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A minimal model of interaction between climate and vegetation: qualitative approach

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Abstract

A so-called "minimal model" is presented for the qualitative description of the interaction between climate and vegetation. Such conceptual simple models are important to give a better understanding of the fundamental feedback mechanisms acting between geo- and biosphere. This spatially one-dimensional model is analyzed in respect to the uniform biosphere and the existence of possible diffusive instabilities. It is proved that there do not exist any non-uniform equilibrium solutions, but the evolution of non-uniform initial perturbations is quite interesting: depending on the spatial form of the perturbations propagating non-linear waves are observed.

Keywords: Climate; Minimal models; Nonlinear waves; Stability; Vegetation

1. Introduction

Today climatic change is a much debated topic: there is a huge amount of publications, which grow faster than exponential curve, speculations, etc. which mask (very successfully) on one simple fact: we hardly conceive ourselves how the "Biosphere machine" is operating. Therefore the role of simple (and simplest) models, so-called "minimal models" (Moiseev and Svirezhev, 1979) can be very helpful, if they are sufficiently simple for understanding and possess a significant amount of the qualitative properties of the investigated system.

The analytical as well as the numerical analysis of such a conceptual model for description of the interaction of climate and vegetation is the aim of our paper. The first attempts in this type of vegetation–climate modelling were done by Vernadsky (1926) and then by Watson and Lovelock (1983) ("Daisy-world" as a model of some hypothetical planet) and extented to 2-dimensional structures (Schellnhuber et al., 1993).

The paper is organized as follows: firstly we will give a description of the vegetation–climate model, then an analysis of the uniform biosphere and of possible diffusive instabilities is presented. Finally the progagation of waves in the spatial model is analyzed in respect to the dependence on the initial conditions.

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2. The model description

We suppose the climate of our world is determined by the temperature $T(x,t)$ only, and the dynamics is described by the equation

$$C\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \hat{S}(1-A) - \hat{\sigma}T^4, \quad x \in \Omega, \tag{1}$$

where Ω is the world "area", A is the albedo of planet surface at a given point x, $\hat{S}$ is the solar constant, so that the term $\hat{S}(1-A)$ is the insolation, and the term $\hat{\sigma}T^4$ (where $\hat{\sigma}$ is the Stephan–Boltzmann constant) is the erradiation. The latters can be presented in the Budyko's form (Budyko, 1969) also: $a + bT$, but it is not principally. The Budyko form together with the empirical constants a, b would give us a better agreement with the real radiation data of the earth. For a review of the so-called energy-balance models see, e.g., Henderson-Sellers and McGuffie (1990). Let us rewrite Eq. 1 in the form:

$$\frac{\partial T}{\partial t} = D_T\frac{\partial^2 T}{\partial x^2} + \Psi - \sigma T^4, \quad \text{where } D_T = k/C,\ \sigma = \hat{\sigma}/C,\ \Psi = \frac{\hat{S}}{C}(1-A) \tag{2}$$

Further we suppose that the planet surface can be covered by the vegetation with the density $N(x,t)$, its dynamics is described by a logistic equation:

$$\frac{\partial N}{\partial t} = D_N\frac{\partial^2 N}{\partial x^2} + \alpha N - \gamma N^2 \tag{3}$$

where α is the Malthusian parameter (Malthus, 1798) of the growth function.

Now we formulate two hypotheses, which are the description of feedbacks between the climate and vegetation of our world.

Hypothesis 1. *The albedo A depends only on the vegetation density N, so that $A = A(N)$ is a monotone decreasing function of N (see Fig. 1).*

Then the function $\Psi(N)$ will be the following (see Fig. 2). Obviously that $\Psi'_N(N) \geq 0$.
We assume $\Psi'_N > 0$ for any $N > 0$.

Hypothesis 2. *The growth function α depends only on T; α is an unimodal function of T (see Fig. 3).*

Obviously that $\alpha'_T > 0$ if $T < T_{\text{opt}}$, and $\alpha'_T < 0$ if $T > T_{\text{opt}}$, $\alpha(T) > 0$ if $T \in (T_{\min}, T_{\max})$.

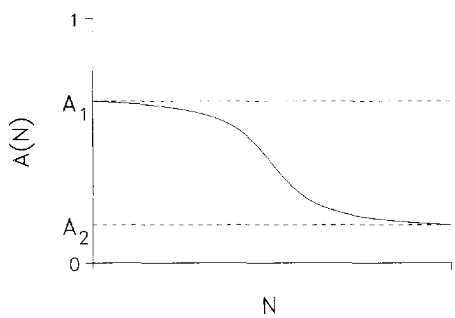

Fig. 1. Albedo A as a function of vegetation density N according to hypothesis 1. A_1 is the albedo of the naked surface, A_2 of surface fully covered by vegetation.

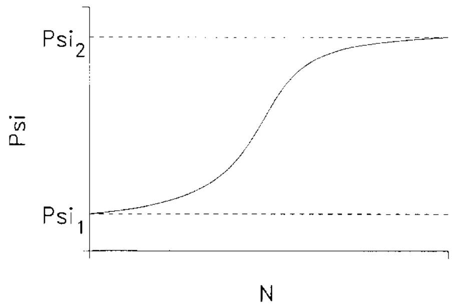

Fig. 2. The function $\Psi(N) = (\hat{S}/C)(1 - A(N))$ in (2).

On the ecological point of view, the $\alpha(T)$ describes the *ecological niche*, defined as the interval of T with $\alpha(T) > \gamma$, for vegetation in the space of climatic factors.

And finally, Eqs. 2 and 3 together with the functions $A(N)$ and $\alpha(T)$ and the corresponding initial and boundary conditions make up the biosphere for our world.

3. Preliminary analysis: uniform biosphere

First we will consider solutions, which do not depend on the spatial coordinate x. These solutions correspond to the so-called "uniform biosphere", i.e. the biosphere with characteristics which are identical for any point of the planet. The evolution in time of the uniform biosphere derived from Eqs. 2 and 3 is determined by the following nonlinear system:

$$\frac{\partial T}{\partial t} = \Psi(N) - \sigma T^4,$$
$$\frac{\partial N}{\partial t} = \alpha(T)N - \gamma N^2. \tag{4}$$

Further we can see that the most important properties of the general problem can be reduced from analysis of the system 4.

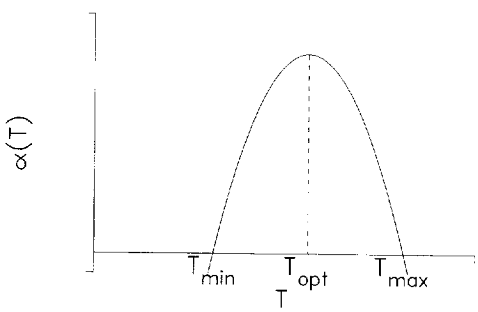

Fig. 3. The Malthusian growth function $\alpha(T)$ according to hypothesis 2.

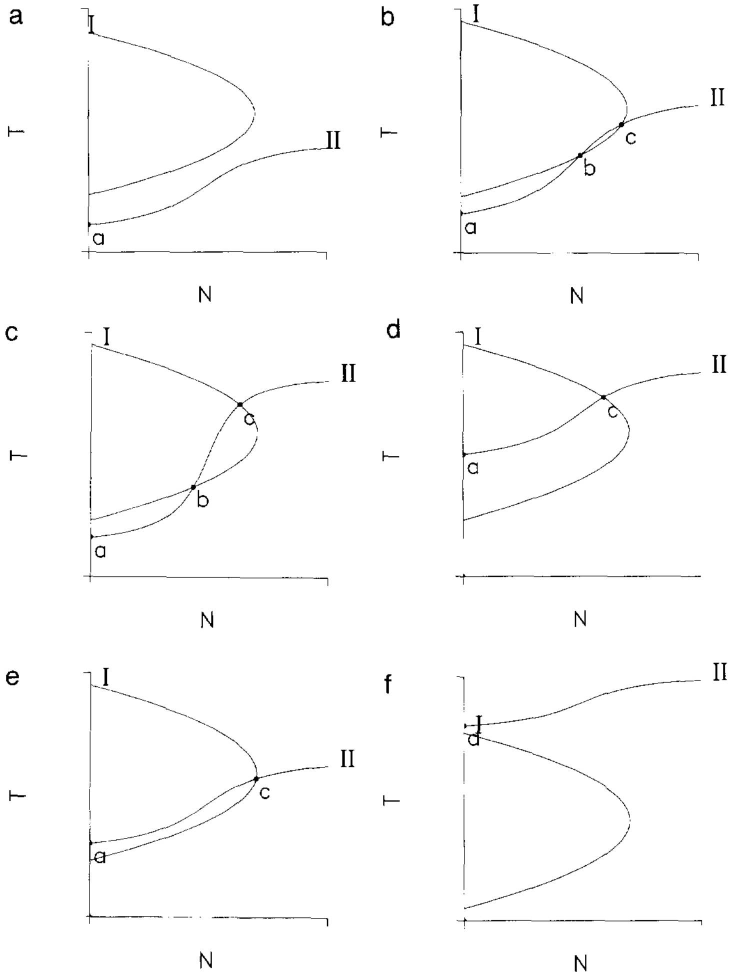

Fig. 4. Graphical representation of Eq. 5 at different $\hat{S}$. I: $N=\alpha(T)/\gamma$, II: $T=(\Psi(N)/\sigma)^{1/4}$. The different stationary points (T^*,N^*) are denoted by letters a–c.

Let us consider the equilibrium points $T(t)=T^*=$ const, $N(t)=N^*=$ const. of Eq. 4. Then they must satisfy the equations:

$$T^*=(\Psi(N^*)/\sigma)^{1/4},\quad N^*=0 \quad\text{or}\quad N^*=\alpha(T^*)/\gamma. \tag{5}$$

It is better to consider them graphically on the plane $\{N,T\}$. Depending on the value $\hat{S}$, six cases can be differentiated in respect to the number and position of the intersections between curve (I): $N=\alpha(T)/\gamma$ and (II): $T=(\Psi(N)/\sigma)^{1/4}$ (see Fig. 4a–f). These intersections fulfill Eq. 5 and are equilibria of Eq. 4.

Up to three equilibria denoted as $a=\{N_a^*,T_a^*\}$, $b=\{N_b^*,T_b^*\}$, and $c=\{N_c^*,T_c^*\}$ can be found. The points a in Fig. 4a–f are *semi-trivial* equilibria, where

$$N^*=0 \quad\text{and}\quad T^*=(\Psi/\sigma)^{1/4}. \tag{6}$$

If $T^* < T_{min}$, then we can speak about our planet as the "cold desert", if $T^* > T_{max}$, then about the "hot desert".

Let us calculate the eigenvalues $\lambda_{1,2}$ of the Jacobi matrix

$$J = \begin{pmatrix} \frac{\partial}{\partial T}(\Psi(N) - 4\sigma T^4) & \frac{\partial}{\partial N}(\Psi(N) - 4\sigma T^4) \\ \frac{\partial}{\partial T}(\alpha(T)N - \gamma N^2) & \frac{\partial}{\partial N}(\alpha(T)N - \gamma N^2) \end{pmatrix} = \begin{pmatrix} -4\sigma T^{*3} & \Psi_N'(N) \\ \alpha_T'(T)N & \alpha(T) - 2\gamma N \end{pmatrix} \tag{7}$$

for Eq. 4:

$$\lambda_{1,2} = \frac{1}{2}\left\{-(\gamma N^* + 4\sigma T^{*3}) \pm \sqrt{(\gamma N^* + 4\sigma T^{*3})^2 - 4N^*(4\gamma\sigma T^{*3} - \alpha_T'(T^*)\Psi_N'(N^*))}\right\} \tag{8}$$

The value of the λ_i characterizes the behaviour of the system in the vicinity of the equilibria $\{N^*, T^*\}$:

1. $\lambda_{1,2} < 0$, $\lambda_{1,2} \in \mathbf{R}$: $\{N^*, T^*\}$ is an attractor and is called a *stable node.*
2. $\mathrm{Re}(\lambda_{1,2}) < 0$, $\lambda_{1,2} \in C$: $\{N^*, T^*\}$ is called a *focus.* The local phase portrait is a spiral that winds into the node.
3. $\lambda_1\lambda_2 < 0$: $\{N^*, T^*\}$ is a repeller and is called a *saddle* point with one stable direction.
4. $\lambda_{1,2} > 0$: $\{N^*, T^*\}$ is a repeller and is called a *unstable node.*

If one of the λ_i is equal to zero, then an analysis of next order is neccessary. A comprehensive introduction to the analysis of ordinary differential equations and dynamical systems can be found in Arnol'd (1973), Jordan and Smith (1977), and McCauley (1993).

For the "semi-trivial" points with $N^* = 0$ we have

$$\lambda_1 = \alpha(T^*), \quad \lambda_2 = -4\sigma T^{*3} \tag{9}$$

and if either $T^* < T_{min}$ or $T^* > T_{max}$, the equilibrium $\{N^* = 0, T^*\}$ is a stable node ($\lambda_{1,2} < 0$), if $T_{min} < T^* < T_{max}$, then the equilibrium is a saddle point ($\lambda_1\lambda_2 < 0$).

For the "non-trivial" equilibrium with $N^* = \alpha(T^*)/\gamma > 0$ we have the following:

- if $\alpha_T'(T^*)\Psi_N'(N^*) < 4\sigma\gamma T^{*3}$, then the point $\{N^*, T^*\}$ is either a stable node or focus,
- if $\alpha_T'(T^*)\Psi_N'(N^*) > 4\sigma\gamma T^{*3}$, then this point is a saddle point.
- if

$$(4\sigma T^{*3} - \gamma N^*)^2 + 4N^*\alpha_T'(T^*)\Psi_N'(N^*) < 0, \tag{10}$$

then the point $\{N^*, T^*\}$ is a focus, in the opposite case we have either node or saddle.

According to the Poincaré–Bendixson theorem the system ends up either in a stable node or a limit cycle (a one-dimensional attractor) because we have only two phase variables T and N. If we apply to the system 4 the Dulac criterion as a proof of the absence of limit cycles (see, e.g., Hale and Kocak (1991)) in the form

$$\mathcal{F}(\mathcal{T}, \mathcal{N}) = \frac{\partial}{\partial T}\left\{\frac{1}{N}(\Psi(N) - \sigma T^4)\right\} + \frac{\partial}{\partial N}\left\{\frac{1}{N} \cdot N(\alpha(T) - \gamma N)\right\} = -\frac{4\sigma T^3}{N} - \gamma < 0,$$

then we can see that there is even not a limit cycle inside the positive quadrant $N > 0$, $T > 0$. Therefore we conclude:

The structure of phase plane for the system 4 is sufficiently simple.

Let us come back to Fig. 4a–f: now phase portraits of system 4 are plotted for parameters equivalent to the corresponding Fig. 4 in respect to the number and positions of the equilibria. Their stability is analyzed in the following.

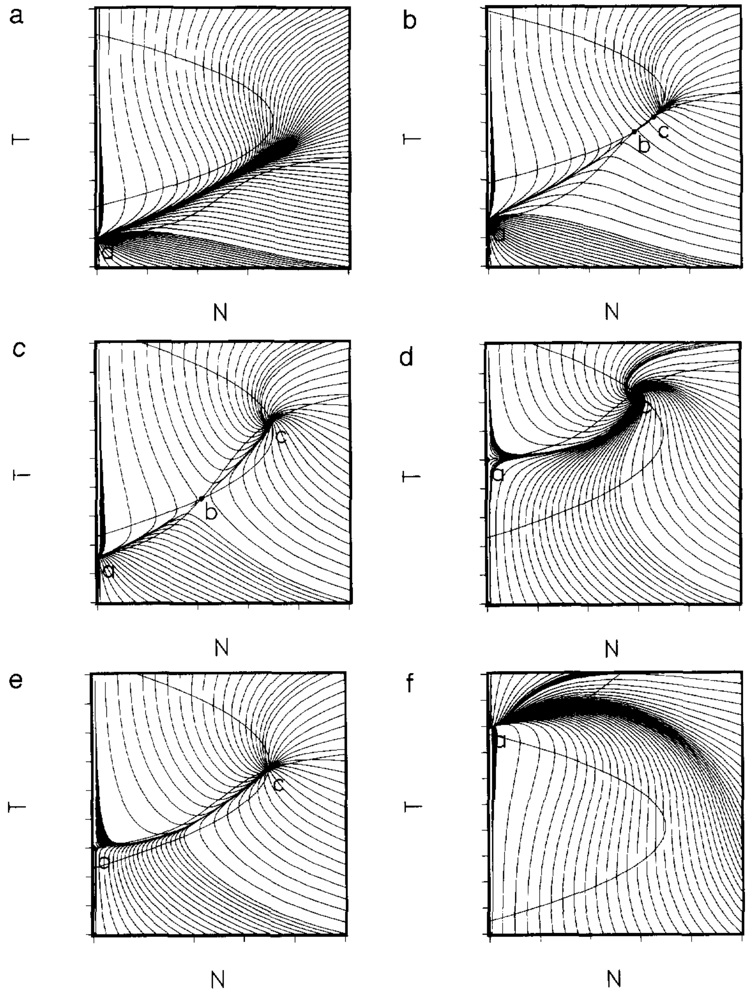

Fig. 5. (a) Phase portrait of system 4 corresponding to Fig. 4a with $T_a^* < T_{\min}$ ("cold desert"). (b) Phase portrait of system 4 with $T_c^* < T_{\text{opt}}$. (c) Phase portrait of system 4 with $T_c^* < T_{\text{opt}}$. (d,e) Phase portrait of system 4 with $T_a^* > T_{\min}$. (f) Phase portrait of system 4 with $T_a^* > T_{\max}$ ("hot desert").

In Fig. 5a the point a is a stable node, the final state of this planet (for any initial conditions) is the "cold desert" ($N^* = 0$, $T^* < T_{\min}$, i.e. no vegetation can occur). It exists always if

$$T_{\min} > (\Psi_1/\sigma)^{1/4}. \qquad (11)$$

If $T_{\min} > (\Psi_1/\sigma)^{1/4}$ as before, but $T_c^* < T_{\text{opt}}$, we have the following picture (Fig. 5b). In this case the point a is a stable node, the point b is a saddle and the point c is a stable node, since $\alpha'_T(T_c^*) > 0$.

It is interesting that there are two stable final states (point a and c), and the singular trajectory of saddle point b divides the quadrant $N > 0$, $T > 0$ into two domains of attractivity. The initial condition of temperature and amount of vegetation determines whether vegetation can exist or not.

Let $T_c^* > T_{\text{opt}}$ (Fig. 5c), and, in addition, the inequality 10 is valid (note if $T_c^* = T_{\text{opt}}$, this inequality is not valid), then the point c is a stable focus. As before, the point a is a stable node, the point b is a saddle. If

$T_a^* > T_{\min}$ the phase picture (Fig. 5d,e) changes: the point a becomes a saddle point, the point c is, as before, either a stable node or a stable focus. In Fig. 5f the point a is a stable node that corresponds to the "hot desert" ($N_a^* = 0$, $T_a^* > T_{\max}$).

4. Do dissipative structures exist? Diffusive instability

The local analysis of singular points and their stability can not answer us the question: do dissipative structures exist here, i.e. do other solutions, except constants, exist in this problem? In order to get the answer, we have to test them in relation to the diffusive instability (see Levin, 1976, and Svirezhev, 1978).

The validity of the following inequality (at the corresponding equilibria determined in the previous section) is a necessary and sufficient condition for the diffusive instability:

$$D_T \Phi'_N + D_N F'_T > 2\sqrt{D_T D_N (F'_T \Phi'_N - F'_N \Phi'_T)}\,, \tag{12}$$

where $\Phi = \alpha(T)N - \gamma N^2$; $F = \Psi(N) - \sigma T^4$. From Eq. 12 we have

$$(\alpha(T^*) - 2\gamma N^*) D_T - 4\sigma T^{*3} D_N > 2\sqrt{D_T D_N (F'_T \Phi'_N - F'_N \Phi'_T)}\,, \tag{13}$$

where $F'_T = -4\sigma T^{*3}$; $\Phi''_N = \alpha(T^*) - 2\gamma N^*$; $F'_N = \Psi'_N(N^*)$; $\Phi'_T = \alpha'_T(T^*)N^*$.

If $N^* = 0$, then from Eq. 13 we get

$$\alpha(T_0^*) D_T - 4\sigma T_0^{*3} D_N > 2\sqrt{D_T D_N \left(-4\sigma T_0^{*3} \cdot \alpha(T_0^*)\right)}\,. \tag{14}$$

Obviously that $\alpha(T_0^*)$ must be negative, i.e. this equilibrium is a stable node, but in this case the inequality is not valid for all values of T_0^*, D_T and D_N.

If the equilibrium is non-trivial, i.e. $N^* > 0$, then

$$-\gamma N^* D_T - 4\sigma T^{*3} D_N > 2\sqrt{D_T D_N N^* \left(4\gamma\sigma T^{*3} - \alpha'_T \Psi'_N\right)} \tag{15}$$

We can see in this case that the diffusive instability does not exist either.

And finally we can say that in this problem there are not the spatially non-uniform solutions (differing out of constant) like dissipative structures.

5. The waves: propagation of perturbations

"Revenons à nos moutons", i.e. let us consider the original problem again: there is the system of two nonlinear parabolic equations:

$$\begin{aligned} \frac{\partial T}{\partial t} &= D_T \Delta T + \Psi(N) - \sigma T^4 \\ \frac{\partial N}{\partial t} &= D_N \Delta N + \alpha(T) N - \gamma N^2 \end{aligned} \tag{16}$$

Since inhomogeneous equilibrium solutions do not exist, we can restrict to wave solutions of Eq. 16, i.e. solutions in the form

$$\begin{aligned} T(x,t) &= T(x + vt) \\ N(x,t) &= N(x + vt), \end{aligned} \tag{17}$$

where $v = \text{const}$ is the velocity of wave, to study the propagation of perturbations and, secondly, to special class of initial conditions, which will generate these waves: $T(x,0) = T_0(x)$ and $N(x,0) = N_0(x)$ must be finite functions, i.e. they differ out of zero only on a finite interval.

If we substitute Eq. 17 into Eq. 16, then we get (let $\xi = x + vt$):

$$\begin{aligned} vT' &= D_T T'' + \Psi(N) - \sigma T^4, \\ vN' &= D_N N'' + \alpha(T) N - \gamma N^2. \end{aligned} \tag{18}$$

This 2-dimensional system of second-order ordinary differential equations can be transformed into a 4-dimensional first-order system if we add two phase variables p, q:

$$\begin{aligned} \frac{dT}{d\xi} &= p, \\ \frac{dp}{d\xi} &= \frac{v}{D_T} p + \frac{\sigma T^4}{D_T} - \frac{\Psi(N)}{D_T}, \\ \frac{dN}{d\xi} &= q, \\ \frac{dq}{d\xi} &= \frac{v}{D_N} q - \frac{\alpha(T) N}{D_N} + \frac{\gamma N^2}{D_N} \end{aligned} \tag{19}$$

At the equilibria of Eq. 19 p and q must be zero. Then the equilibria of Eq. 4 are also equilibria of the two remaining equations of Eq. 19. But mention that the system dimensions are distinct.

In the remaining part of our paper we will focus on the development of such propagating waves under different initial conditions of T and N. It is necessary to solve Eq. 19 explicitly, which cannot be done analytically due to the nonlinearity of the system. Therefore further analysis of the system is carried out with numerical methods.

The original system 16 was solved for different initial conditions of the form:

$$\begin{aligned} T(x,0) &= T_0 = (\Psi(0)/\sigma)^{1/4} \\ N(x,0) &= R_w(x) = \begin{cases} 1 & \text{if } |x| \le w/2 \\ 0 & \text{if } |x| > w/2 \end{cases} \end{aligned} \tag{20}$$

$N(x,0)$ is defined on a finite carrier $[-w/2, w/2]$ and could be member of the class of initial perturbations generating nonlinear waves.

From now on $N(x,t)$, $T(x,t)$ are defined on a finite world $\Omega = [-L, L]$. Therefore the behaviour of the model must be defined at the boundaries $x = -L, L$. We choose periodic boundary conditions

$$\begin{aligned} N(-L,t) &= N(L,t) \\ T(-L,t) &= T(L,t) \end{aligned} \tag{21}$$

defining a ring topology on Ω.

The numerical calculations were done with an adaptive Runge–Kutta scheme after transforming the partial differential equations by a finite difference method into a system of ordinary ones. For a short description of the applied algorithms see, e.g., Press et al. (1988). These calculations were repeated for a set of parameters w and T_0, i.e. different insolation $\hat{S}$, on the domain Ω and the following was observed:

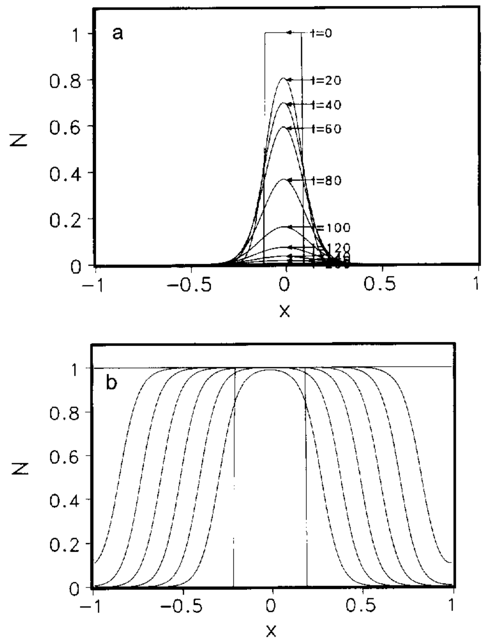

Fig. 6. Evolution in time of the vegetation $N(x,t)$ for a rectangular perturbation with (a) $w < w_{crit}$, (b) $w > w_{crit}$. The labels at the curves in (a) denote the different times t. The equidistant curves in (b) indicate a constant propagation velocity of the nonlinear waves.

If $T_0 < T_{min}$, the evolution in time depends on the parameter w (the width of the rectangular initial condition for N as defined in Eq. 20) (see Fig. 6a,b). If w is below a critical value w_{crit}, then the solution $N(x,t)$ vanishes in time, i.e.:

$$\lim_{t\to\infty} N(x,t) = 0 \quad \text{if } w < w_{crit} \tag{22}$$

For $w > w_{crit}$, however, a propagation of the initial perturbation with a constant velocity v can be observed (see Fig. 6b).

An explanation of this behaviour can be found if we compare the system with the equivalent system of the uniform biosphere: at the chosen T_0 two equilibria are stable with $N^* = 0$ and $N^* > 0$ (see Fig. 5b). It depends on the initial value of N which of the two equilibria are reached. The diffusive system exhibits a similar behaviour because the final state depends on the width of the initial perturbation.

As a first guess, the total amount of vegetation N, described by

$$N_{total} = \int_{\Omega} N(x,0)\mathrm{d}x \tag{23}$$

seems to be an appropriate criterion for the development of propagating waves. However, this is not valid, the geometrical arrangement must also be taken into account. To prove this a composition of two rectangular perturbations ($w < w_{crit}$, $2w > w_{crit}$) separated by a distance δ was used as an initial configuration:

$$\begin{aligned} N(x,0) &= R_w\left(x - \frac{\delta}{2}\right) + R_w\left(x + \frac{\delta}{2}\right), \quad w < w_{crit} \\ N_{total} &= \int_{-L}^{L} N(x,0)\mathrm{d}x = 2w > w_{crit} \end{aligned} \tag{24}$$

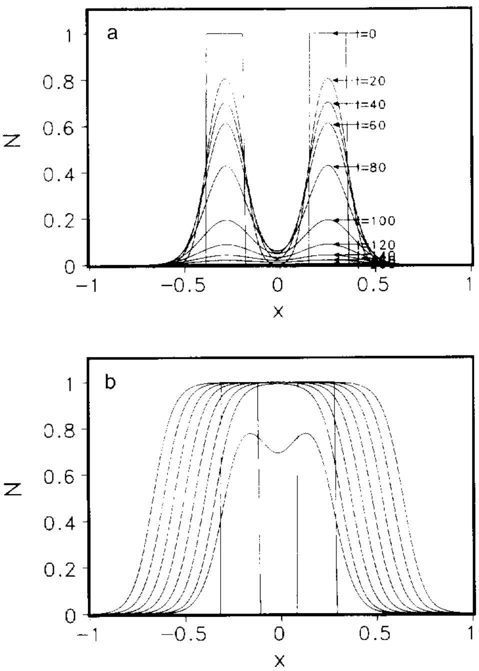

Fig. 7. Development of the initial perturbation according to Eq. 24 for (a) large δ and (b) small δ.

Note that each of the rectangular perturbations R_w alone ($\delta \to \infty$) goes to zero for $T_0 < T_{\min}$. For $\delta = w$, however, we have

$$N(x,0) = R_w(x - w/2) + R_w(x + w/2) = R_{2w}(x), \qquad (25)$$

which propagates in time with nonlinear waves. A development of propagating waves occurs even for $\delta > w$ (Fig. 7b), while for large δ the perturbation vanishes in time (Fig. 7a).

6. Conclusion

It was shown that the proposed vegetation–climate model has a topological simple phase plane, i.e. all trajectories end up in equilibrium points. Analysis of the stability of the equilibria indicates an existence of two stable nodes for a certain interval of $\hat{S}$. Depending on the initial point in the phase plane $\{N,T\}$ it will either reach the point with $N^* = 0$ or $N^* > 0$. These two equilibria coincide with the occurrence of vegetation in our climate–vegetation model.

Diffusive instabilities do no exist either. Spatial perturbations of the system, however, lead to the development of propagating nonlinear waves. Their occurrence depends in a non-trivial way on the initial geometrical configuration. We can say that the spatial form of an ecosystem leads either to an extinction or a growth by developing "travelling waves".

In the next step the numerical simulations will be extended to spatial two-dimensional structures in order to see the evolution of spatio-temporal patterns. A second goal is the incorporation of a carbon cycle into this fairly simple model. Such an extension will allow us to have a more complex structure of the phase plane.

References

Arnol'd, V.I., 1973. Ordinary Differential Equations. MIT Press, Cambridge, MA.

Budyko, M.I., 1969. The effect of solar radiation variations on the climate of the earth. Tellus, 21: 611–619.

Hale, H. and Kocak, H., 1991. Dynamics and Bifurcation. Springer Verlag, Berlin, 373 pp.

Henderson-Sellers, A. and McGuffie, K., 1990. A Climate Modelling Primer. John Wiley & Sons, Chichester.

Jordan, D.W. and Smith, P., 1977. Introduction to Nonlinear Ordinary Differential Equations. Clarendon, Oxford.

Levin, S.A., 1976. Population dynamics in heterogeneous environments. Annu. Rev. Ecol. Syst., 7: 287–310.

Malthus, T.B., 1798. First Essay on Population. Reprinted 1926, Macmillan, London.

McCauley, J.L., 1993. Chaos, Dynamics and Fractals. Cambridge University Press, Cambridge.

Moiseev, N.N. and Svirezhev, Y.M., 1979. Systems analysis of the biosphere dynamics: minimal model of the biosphere. USSR Acad. Sci. Herald, 2: 47–58.

Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T., 1988. Numerical Recipes in C. Cambridge University Press, Cambridge.

Schellnhuber, H.J. and von Bloh, W., 1993. Homöostase und Katastrophe. In: H.J. Schellnhuber and H., Sterr (Editors), Klimaänderung und Küste. Springer Verlag, Berlin, pp. 11–27.

Svirezhev, Y.M., 1978. Nonlinear Waves, Dissipative Patterns and Catastrophes in Ecology. Nauka, Moscow (in Russian).

Vernadsky, V.I., 1926. The Biosphere. Nauchtechizdat, Leningrad.

Watson, A.J. and Lovelock, J.E., 1983. Biological homeostasis of the global environment: the parable of Daisyworld. Tellus, 35B: 284–289.