detail.

If $T_0^* \notin (T_1, T_2)$, i.e. the equilibrium temperature lies outside the tolerance interval for photosynthesis, then this equilibrium is always stable. If $T_0^* \in (T_1, T_2)$ then the equilibrium is stable, if

$$g_T(T_0^*) < \frac{m}{P_m f(0)} = \frac{m}{4P_m} A \tag{3.16}$$

or

$$g_T \left\{ \sqrt[4]{\frac{1}{\sigma} \Phi(0)} \right\} < \frac{m}{4P_m} A \tag{3.17}$$

Suppose the reciprocal operator g^{-1} exists, so that

$$\sqrt[4]{\frac{\Phi(0)}{\sigma}} < g^{-1} \left\{ \frac{m}{4P_m} A \right\} \tag{3.18}$$

Since

$$\Phi(0) = \frac{S(1 - \alpha_1)}{\varphi_A} \tag{3.19}$$

then the left part of Eq. (3.18) depends only on the characteristics and total amount of carbon in the system. The right part of Eq. (3.18) depends only on the biotic characteristics of the planetary vegetation and also the total carbon. If we assume that the equilibrium temperature for the ‘naked’ planet is fixed—the total carbon is fixed also—but we can change (e.g. in an evolutionary way) the vegetation characteristics, then we can pass from a stable ‘naked’ equilibrium to an unstable one.

Let us consider Fig. 9: from this picture we can see that the condition $T_0^* \in [T_1, T_2]$, i.e. the condition that the equilibrium temperature belongs to the photosynthesis tolerance interval, is not sufficient in order to pass to the instability of a ‘naked’ equilibrium, i.e. for the ‘origin of life’. It is necessary that the temperature $T_0^* \in (T_1, T_2)$, then the ‘naked’ equilibrium becomes unstable and ‘life’ can occur. This interval depends on such biotic characteristics as the residence time of carbon in the biota ($\tau = 1/m$) and its maximum productivity P_m . We call the interval (T_1, T_2) the ‘vegetation tolerance interval’.

Let us consider the change of the product βA . Its increase, which corresponds either to the decrease of carbon residence time in the biota or to a decrease of the maximal productivity of photosynthesis, leads to a reduction of the vegetation interval. And, vice versa, the decrease of βA , because of the increase of residence time or increase of maximal productivity, increases the vegetation interval.

It is obvious that if $\beta A > \max_T g_T = 1$, then the ‘naked’ equilibrium is stable for any T_0^* , and life cannot arise in the vicinity of this equilibrium. In other words, there is some critical combination from P_m , m (or τ), and A :

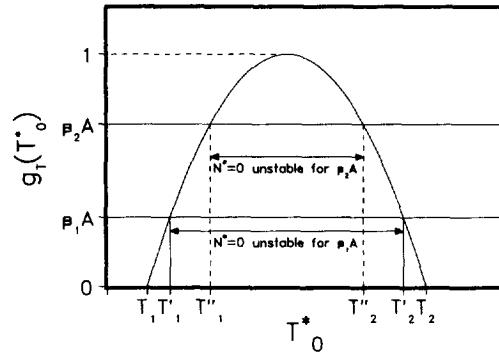


Fig. 9. To the problem of stability for ‘naked’ equilibrium: in the interval $[T_1, T_2]$ with $g_T < \beta_1 A$ it is unstable ($[T_1', T_2']$ or $\beta_2 A$). For $\beta A = 1$ this interval is reduced to a point, above 1 the equilibrium is stable for any T .

$$\frac{mA}{4P_m} < 1 \tag{3.20}$$

or

$$A < 4P_m\tau \tag{3.21}$$

On the other hand

$$T_0^* = \left\{ \sqrt[4]{\frac{\Phi(0, A)}{\sigma}} \right\} = \left\{ \sqrt[4]{\frac{S(1 - \alpha_1)}{\sigma\varphi A}} \right\} \tag{3.22}$$

is the monotonous increasing function of A , since φ_A monotonously decreases with the growth of A . Since $\varphi_A \rightarrow \varphi_\infty = \text{const}$, then $T_0^*(A) \rightarrow (T_0^*)_{\text{max}} = \text{const}$, if $A \rightarrow \infty$.

In fact there are two bifurcation parameters: β and A . Note that $T_0^*(A) \rightarrow (T_0^*)_{\text{min}}$, if $A \rightarrow 0$, where

$$(T_0^*)_{\text{min}} = \sqrt[4]{\frac{S(1 - \alpha_1)}{\sigma}} \tag{3.23}$$

since $\varphi(A = 0) = 1$.

Let us assume that $(T_0^*)_{\text{min}} < T_1$, and $(T_0^*)_{\text{max}} > T_2$, then we have the stability diagram as in Fig. 10. The border in the (β, A) domain is determined by the function

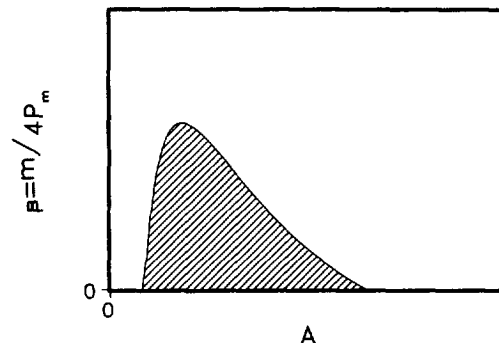


Fig. 10. Stability border for $N^* = 0$ in the $\{\beta, A\}$ domain: $g_T(T_0^*) = \beta A$, $T_0^* = T_0^*(A)$. The shaded area indicates the area of instability of $N^* = 0$ where life can arise in the vicinity of the ‘naked’ equilibrium.

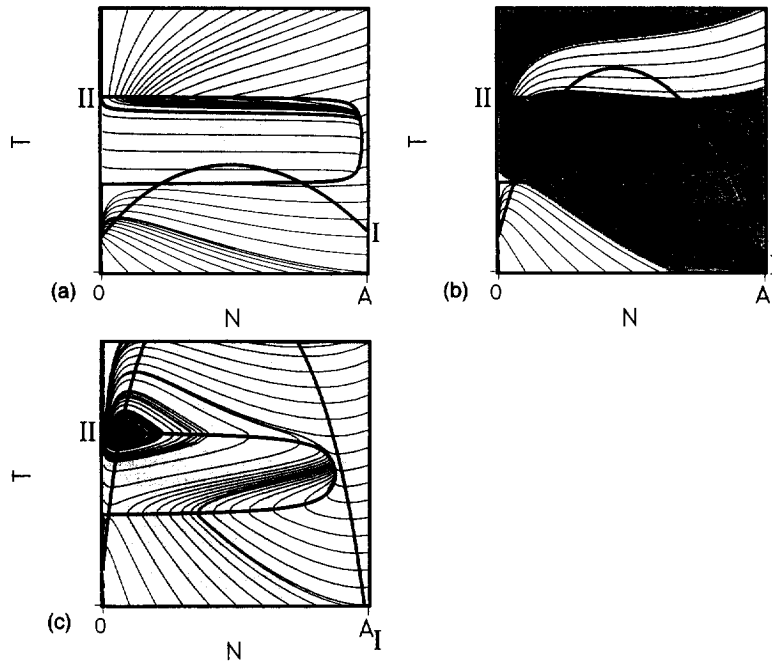


Fig. 11. Phase portraits for increasing A and fixed T_0^* . (a): ‘cold desert’ and ‘cold’ planet are stable equilibria, (b) ‘cold desert’, ‘cold’ and ‘hot’ planet are stable, (c) ‘cold desert’ and ‘hot’ planet are stable. Curve I: $T^* = \sqrt[4]{\Psi(N^*)/\sigma}$, and II: $N^* = A(1 - \beta A/g_T(T^*))$. The different shaded areas denote the basins of attraction.

$$\beta_{crit}(A) = \frac{g_T(T_0^*(A))}{A} = \frac{g_T((T_0^*)_{min}/\sqrt[4]{\varphi A})}{A} \tag{3.24}$$

whereas for $\beta < \beta_{crit}$ the ‘naked’ equilibrium is unstable.

4. About the numerical estimation for model parameters

Based on the analytical analysis, the system is analysed numerically in respect to the positions and number of stable equilibria. Phase portraits showing trajectories in the $\{T, N\}$ -domain give a visual impression of the system behaviour. The equilibria of the system are easily obtained by plotting the two functions I: $T^* = \{\Phi(N^*)/\sigma\}^{1/4}$ Eq. (2.11) and II: $N^* = A(1 - \beta A/g_T(T^*))$ Eq. (3.11) and determining their intersection points in the phase space. The maximum number of possible equilibria is restricted to five, because both functions I and II are unimodular ones and therefore intersect at a maximum of four points in $\{T, N\}$. Together with the ‘naked’ equilibrium one gets five as an upper boundary for the number of stationary solutions.

For a certain set of parameters we have a phase portrait as in Fig. 11a: two of the three possible equilibria are stable, the ‘cold desert’ with $N=0$ and the ‘cold’ planet with $N>0$. The unstable equilibrium separates the two basins of attraction. If A is increased, but T_0 is fixed, the phase portrait changes according to Fig. 11b, where three stable equilibria exist (‘cold desert’, ‘cold’ planet, and ‘hot’ planet). A further increase of A reduces their number to two (Fig. 11c).

Since we would like our planet to be similar to the earth, we shall use real values for estimation of the model parameters.

It is known for earth that $S = 340 \text{ W/m}^2$, $k = 1 - 2 \times 10^7 \text{ J/m}^2 \text{ per K}$ for land and $k = 20 \times 10^7 \text{ J/m}^2 \text{ per K}$ for ocean. Since our planet has no ocean, without loss of generality, we can put $k = 3 \times 10^7 \text{ J/m}^2 \text{ per K}$. The time step of the model is equal to 1 year $\approx 3 \times 10^7 \text{ s}$, then $k \approx 1$ in the corresponding units. The albedo for 'naked' earth (white sands) is $\alpha_1 \approx 0.4$, and for 'green' earth $\alpha_2 \approx 0.1$ (Robock, 1980).

About the role of atmospheric carbon: the 'greenhouse' effect. In agreement with (Petoukhov, 1995), $\varphi_\infty = 0.6$ and $k_c = 355 \text{ ppm}$ or $k_c = 750 \text{ Gt}$, so that

$$\varphi(C) = 1 - \frac{0.4C}{750 + C} \quad (4.1)$$

if C is measured in Gt. Note that $\sigma = 5.67 \times 10^{-8} [\text{W/m}^2\text{K}^4]$.

The temperature of the 'cold desert' for earth is determined by

$$T_0^* = \sqrt[4]{\frac{\Phi(0)}{\sigma}} = \sqrt[4]{\frac{s_1}{\sigma}} \approx -28^\circ\text{C} \quad (4.2)$$

Here $s_1 = (1 - \alpha_1)S$. On the other hand, the contemporary temperature is $T^* = 14^\circ\text{C}$. Note that in this case the mean albedo $\alpha_0 = 0.3$, so that $s_0 = 238 \text{ W/m}^2$, and the equilibrium temperature for a planet without atmosphere would be

$$(T_0^*)' = \sqrt[4]{\frac{s_0}{\sigma}} \approx -18.5^\circ\text{C} \quad (4.3)$$

Containing CO_2 and H_2O , the atmosphere increases the temperature (the 'greenhouse effect'). This increase can be obtained as a result of the reduction of the coefficient σ ; when we introduce some 'effective' σ denoted as σ' :

$$\sigma' = \varphi_C(C)\varphi_W(W)\sigma \quad (4.4)$$

where $\varphi_C(C)$ is a decreasing function, taking into account the role of CO_2 , $\varphi_W(W)$ is the similar function for water vapour with concentration W . Obviously, $\varphi_C(0) = \varphi_W(0) = 1$. For $C = 610 \text{ Gt}$ the corresponding value is $\varphi_C = 0.75$.

If we neglect the water vapour contribution, then

$$(T_0^*)'' = \sqrt[4]{\frac{s_0}{0.75\sigma}} \approx 0.5^\circ\text{C}$$

just as the real temperature is $T^* = 14^\circ\text{C}$. It shows that this contribution is very important, and we must include it in our consideration (in some implicit changing σ). From this condition

$$(287 \text{ K})^4 = \frac{s_0}{0.75\varphi(W)\sigma} \quad (4.5)$$

we get $\varphi_W = 0.825$, and $\sigma' = 0.75 \times 0.825\sigma$. But only considering CO_2 in the atmosphere, then

$$\sigma' = \varphi_C(C)\sigma''$$

where

$$\sigma'' = 4.68 \times 10^{-8} \text{ W/m}^2\text{K}^4 \quad (4.6)$$

and

$$(T_0^*)'' = \sqrt[4]{\frac{s_1}{\sigma''}} \approx -16^\circ\text{C} = T_{\min}^* \quad (4.7)$$

Since

$$\Phi(N) = (T_{\min}^*)^4 \left\{ \frac{k_x + \mu N}{k_x + N} \frac{k_c + C}{k_c + \varphi_\infty C} \right\} \quad (4.8)$$

where $\mu = s_2/s_1 = 1.5$, then

$$T^* = \sqrt[4]{\frac{\Phi(N)}{\sigma^n}} > T_{\min}^* \quad (4.9)$$

We assume that the contemporary state of the biosphere is an equilibrium.

We consider the current productivity of the biosphere as the equilibrium one. Here and hereafter we use the data from (Krapivin et al., 1982).

The ‘soil’ compartment was omitted in our model. Its influence can be described by an increase of the residence time of carbon in the biota. On the other hand, the relatively large part of dead organic matter (≈ 60 – 65%) is returned very rapidly to the atmosphere. For this reason, as an initial approximation, we can forget about the ‘soil’ compartment and consider the following estimations:

$$\begin{aligned} N^* &= 750 \text{ Gt}, \\ P^* &= 60 \text{ Gt/year}, \\ A^* &= N^* + C^* = 1360 \text{ Gt}, \\ m &= P^*/N^* \approx 0.08 \text{ 1/year} \end{aligned} \quad (4.10)$$

so that the residence time of carbon in the biota is equal to 12.5 years.

The tolerance interval for photosynthesis is $[T_1 = 5^\circ\text{C}, T_2 = 40^\circ\text{C}]$, and $T_{\text{opt}} \approx 23^\circ\text{C}$ (for a parabolic approximation of the growth function). Since $T^* = 14^\circ\text{C}$, then

$$g_T^* = \frac{4}{(\Delta T)^2} (T_2 - T^*)(T^* - T_1) \approx 0.764 \quad (4.11)$$

And from

$$G_N^* = \frac{4}{(A^*)^2} N^*(A^* - N^*) = 0.99 \quad (4.12)$$

we get $P_m = P^*/(G_N^* g_T^*) \approx 80 \text{ Gt}$. And, finally, from

$$\beta = \frac{m}{4P_m} = 2.5 \times 10^{-4} \text{ 1/Gt} \quad (4.13)$$

we get $A_{\text{cr}} = 1/\beta = 4 \times 10^3 \text{ Gt}$.

We have for our planet: $A = 1.36 \times 10^3 \text{ Gt} < A_{\text{cr}}$, i.e. the total amount of carbon is less than three times compared with its critical value and equilibria with $N^* > 0$ can exist. Remembering the formula for $\Phi(N)$, we can test the value T^* . It is equal to 14°C (for $k_x = 600 \text{ Gt}$). And finally we must note that all values N , C , etc. are measured in Gt (10^9 tons).

Fig. 12 plots the phase portrait corresponding to the parameter setting for earth summarized in Table 1. The phase portrait is similar to Fig. 11a, where two equilibria (the cold desert and the cold planet with vegetation) are stable for the given A . But note that for the present state of the earth the planet without vegetation $N^* = 0$ for any T is stable preventing life arising from the ‘dead’ planet state. Leaving the basin of attraction of $N^* > 0$ by, e.g. perturbation of N to a value below a critical threshold leads to a complete extinction of vegetation on earth.

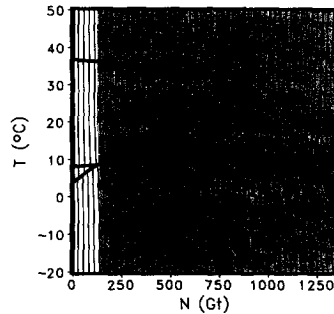


Fig. 12. Phase portrait for the system with parameter settings according to real earth scenario.

5. Conclusion

The presented climate-vegetation model including a carbon cycle shows some interesting features: the performed stability analysis of the equilibria of the system give us up to three stable points. Depending on the initial condition of our planet, the ‘cold’ desert without any vegetation will be stable, the ‘cold planet’ with a large amount of vegetation will exist, or a ‘hot’ planet with a small amount on vegetation compared with the cold planet will be reached. The initiation of life on our virtual planet depends on the value of two bifurcation parameters, the total amount of carbon A in the system and a combination of biotic characteristics β of the vegetation.

The stability of the different stationary states can be dramatically changed if we increase the total amount of carbon by, e.g. anthropogenic emissions. It is possible that the cold planet becomes unstable and a new equilibrium such as the hot planet is realized. Such a transition forces a drastic decrease in the vegetation of the planet.

In a next step we will introduce a water cycle into the model. This will increase the number of bifurcation points of the planet. A second goal is the determination of transition times from one stable equilibrium to another due to stochastic perturbations of the system.

Table 1
Model parameters for real earth scenario

Parameter	Value	Units
S	340	W/m^2
k	3×10^7	$\text{J/m}^2 \text{ K}$
σ	5.67×10^{-8}	$\text{W/m}^2 \text{ K}^4$
α_1	0.4	
α_2	0.1	
φ_∞	0.6	
k_c	750	Gt
k_x	600	Gt
T_1	5	$^\circ\text{C}$
T_2	40	$^\circ\text{C}$
A	1360	Gt
m	0.08	1/year
P_m	80	Gt/year

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