Model Ensembles for Natural Resource Management: Extensions of Qualitative Differential Equations Using Graph Theory and Viability Theory

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For Gerhard

Preface

Anecdote Concerning the Lowering of Productivity

A fisherman is dozing in the sun in his rowing boat. A tourist asks: "The weather is great and there's plenty of fish, so why are you lying around instead of catching more?"
The fisherman replies: "Because I caught enough this morning."
"But just imagine," the tourist says, "you could go out three or four times a day and bring home three or four times as much fish! And then you know what could happen?" – The fisherman shakes his head.
"After a year you could buy yourself a motorboat, and after two years a second one. One day you might be able to build a freezing plant." – "And then?" asks the fisherman.
"Then you could spend your time at the harbour, dozing in the sun and looking at the beautiful ocean."
"But that is exactly what I am already doing," says the fisherman. (based on Böll 1986)

This work is motivated by concerns about the way our society interacts with its natural environment. The story about the fisherman illustrates how social processes shape the state of natural resources. I'm convinced that mathematical methods can help designing sustainable futures – if they are used with a respectful attitude towards other domains of knowledge. I hope my work contributes to this task.

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Klaus Eisenack
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Contents

Notations and Concepts

$\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-$	real numbers, non-negative and non-positive real numbers
$\mathbb{N}$	natural numbers (including 0)
$X \subseteq \mathbb{R}^n$	state space
$C(X,Y), C^1(X,Y)$	space of continuous and continuously differentiable functions from X to Y
$x(\cdot), (x_i)$	trajectory or solution on $\mathbb{R}_+$, sequence
$\dot{x}(\cdot) = \frac{d}{dt}x(\cdot)$	derivative with respect to time
$D_x f = \frac{\partial f}{\partial x}, D_j f_i = \frac{\partial f_i}{\partial x_j}$	partial derivatives
$\mathcal{J}(f), \mathcal{P}(X)$	Jacobian of a function f, power set of a set X
D^c	complement of a set D
$\langle x_1, x_2, \ldots, x_n \rangle$	ordered set
$(x_1\, x_2\, \ldots\, x_n)^t$	transpose of a vector

Model Ensembles

$\mathcal{M}$	model ensemble
$\mathcal{E}$	space of admissible trajectories
$\mathcal{M}(\Sigma)$	monotonic ensemble (DEF. 1, p. 22)
$\mathcal{M}(\mu, C)$	monotonic landmark ensemble (DEF. 4, p. 29)
$\mathcal{S}_{\mathcal{M}}(\cdot)$	set-valued solution operator

Qualitative Differential Equations

$\mathcal{A}$, $\mathcal{A}_*$	domain of signs and of extended signs $[-], 0, [+], [?]$
$\sigma, \sigma_i, \sigma_{i,j}$	(extended) sign
$[\cdot], [\cdot]_\lambda$	sign operators, extended to vectors and matrices (p. 22, 28)
$\approx$	consistency of signs, sign vectors or sign matrices (p. 22)
Σ	matrix of extended signs
v, w	sign vectors, qualitative states or vertices of a state-transition-graph
$\tilde{x}$	(landmark) abstraction of a reasonable function $x(\cdot)$ (DEF. 2, p. 23, DEF. 5, p. 29)
$\mathcal{S}_{\mathcal{M}(\Sigma)}(\cdot)$	solution operator of a monotonic ensemble (p. 22)
λ, Λ	landmark, landmark vector (p. 28)
Q, S	quantity space, qualitative state space (p. 28)
$\mathrm{qmag}(v), \mathrm{qdir}(v), \mathrm{qval}_i(v)$	qualitative magnitude, qualitative direction, qualitative value of a qualitative state v (Eq. 2.8, Eq. 2.10, Eq. 2.6, p. 28)
$\mathcal{S}_{\mathcal{M}(\mu,C)}(\cdot)$	solution operator of a monotonic landmark ensemble (p. 29)
μ	map from a quantity space $Q \to \mathcal{A}_*^{n\times n}$
$C = \{C_1, \ldots, C_m\}$	set of constraints on a qualitative state space
$a_\Lambda(x, \dot{x})$	state abstraction with respect to a landmark vector Λ (p. 28)
$Z_0(v)$	vanishing indices of a sign vector v (p. 24)
$v \wedge w$	intermediate state of qualitative states v, w (p. 25, 30)
$\pi_I(G)$	simple projection of a state-transition graph G with respect to an index set I (DEF. 11, p. 39)

Differential Inclusions and Viability Theory

$F : X \rightsquigarrow Y$	set-valued map from X to the power set $\mathcal{P}(Y)$ (p. 42)
$\mathrm{Dom}(F), \mathrm{Graph}(F)$	domain and graph of a set-valued map F (p. 42)
$\mathcal{S}_F(\cdot)$	solution operator of a differential inclusion given by a set-valued map F (p. 43)
$\mathrm{Viab}_F(K, C)$	viability kernel of K with target C (under a set-valued map F, DEF. 13, p. 46)
$\mathrm{Inv}_F(K, C), \mathrm{Abs}_F(K, C)$, $\mathrm{Capt}_F(K, C)$	invariance kernel, absorption basin and capture basin of K with target C (DEF. 13, p. 46)

Graph Theory

G	directed graph
$V(G), E(G)$	set of vertices and edges of a graph G
v, w and e, f	vertices and edges
$v_1, \dots, v_m$	path in a graph
$\Gamma(v), \Gamma^*(v)$	successors of vertex v in G and in its transitive closure G^*
G^*, G^{-1}	transitive closure, reversal of a graph G
G^-	bi-directed subgraph of G (p. 38)

For a directed graph G with **vertices** $V(G)$, $E(G) \subseteq V(G) \times V(G)$ is the set of **edges**. Since we deal only with undirected graphs, we simply call them graphs. For an edge $e = (v, w)$, v is called the **source** of e, and w its **target**. The set of **successors** of a state $v \in V(G)$ is denoted by $\Gamma(v) := \{w \in V(G) \mid (v, w) \in E(G)\}$. The number $|\Gamma(v)|$ is the **degree** of the vertex v. A graph G is **loop free** if $\forall v \in V(G) : (v, v) \notin E(G)$. A finite sequence of vertices in $V(G)$, written $v_1, \dots, v_m$, is called a **path** of length $(m - 1)$ if $\forall i = 1, \dots, m - 1 : v_{i+1} \in \Gamma(v_i)$, and no state occurs more than once in the sequence. The only exception to the last restriction is if $v_m = v_1$, when the path is called a cycle. The restriction to finite sequences is sufficient for this thesis. The **transitive closure** G^* of G is defined by $V(G^*) := V(G)$ and

$$E(G^*) := \{(v, w) \in V(G) \times V(G) \mid \text{ there is a path } v, \dots, w \text{ in } G\}.$$

A graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is a **subgraph** of the **supergraph** G. A graph H is induced by a subgraph H' of G if $V(H) = V(H')$ and

$$E(H) = \{(v, w) \in E(G) \mid v, w \in V(H')\}.$$

A subgraph H of G is **strongly connected**, if for all $v, w \in V(H)$, $v \neq w$, there is a path $v, \dots, w$ and a path $w, \dots, v$. It is called a **strongly connected component** if it is a maximal strongly connected subgraph with respect to inclusion of vertices. For two graphs G, H, the **intersection** $G \cap H$ is defined by $V(G \cap H) = V(G) \cap V(H)$, $E(G \cap H) = E(G) \cap E(H)$ and the **union** $G \cup H$ by $V(G \cup H) = V(G) \cup V(H)$, $E(G \cup H) = E(G) \cup E(H)$. For an edge $(v, w) \in E(G)$, its reversal is $(w, v) \in V(G) \times V(G)$. The **reversal** G^{-1} of G is obtained by reversing all edges, i.e. $V(G^{-1}) := V(G)$,

$$E(G^{-1}) := \{(v, w) \in V(G) \times V(G) \mid (w, v) \in E(G)\}.$$

An edge $e \in E(G)$ is **bi-directed** if $E(G)$ also contains its reversal.

Chapter 1

Introduction

This thesis is devoted to the study of infinite ensembles of ordinary differential equations with common monotonicity properties. Such ensembles arise in the research about sustainable development, which aims to meet current human needs while maintaining the environment and natural resources for future generations (WCED 1987). In the domain of sustainability science, uncertainties about dynamic social-ecological systems pose major challenges, and typologies of such systems are an important research field.

I present qualitative differential equations (QDEs) and differential inclusions as important methods for this task and introduce the general framework of *model ensembles* which includes both methods as special cases. They are complemented with concepts from viability theory to combine reasoning about dynamics with the evaluation of possible development paths. Due to the general nature of infinite ensembles, they do not produce unique trajectories but a large set of solutions. I develop several new techniques to evaluate and simplify such solutions sets in this thesis. The strengths of the new methods, as well as the strengths of QDEs in general, are demonstrated with several applications from natural resource management.

There are many instances where natural resources degrade even if they are managed in some way. There is an urgent need to assess management regimes and design sustainable options, as an important goal of sustainability science which "seeks to understand the fundamental character of interactions between nature and society. Such an understanding must encompass the interaction of global processes with the ecological and social characteristics of particular places and sectors" (Kates et al. 2001).

Syndrome Research and Management of Renewable Resources

The methods I develop and present in this thesis are motivated by and applied to problems from natural resource overuse by agriculture, fishery and water management.

Soils and water are crucial for agriculture. However, approximately 15% of the terrestrial surface is covered by soils which are degraded by human activities. These results of intensive industrial agriculture or overuse of marginal land – probably under the additional pressure of climate change – are most visible in the poor regions of the world (WBGU 1994). Management tries to change land-use practices and the forces driving overuse, for example by development programs.

Marine fish stocks are under extreme pressure worldwide. Approximately 50% of the world's fish resources are fully exploited, 20% overexploited, and 10% depleted (FAO 2004). Even though over-fishing has been a fact since historical times (Jackson et al. 2001), the problem has gained a new quality due to the industrialisation of commercial fishing, which has reduced community biomass by 80% within 15 years of exploitation by highly capitalised fishing fleets (Myers and Worm 2003). Common management approaches include catch quotas and effort limitations.

Fresh water is essential for basic human needs, health, and food production. It is also an indispensable element to maintain biodiversity and ecosystems, which, in turn, provide fresh water. Human withdrawal of water is mainly for agriculture (69%), but also for industrial uses (23%) and households (8%) (WBGU 1997). Management efforts are centred around the task of sustaining water quantity and quality, the latter being, among other influences, threatened by eutrophication.

Patterns of non-sustainable use emerge from interactions between society and ecosystems. Although they may be coupled via global processes, they occur mostly on a regional or local scale. However, they can be observed at many places on Earth in a more or less similar way. This qualifies them as so called *syndromes of global environmental change* (Schellnhuber et al. 1997). The syndrome approach seeks to analyse the interactions of the most relevant processes of global change while preserving local and regional peculiarities by considering case studies. To provide a global view of local and regional dynamics of environmental degradation, it aims at the identification of typical functional patterns of social-ecological dynamics (Lüdeke et al. 2004). This poses various research challenges. A typology of patterns requires an adequate notion of similarity between instances of problems, and strategies to mitigate or prevent syndromes have to be developed (Petschel-Held et al. 1999).

Methodological Challenges

Causal loop diagrams and mathematical models are important tools to classify and understand syndromes (Schellnhuber et al. 2002). Once a set of variables which describes interconnected relevant processes in a particular case study is chosen, these diagrams are a well-established tool to formulate their interactions (Petschel-Held et al. 1999; Stave 2002). In its simplest form, a causal loop diagram is a directed graph with marked edges. Each vertex represents a variable, and each edge an influence of the source variable on the target variable which can be marked as positive or as negative (for an example, see Fig. 1.1). The meaning of the edges is vague at this stage, but needs to be made explicit for modelling purposes (see below). In traditional systems dynamics modelling (Forrester 1968; Sterman 2000), the causal loop diagram is a starting point to develop a quantitative model, usually in the form of an ordinary differential equation (ODE). In contrast, within the context of syndrome research we do not "refine" the diagram to that point, so that it still subsumes a broad range of systems which share only the structure expressed by the diagram. It is an "archetype" for a generalised pattern of global environmental change.

In addition to finding generalised patterns, modelling in the domain of syndrome research and sustainability science has to face various challenges, including the following (Eisenack

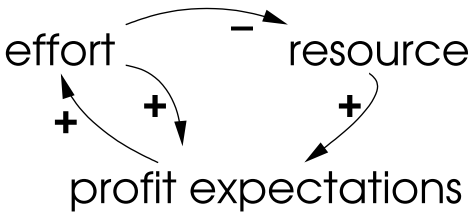

Figure 1.1: Example for a causal loop diagram: simplified version of the Overexploitation Syndrome (Cassel-Gintz and Petschel-Held 2000; Kropp, Eisenack, and Scheffran 2006).

et al. 2002):

- Generality: models should provide insights for single applications, but should also apply to a broader set of cases with general features in common. Such communalities between different cases are important to obtain a global overview, to classify different instances, and are the base for transferring best practices.
- Uncertainty: many interactions of social-ecological systems are not known quantitatively, knowledge about the processes is often limited, there are data gaps or unpredictable future influences, e.g. depending on strategic political choice. Under such conditions, the modeller cannot discriminate between reasonable alternatives, but if urgency to solve a problem is high, the analysis and the management strategies should be robust.
- Complexity: social-ecological systems tend to be composed of many tightly coupled, non-linear subsystems and interactions which are often difficult to disentangle.
- Normativity: it is necessary to classify patterns of interactions as problematic or desirable, e.g. to distinguish syndromes from other functional patterns. This involves value judgements which cannot be made by science. However, normative knowledge has to be considered in a transparent way and research can contribute to the assessment and development of management practices.
- Non-quantitative knowledge: to understand social-ecological systems, knowledge from different disciplines and with different degrees of quantification has to be integrated.

Qualitative Reasoning with Model Ensembles

By considering not a single model but a whole ensemble of models, some degree of generality can be obtained and a variety of possible system configurations which we can think of under uncertainty can be covered. In contrast to so called ensemble runs, where solutions for a finite set of parameterisations of a model are computed to determine, e.g. sensitivity to parameters (e.g. Stainforth et al. 2005), we will deal with infinite ensembles. I introduce the concept of a model ensemble as a (possibly infinite) set $\mathcal{M}$ of functions, where each $f \in \mathcal{M}$ constitutes an ordinary differential equation $\dot{x} = f(x, t)$. It is clear that such an approach does not yield unique solutions in general. The task is to describe $\mathcal{M}$ in a concise way which only considers knowledge which is certain to a high degree and which allows for an aggregated computation of the resulting (infinite) solution set.

This systematic framework can be used to define differential inclusions, QDEs and traditional methods on a common base. Differential inclusions represent an important approach to account for generality and uncertainty (Aubin and Cellina 1984), since contingent dynamics can be computed even if no probabilistic knowledge is available. In contrast, qualitative differential equations (Kuipers 1994) are deterministic but subsume a broad set of possible configurations. For the assessment of large solution sets, viability theory provides a powerful way to consider normative issues in a formal way by investigating whether solutions satisfy prescribed state constraints (Aubin 1991). Thus, some of the new methods to deal with model ensembles I develop in this thesis draw on viability theory.

A causal loop diagram is a third way to define a model ensemble. I will present a precise definition which specifies a criterion for an ODE to be consistent with a causal loop diagram. Since the diagram only contains qualitative information, there is an infinite number of such ODEs for a given diagram. Hence, $\mathcal{M}$ is defined to cover every consistent ODE, because *all* of them should be considered as possible realisations.

We will see that QDEs provide all solutions of such a model ensemble. In the graph theoretical terminology I propose, they allow for the computation of a state-transition graph which covers all trajectories consistent with a causal loop diagram. More complex model ensembles can also be treated. In many cases relevant conclusions can be drawn from a state-transition graph. However, due to the generality of the diagram, common qualitative reasoning has its limitations because we usually obtain a very large state-transition graph. Dealing with this problem is the main content of this thesis. I pursue two general strategies:

- Automated detection of structures in large state-transition graphs which are relevant for sustainability issues.
- Simplification of state-transition graphs by posing additional model assumptions without restricting the model ensemble to a singular ODE.

Outline

The thesis is motivated by the following hypotheses:

- Current qualitative reasoning methods can be substantially improved for applications.
- Model ensembles are the common root of methods operating under uncertainty and generality, subsuming QDEs, differential inclusions and causal loop diagrams. They meet central challenges of sustainability science.
- Even under uncertainty and generality, robust properties of social-ecological systems can be found. This contributes to the management of natural resources.
- QDEs can be used to design viable control strategies.

In Chapter 2 I begin with the formalisation of the idea of model ensembles. Using this concept, QDEs are introduced and partially re-formulated using graph theory. An overview is given of common strategies to handle large state-transition graphs from a new and systematic perspective which draws on model ensembles and the graph theoretical formulation. This perspective is also used in later chapters. Finally, basic results from differential inclusions and viability theory are summarised. In Chapter 3 I present new techniques to analyse and simplify state-transition graphs of large QDEs by introducing new types of formal assumptions which are motivated by problems from sustainability science. The first technique identifies normative relevant subgraphs by transferring concepts from viability theory to graph theory. The second eliminates edges of limited importance from the state-transition graph. The third refines model ensembles by making assumptions on the order of the coefficients of the Jacobian of a system. Finally, I combine QDEs, differential inclusions and viability theory to investigate how knowledge about intervals which enclose partial derivatives yields stronger results. All these methods are applied in Chapter 4 to problems of natural resource management. At first, interventions which may stop the so called impoverishment-degradation spiral which can be observed in developing countries are presented. Then, I investigate several problems of marine fisheries: the problematic capital dynamics in the unregulated case, and conditions under which a participatory process to determine catch restrictions is effective. Finally, options for a lake manager to prevent eutrophication are compared. Chapter 5 concludes the thesis with a summary of the scope and limits of the methods developed in the first chapters with respect to the experience from Chapter 4.

Chapter 2

Qualitative Reasoning with Model Ensembles

This chapter introduces qualitative differential equations (QDEs) from a new perspective and recalls basic results from differential inclusions and viability theory. In the first section, I develop a novel formal framework which includes QDEs, differential inclusions, causal loop diagrams and other modelling approaches common in sustainability science as special cases. These approaches are highly relevant to account for challenges of uncertainty and generality (cf. Chapter 1). Called framework of model ensembles, common methods like ensemble runs in climate change research are generalised. More importantly, the advantages of QDEs as a modelling tool for syndrome research and natural resource management are clarified. The framework is a guiding principle for the whole thesis. In contrast to traditional approaches to QDEs, I explicitly use graph theoretical concepts to simplify definitions and proofs. Differential inclusions, recalled in the third section, are a generalisation of ordinary differential equations (ODEs). They take account of uncertainties in a way which is complementary to QDEs, and they are the basis for introducing viability theory in the fourth section. This inverse method is, in particular, well-suited to assess model ensembles and to design control strategies. It thus integrates issues of normativity transparently into the analysis. Together with the graph theoretical approach to QDEs, differential inclusions and viability theory will be fundamental for the new techniques I develop in Chapter 3.

2.1 Model Ensembles and Set-Valued Solution Operators

Rather than considering a single model, e.g. an ODE system, the basic idea of the framework of model ensembles I define in this thesis is to consider a whole set of models systematically. This is necessary to investigate real-world systems which cannot be formally described in a unique way due to uncertainty or generality. Although not formalised from the systematic perspective I develop here, such a style of reasoning is common (not only) in sustainability science, e.g. for parameter variation (e.g. Stainforth et al. 2005), model comparison (e.g. Gregory et al. 2005), scenario development (e.g. Nakićenović et al. 2000; Millennium Ecosystem Assessement 2005; Swart et al. 2004), and to characterise patterns of global environmental change (e.g. Schellnhuber et al. 2002). Also, causal loop diagrams as introduced in Chapter 1 can be seen as a general description of a broad range of different systems. The concise framework proposed in this thesis generalises these ideas.

A **model ensemble** $\mathcal{M}$ is defined as a set of functions $f : X \times \mathbb{R}_+ \to \mathbb{R}^n$ on a **state space** $X \subseteq \mathbb{R}^n$. These functions are called **models**, each describing a possible configuration of a real-world system under investigation or one example of the pattern to be analysed. The set $\mathcal{E}$ contains functions $x(\cdot) : \mathbb{R}_+ \to X$, being the space of **admissible trajectories** of the systems, e.g. $\mathcal{E} = C^1(\mathbb{R}_+, X)$. Each model $f \in \mathcal{M}$ defines a family of initial value problems

$$\begin{aligned} \dot{x} &= f(x,t), \\ x(0) &= x_0, \end{aligned}$$

with $x_0 \in X$. We call the set of all initial value problems given by a model ensemble the **systems of the model ensemble** $\mathcal{M}$. It is also possible to consider model ensembles which only contain autonomous models.

Of course, the systems of the model ensemble have (in general) different solutions. Thus, a *set* of trajectories must be assigned to each initial value x_0. The set-valued **solution operator** $\mathcal{S}_{\mathcal{M}}(\cdot) : X \to \mathcal{P}(\mathcal{E})$ (of a model ensemble $\mathcal{M}$ with respect to a state space X and admissible trajectories $\mathcal{E}$), assigning to an initial state a subset of $\mathcal{E}$, is defined by

$$\mathcal{S}_{\mathcal{M}}(x_0) := \{x(\cdot) \in \mathcal{E} \mid x(0) = x_0, \exists f \in \mathcal{M} \forall t \in \mathbb{R}_+ : \dot{x}(t) = f(x(t), t)\}.$$

Depending on $\mathcal{E}$ it may be sufficient that the ODE only holds almost everywhere. We call the elements of $\mathcal{S}_{\mathcal{M}}(X)$ the **solutions of the model ensemble** $\mathcal{M}$ (with respect to a state space X and admissible trajectories $\mathcal{E}$). If an application requires a very general model, $\mathcal{M}$ is the collection of all cases which have to be analysed. Similarly, in the case of uncertainties, $\mathcal{M}$ is defined to subsume all systems which must be considered. These are given by the part of the knowledge base which is certain to a high degree, while variation is admitted for uncertain parameters, functions or processes. The solution operator is closely related to the concept of an evolutionary system as defined by Aubin (2001). The main challenge in reasoning with model ensembles is to find relevant structure in $\mathcal{S}_{\mathcal{M}}(X)$. This includes

1. Representing a model ensemble in a way which is adequate to the modeller and allows for a formal treatment,

2. Efficient algorithms to determine $\mathcal{S}_{\mathcal{M}}(X)$ from a (possibly infinite) model ensemble $\mathcal{M}$,

3. Detecting structural features of the solutions of the model ensemble.

An example for the latter is to introduce normative settings in a formalised way by a partition $X = A \cup B$ of the state space into a preferable region A and a problematic region B. If for *all* $x(\cdot) \in \mathcal{S}_{\mathcal{M}}(x_0)$ and for all $t \geq 0$ the relation $x(t) \in B$ holds such that the system is "locked in" B due to its intrinsic dynamic interactions, it may be said that "a catastrophic outcome is unavoidable". Or if $\exists x(\cdot) \in \mathcal{S}_{\mathcal{M}}(x_0), t \geq 0 : x(t) \in A$, then "it is possible to sustain preferable conditions". Such features are very robust in that they hold for a whole model ensemble and not just one model. They are introduced more formally in section 2.4 (p. 45). We now provide some examples for model ensembles.

EXAMPLE 1: Let $\mathcal{M}$ contain only one function $f : X \times \mathbb{R}_+ \to \mathbb{R}^n$ which is Lipschitz on X, and let the admissible trajectories be $\mathcal{E} = C^1(\mathbb{R}_+, X)$. Then, $\mathcal{S}_\mathcal{M}(x_0)$ contains the usual solutions of the initial value problem with $x(0) = x_0$ which exist on $\mathbb{R}_+$. □

EXAMPLE 2: Given a function $f' : X \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n, (x, t; p) \mapsto f'(x, t; p)$, depending on a parameter vector p, and a finite set P of possible parameterisations, define the finite model ensemble

$$\mathcal{M} := \{f \in C(X \times \mathbb{R}_+, \mathbb{R}^n) \mid f(x, t) = f'(x, t; p), p \in P\}.$$

Then, the solution operator with respect to a set of admissible trajectories provides all "scenario runs" for the different parameterisations. □

EXAMPLE 3: For a given autonomous measurable function $f' : X \times U \to \mathbb{R}^n, (x, u) \mapsto f'(x, u)$, where $U \subseteq \mathbb{R}$ is a given interval of control values, we define the infinite model ensemble

$$\mathcal{M} := \{f : X \times \mathbb{R}_+ \to \mathbb{R}^n \text{ measurable } \mid f(x, t) = f'(x, u(t)), u(t) \in U\}.$$

Taking absolutely continuous functions as admissible trajectories, the solution operator $\mathcal{S}_\mathcal{M}(x_0)$ describes all trajectories starting from x_0 which result from any measurable open-loop control $u(\cdot) : \mathbb{R}_+ \to U$. □

EXAMPLE 4: Another example (which will be formalised in detail in section 2.2 and section 2.3, p. 42) are causal loop diagrams. I adopt a straightforward way to interpret a causal loop diagram (cf. Richardson 1986). Suppose that the diagram is given by a directed graph where edges are marked with signs. The vertices represent real variables, say $x_1, \ldots, x_n$. An ODE $\dot{x} = f(x)$ with state vector $x = (x_1 \ldots x_n)^t \in \mathbb{R}^n$ is said to be consistent with the diagram if the signs of the partial derivatives $\operatorname{sgn}(D_j f_i(x))$ correspond to the signs of the edges. If there is no edge between two variables, the partial derivative vanishes. For example, if there is a positive edge from profit expectations x_1 to effort x_2 (cf. Fig. 1.1, p. 13), then $D_1 f_2(x) > 0$, i.e. high profit expectations increase the change of effort. Since such conditions are fulfilled by multiple functions, this provides a collection of ODEs which are all consistent with the influence diagram, and thus a model ensemble $\mathcal{M}$. □

Within the framework of model ensembles, further questions can be posed which are highly relevant for sustainability science. If it is not possible find relevant features common to all solutions of a model ensemble $\mathcal{M}$ we can try to identify subsets $\mathcal{M}' \subseteq \mathcal{M}$ for which such robust properties can be determined. The characterisation of $\mathcal{M}'$ is associated with the discovery of structural features which bring about problematic or desirable system behaviour. In other terms, conditions under which certain (sub)pattern evolve are found. If $\mathcal{M}$ is partially determined by certain control measures imposed on the system, and $\mathcal{M}'$ by alternative control measures, the differences between the solution operators $\mathcal{S}_\mathcal{M}(X)$ and $\mathcal{S}_{\mathcal{M}'}(X)$ are of interest. This kind of reasoning is used intensively in Chapter 4. Moreover, there are cases where solutions of a model ensemble are artifacts from the assumptions the modeller made. Then it is important to restrict $\mathcal{M}$ so that the artifacts are eliminated. Very "unlikely" or "irrelevant" cases which cannot be refuted on base of the original model ensemble are further reasons to restrict $\mathcal{M}$ or even $\mathcal{E}$. Generally, a **restriction** means a restriction of the model ensemble to some $\mathcal{M}' \subseteq \mathcal{M}$, of the admissible trajectories to some $\mathcal{E}' \subseteq \mathcal{E}$, or of the state space to some $X' \subseteq X$. This is the organising principle for the new techniques developed in Chapter 3, where $\mathcal{E}$ and $\mathcal{M}$ are increasingly restricted step by step.

2.2 Qualitative Differential Equations

In this section I reformulate the concept of qualitative differential equations (QDEs) from a new graph theoretical perspective which differs from the algorithmic view in the original work of Kuipers (1994). This allows for a straightforward introduction, for some new propositions, and paves the road for the development of advanced techniques in Chapter 3. Moreover, I present QDEs as an example for model ensembles, making the introduction more systematic. This embeds the theory into the broader framework developed in section 2.1 and provides a common ground for the comparison to other methods, e.g. differential inclusions (cf. section 2.3, p. 42). The core idea of QDEs is to scan the state and velocity space of the systems of a model ensemble, i.e. a set of ordinary differential equations which share common monotonicity properties: we assume that the dynamics of the system are governed by an ODE $\dot{x} = f(x), f : \mathbb{R}^n \to \mathbb{R}^n$, where only the signs of the partial derivatives $D_i f_j, i, j = 1, \ldots, n$ are known. This approach, called qualitative reasoning, was developed in the 1980s in the field of artificial intelligence. It was made to resemble commonsense reasoning, to model with incomplete knowledge, and to increase the efficiency of automatic reasoning algorithms by abstracting from details which are not necessary to perform a given task. In the context of sustainability science QDEs are very valuable to analyse causal loop diagrams and to deal with uncertainty, generality and non-quantitative knowledge (cf. Chapter 1). As information about the signs of the partial derivatives is not sufficient to set up a unique ODE, we have to deal with a model ensemble and its set-valued solution operator. The latter contains a broad variety of solutions, which nevertheless share some common properties. These are derived by deducing all possible sequences of sign vectors $\operatorname{sgn}(\dot{x})$ just from the signs of the partial derivatives – without solving any system of the model ensemble. The full theory of QDEs permits restricting the model ensemble by requiring more than prescribed signs, e.g. algebraic relations on the state space, which will be introduced below. However, already the signs of the partial derivatives offer an interesting way to interpret a causal loop diagram.

The theory of QDEs is a prominent approach to qualitative reasoning. Other related approaches are confluences (de Kleer and Brown 1984) and qualitative process theory (Forbus 1984). Confluences are equations on the domain of signs. Thus, positive and negative relations between variables of a model can be formulated, and conclusions about their signs can be drawn. The challenge here – and also in the other approaches – are the ambiguities of sign algebra (Williams 1991). In a very simple example we do not know the sign of $x + y$ if $x > 0$ and $y < 0$. To close sign algebra, in addition to $\{[+], 0, [-]\}$, an element $[?]$, denoting an unspecified sign, has to be introduced. This gives rise to different interpretations of confluences, e.g. whether $x = y$ is valid for $x = [+]$ and $y = [?]$ or not.

Qualitative process theory (QPT) explicitly characterises dependencies of variables by four basic types: positive/negative influence of one variable x on the *change* $\dot{y}$ of another, and positive/negative relationships between variables x, y. Thus, dynamics come into the reasoning process: we can infer under which conditions a variable increases or decreases. In this area, much work was done on building models automatically from libraries of model fragments (Falkenhainer and Forbus 1991).

QDEs extend this approach by introducing so called landmarks, i.e. critical values of the state variables where signs of partial derivatives change (Kuipers 1984). Meanwhile, several other extensions have been made (Kuipers 1994; Kuipers 2001), some of which will be reviewed below in section 2.2.4 (p. 36). In parallel, a broad field of applications has emerged, e.g. in finance (Benaroch and Dhar 1995), epidemiology and genetics (Heidtke and Schulze-Kremer 1998; Trelease et al. 1999), chemistry (Juniora and Martin 2000; Syed et al. 2002), ecology (Guerrin and Dumas 2001; Bredeweg and Salles 2003), the automotive industry (Price and Snooke 1997; Sachenbacher 2001) and sustainability science (Petschel-Held et al. 1999; Petschel-Held and Lüdeke 2001; Eisenack and Kropp 2001; Kropp et al. 2002; Sietz et al. 2005; Eisenack et al. 2006).

In the domain of mathematics, the works of Dordan (1992, Dordan (1995), Aubin (1996), and Hüllermeyer (1997) introduce the concept of a monotonic cell, consisting of all states x such that $f(x)$ has a given sign vector. A trajectory can be described qualitatively by the sequence of monotonic cells it visits. By imposing additional restrictions on f, they investigate the issue of the existence of solutions of a QDE more seriously than in the literature from computer science. They also generalise the approach to other partitions of the state space than by signs, called qualitative frames. However, the approach is more restrictive in that only single ODEs are considered. This is interesting in itself, but not sufficient for our purposes where generality of models and uncertainties have to be taken into account.

In the first subsection the basic concepts of QDEs will be reformulated using graph theoretical concepts and the framework of model ensembles. Then, the basic concepts are extended by introducing landmarks. In the third subsection, the QSIM algorithm is outlined. Finally, I give an overview of advanced QDE techniques to tackle large solutions of QDEs.

2.2.1 Basic Qualitative Differential Equations

For the sake of simplicity I start with a version of QDEs which is closer to QPT. It omits algebraic constraints, e.g. on the signs of the right-hand sides of ODEs, and only considers influences on the change of variables; the velocity, but not the state space is investigated. This will be extended in the next subsection. There is one difference to the original work of Kuipers (1994): the focus is on qualitative states which persist for time *intervals* – but intermediate states occurring only for a point in time (so called time-point states) are not represented explicitly. This has the advantage that solutions of QDEs can be displayed in a much more accessible form (Eisenack and Petschel-Held 2002).

At first we specify the kind of model ensemble which constitutes a QDE (cf. section 2.1, p. 17). For this some elementary sign algebra is needed (Williams 1991). By $\mathcal{A} := \{[+], 0, [-]\}$ we denote the domain of signs, and by $\mathcal{A}_* := \{[+], 0, [-], [?]\}$ the domain of extended signs, where $[?] = \{[+], 0, [-]\}$ denotes an unknown sign. A tolerance relation, denoted here by $\approx$, is defined for $\sigma_1, \sigma_2 \in \mathcal{A}_*$ by

$$\sigma_1 \approx \sigma_2 \text{ iff } \sigma_1 = \sigma_2 \text{ or } \sigma_1 \in \sigma_2 \text{ or } \sigma_2 \in \sigma_1.$$

addition	[+]	0	[-]	[?]	multiplication	[+]	0	[-]	[?]
[+]	[+]	[+]	[?]	[?]	[+]	[+]	0	[-]	[?]
0	[+]	0	[-]	[?]	0	0	0	0	0
[-]	[?]	[-]	[-]	[?]	[-]	[-]	0	[+]	[?]
[?]	[?]	[?]	[?]	[?]	[?]	[?]	0	[?]	[?]

Table 2.1: Addition and multiplication on the domain of extended signs $\mathcal{A}_*$.

We say that σ_1 and σ_2 are consistent if $\sigma_1 \approx \sigma_2$. All signs are consistent with $[?]$. Addition $+$ and multiplication $\cdot$ of signs are defined in Tab. 2.1. For $\sigma_1, \sigma_2 \in \mathcal{A}_*$ and $0 \neq \sigma_3 \in \mathcal{A}$, rules such as the multiplicative cancellation law

$$\sigma_1 = \sigma_2 \cdot \sigma_3 \text{ iff } \sigma_1 \cdot \sigma_3 = \sigma_2 \tag{2.1}$$

are valid. Additionally, we will use tuples and matrices of (extended) signs ($\mathcal{A}^n$, $\mathcal{A}_*^{n\times n}$). The sign operator $[\cdot] := \operatorname{sgn}(\cdot)$ is extended component wise to vectors and matrices, as well as the relation $\approx$. Now a model ensemble can be defined:

DEFINITION 1: *For a given $n \times n$ matrix of signs $\Sigma = (\sigma_{i,j})_{i,j=1,\dots,n}$, $\sigma_{i,j} \in \mathcal{A}_*$, and a state space $X \subseteq \mathbb{R}^n$ we define the* ***monotonic ensemble***

$$\mathcal{M}(\Sigma, X) := \{f \in C^1(X, \mathbb{R}^n) \mid \forall x \in X : \big[\mathcal{J}(f)(x)\big] \approx \Sigma\},$$

where $\mathcal{J}(f)$ denotes the Jacobian of f. We call a function $x(\cdot) \in C^1([0,T], \mathbb{R}^n)$, possibly $T = \infty$, ***reasonable****, if there is only a finite set of points t with $\dot{x}(t) = 0$ on any bounded interval, and define the space of admissible trajectories $\mathcal{E}$ by all reasonable functions with values in X. We call the systems of the model ensemble $\mathcal{M}(\Sigma, X)$ a* ***QDE****.*

A monotonic ensemble $\mathcal{M}(\Sigma, X)$ is a model ensemble which only contains autonomous models. In most cases the state space of the model is clear from the context, and we simply write $\mathcal{M}(\Sigma)$. Although a set of ODE systems is not an equation we use this designation in analogy to Kuipers (1994). The reason for this is that a QDE can be "solved" by considering a constraint satisfaction problem, i.e. a relational equation over a finite set, which is explained in more detail in section 2.2.3 (p. 32).

Based on DEF. 1, a set-valued solution operator $\mathcal{S}_{\mathcal{M}(\Sigma)}(\cdot)$ is defined (with respect to the state space X and the space of reasonable functions as admissible trajectories $\mathcal{E}$). The set of solutions of the monotonic ensemble $\mathcal{S}_{\mathcal{M}(\Sigma)}(X)$ contains all reasonable solutions of all ODE systems contained in the QDE. It should be noted that the properties of the monotonic ensemble are not sufficient to guarantee a global solution for every $f \in \mathcal{M}(\Sigma)$.

As outlined in the previous section, we are interested in analysing the solution operator. Of course, solving all ODEs is not an option. Therefore, the basic idea of the theory of qualitative differential equations is to determine a certain discretisation of the solutions of the monotonic ensemble directly from Σ. Reasonable functions are chosen as admissible trajectories to allow for a discretisation which tracks the sign vectors $[\dot{x}(t)]$ for each solution. Let $x(\cdot)$ be reasonable. If the relation $\big[\dot{x}_i(t_1)\big] = -\big[\dot{x}_i(t_2)\big]$ holds for one $i \in \{1, \dots, n\}$ and some $t_1 < t_2, t_1, t_2 \in \mathbb{R}_+$, there is a point $t \in (t_1, t_2)$ with $\dot{x}_i(t) = 0$ due to continuity.

If there is one $\tau \in \mathbb{R}_+$ with $\dot{x}(\tau) \neq 0$, there is a maximal open interval J containing τ, such that $\big[\dot{x}(\cdot)\big]$ remains constant over J. Being reasonable guarantees that the set of times t where the sign of $\dot{x}(t)$ changes is discrete and countable. Therefore, the sign vectors of the velocity vector of a solution with increasing t can be written as a sequence.

DEFINITION 2: *For a given reasonable function $x(\cdot)$ on $[0,T]$ we have an ordered sequence of sign jump points (t_j) with $t_0 = 0$ which subsequently contains all boundary points of the closures of all sets $\{t \in [0,T] | [\dot{x}(t)] = v\}$ with $v \in \{[-],[+]\}^n$. We construct a sequence of sign vectors $\tilde{x} = (\tilde{x}_j) := \big(\big[\dot{x}(\tau_j)\big]\big)$, where we arbitrarily choose $\tau_j \in (t_j, t_{j+1})$. If the sequence (t_j) is finite with m elements, we choose $\tau_m \in (t_m, T)$. The sequence $\tilde{x}$ is called* ***abstraction*** *of $x(\cdot)$.*

Note that the abstraction $\tilde{x}$ does not depend the concrete values $\tau_j \in (t_j, t_{j+1}), j \in \mathbb{N}$, since the sign vector $\big[\dot{x}(t)\big]$ is constant on any interval (t_j, t_{j+1}). The sequence (t_j) is chosen from the *boundary points* to have maximal intervals with constant sign vector. They are taken from the *closure* of $\{t \in [0,T] | [\dot{x}(t)] = v\}$ to ignore saddle points where components of $\dot{x}$ vanish for one point in time, but afterwards attain the same sign as before. If there is a sign jump at t, there is a $i \in \{1,\ldots,n\}$ such that the following conditions hold:

$$\left.\begin{array}{rl} (i) & \dot{x}_i(t) = 0, \\ (ii) & \exists \epsilon : \forall t^- \in [t-\epsilon, t) : \ \dot{x}_i(t^-) \neq 0 \text{ or } \forall t^+ \in (t, t+\epsilon] : \ \dot{x}_i(t^+) \neq 0, \\ (iii) & \exists \epsilon : \forall t^- \in [t-\epsilon, t), \, t^+ \in (t, t+\epsilon] : [\dot{x}_i(t^-)] \neq [\dot{x}_i(t^+)]. \end{array}\right\} \tag{2.2}$$

The set of the abstractions of all solutions of a monotonic ensemble is (in general) still infinite. However, they are entailed by a finite graph in the following way:

DEFINITION 3: *Let $\mathcal{M}(\Sigma)$ be a monotonic ensemble, $\mathcal{E}$ the set of reasonable trajectories and $\mathcal{S}_{\mathcal{M}(\Sigma)}(\cdot)$ the corresponding solution operator. We denote the set of the abstractions of the solutions by*

$$\tilde{\mathcal{S}}_{\mathcal{M}(\Sigma)} := \{\tilde{x} \mid \exists x_0 \in X, x(\cdot) \in \mathcal{S}_{\mathcal{M}(\Sigma)}(x_0) : \tilde{x} \textit{ is the abstraction of } x(\cdot)\}.$$

Then, the directed ***state-transition graph*** *G of the monotonic ensemble is defined by the vertices*

$$V(G) := \{v \in \mathcal{A}^n \mid \exists\, \tilde{x} \in \tilde{\mathcal{S}}_{\mathcal{M}(\Sigma)}, j \in \mathbb{N} : \ \tilde{x}_j = v\},$$

called ***qualitative states****, and the edges*

$$E(G) := \{(v,w) \mid \ \exists\, \tilde{x} \in \tilde{\mathcal{S}}_{\mathcal{M}(\Sigma)}, j \in \mathbb{N} : \ \tilde{x}_j = v \textit{ and } \tilde{x}_{j+1} = w\},$$

called ***qualitative transitions****.*

For convenience, the state-transition graph of a monotonic ensemble is also called the state-transition graph of a QDE. Thus, we have defined a directed graph G such that all sequences of $\tilde{\mathcal{S}}_{\mathcal{M}(\Sigma)}$ describe a path in G, i.e. the graph “covers” all reasonable solutions of initial value problems

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x_0 \end{aligned} \tag{2.3}$$

with $f \in \mathcal{M}(\Sigma)$. Note that G is loop free, since subsequent coefficients of the abstraction of a reasonable function are different. The state-transition graph is the basic tool to investigate the structure of $\mathcal{S}_{\mathcal{M}(\Sigma)}(X)$. We are interested in the existence of vertices and edges of G which depends on Σ, and in further properties of G. The state-transition graph of a QDE can be efficiently computed by the so-called QSIM algorithm, which will be outlined in section 2.2.3 (p. 32). Some features of state-transition graphs can already be shown directly.

Which vertices occur in a state-transition graph? Most basically, $\{[-],[+]\}^n \subseteq V(G)$ due to the following reasons: by chain rule $\ddot{x} = \mathcal{J}(f)(x) \cdot \dot{x}$, such that for assumptions about the sign matrix $\left[\mathcal{J}(f)(x)\right]$ not all sign vectors $[\dot{x}]$ are consistent with all sign vectors $[\ddot{x}]$. However, since no claims about $[\ddot{x}]$ are made, no $[\dot{x}] \in \{[-],[+]\}^n$ can be excluded from being a vertex. The situation is more complicated if some $\dot{x}_i \equiv 0$ on (t_j, t_{j+1}), which implies that also $\ddot{x}_i \equiv 0$ on the same interval.

I now present a new necessary criterion for such a vertex to exist. For this, we need the set $Z_0(v) := \{i = 1, \dots, n | v_i = 0\}$, which assigns to a sign vector $v \in \mathcal{A}^n$ the indices of vanishing components.

PROPOSITION 1: *If* $v \in V(G)$*, then for all* $i \in Z_0(v)$

$$\exists j, k \notin Z_0(v), j \neq k : 0 \neq \sigma_{i,j} v_j \approx -\sigma_{i,k} v_k \neq 0$$

or $\forall j \notin Z_0(v) : \sigma_{i,j} = 0$.

PROOF: If $v \in V(G)$ there is an $f \in M(\Sigma)$ and $x_0 \in X$ such that for a solution $x(\cdot)$ to Eq. (2.3) there exists an interval (t_1, t_2) such that $\forall t \in (t_1, t_2) : [\dot{x}(t)] = v$. Thus,

$$\forall i \in Z_0(v) : 0 = \dot{x}_i(t) = \ddot{x}_i(t) = \sum_{j=1,\dots,n} D_j f_i \cdot \dot{x}_j. \tag{2.4}$$

If $\forall j \notin Z_0(v) : \sigma_{i,j} = 0$, then

$$\begin{aligned} \ddot{x}_i(t) &= \sum_{j \in Z_0(v)} D_j f_i \cdot \dot{x}_j + \sum_{j \notin Z_0(v)} D_j f_i \cdot \dot{x}_j \\ &= \sum_{j \in Z_0(v)} D_j f_i \cdot 0 + \sum_{j \notin Z_0(v)} 0 \cdot \dot{x}_j = 0. \end{aligned}$$

If this is not the case, there must be a $j \notin Z_0(v)$ such that $D_j f_i \cdot \dot{x}_j \neq 0$, i.e. $\sigma_{i,j} v_j \neq 0$. Then, Eq. (2.4) can only be true if there is also a $k \notin Z_0(v)$ with $[D_k f_i \cdot \dot{x}_k] = -[D_j f_i \cdot \dot{x}_j]$, i.e. $\sigma_{i,j} v_j \approx -\sigma_{i,k} v_k \neq 0$. □

Additionally, every state-transition graph contains the vertex 0, representing the equilibria of systems of the monotonic ensemble. A sufficient criterion for the existence of vertices is omitted here, since it will be argued in section 2.2.4 (p. 36) that vertices with vanishing components have only limited relevance for applications.

Now, I will prove a characterisation for the existence of edges in the state-transition graph G which cannot be found in the previous literature. It is simplified by considering only vertices with non-vanishing components. When two qualitative states v, w differ only in one component i, there must be a solution of the monotonic ensemble $x(\cdot)$, defined by a model f, which transgresses the main isocline $f_i(x) = 0$ at some time, because this isocline separates the regions of the phase space where $[f(x)] = v$ and $[f(x)] = w$, respectively (the so called monotonic cells). A necessary condition for such a transgression is an appropriate sign of $\ddot{x}_i$ on the main isocline, e.g. if $v_i = [-]$ and $w_i = [+]$, then $[\ddot{x}_i] \approx [+]$ is needed. We define the **intermediate state** $v \wedge w$ for $v, w \in \mathcal{A}$ by

$$(v \wedge w)_i := \begin{cases} v_i & \text{if } v_i = w_i, \\ 0 & \text{if } v_i \neq w_i, \end{cases}$$

Thus, $Z_0(v \wedge w)$ are the indices of the components which change from v to w (or which are constant in one or both states).

PROPOSITION 2: *Let $v, w \in V(G)$, $v \neq w$, and $Z_0(v) = Z_0(w) = \varnothing$. Then, $(v, w) \in E(G)$ iff*

$$\forall i \in Z_0(v \wedge w)\, \exists j \notin Z_0(v \wedge w) : w_i \cdot (v \wedge w)_j \approx \sigma_{i,j}.$$

PROOF: At first, we prove that the existence of an edge $(v, w) \in E(G)$ implies the condition. By DEF. 3 (p. 23), there is only an edge (v, w) if

$$\left.\begin{array}{l} \exists f \in M(\Sigma), x' \in X, t_0 < t_1 < t_2 \in \mathbb{R}_+, \\ \qquad \text{and a solution to Eq. (2.3) with } x_0 = x' \text{ such that} \\ \qquad \forall t \in [t_0, t_1) : [\dot{x}(t)] = v \text{ and} \\ \qquad \forall t \in (t_1, t_2] : [\dot{x}(t)] = w. \end{array}\right\} \tag{2.5}$$

Moreover, $[\dot{x}(t_1)] = (v \wedge w)$ due to continuity of $\dot{x}$. Consider an $i \in Z_0(v \wedge w)$. If $v_i = [+]$ and $w_i = [-]$, the velocity x_i obtains a maximum at t_1. Thus, by continuity, $[\ddot{x}_i(t_1)] = [-] = w_i$. In the case of a minimum, $[\ddot{x}_i(t_1)] = [+] = w_i$. Other cases are not possible since $Z_0(v) = Z_0(w) = \varnothing$. The equality $w_i = [\ddot{x}_i(t_1)] = [(D_1 f_i \ \ldots \ D_n f_i) \cdot \dot{x}(t_1)]$ can only hold if $\exists j \in \{1, \ldots, n\}$ such that $w_i = [D_j f_i(x(t_1)) \cdot \dot{x}_j(t_1)]$. It follows that $\sigma_{i,j} \cdot (v \wedge w)_j \approx w_i$. The index j cannot be an element of $Z_0(v \wedge w)$, because otherwise the sign $\sigma_{i,j} \cdot 0 = 0$ would be not consistent with the sign $w_i \neq 0$. From Eq. (2.1), we conclude that $w_i \cdot (v \wedge w)_j \approx \sigma_{i,j}$.

Now we show that the condition implies the existence of an edge $(v, w) \in E(G)$. Suppose that Σ and $v, w \in V(G)$ are given and the condition is satisfied. Since $Z_0(v) = Z_0(w) = \varnothing$, it holds for all $i \in Z_0(v \wedge w)$ that $w_i = -v_i \neq 0$. It has to be shown that Eq. (2.5) holds. We construct appropriate f, x', t_0, t_1 and t_2.
Choose $c \in \mathbb{R}^n$ such that $\forall i \in Z_0(v \wedge w) : c_i = 0$ and $\forall i \notin Z_0(v \wedge w) : c_i = (v \wedge w)_i \cdot 1$. Let $k(i) \notin Z_0(v \wedge w)$ be an index (depending on i) such that $w_i \cdot (v \wedge w)_{k(i)} \approx \sigma_{i,k(i)}$ for all $i \in Z_0(v \wedge w)$, which exists due to the condition. Define the matrix $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ by $a_{i,k(i)} := n \cdot w_i \cdot (v \wedge w)_{k(i)}$ if $i \in Z_0(v \wedge w)$, and $a_{i,j} := 1 \cdot w_i \cdot (v \wedge w)_j$ if $i \notin Z_0(v \wedge w)$ or $j \neq k(i)$. We set $f(x) := Ax + c$. Clearly, $f \in \mathcal{M}(\Sigma)$ since $[\mathcal{J}(f)(x)] = [A] \approx \Sigma$.
Now choose an arbitrary $t_1 \in \mathbb{R}_+$ and an open interval J_0 containing t_1 such that $x(\cdot) : J_0 \to \mathbb{R}^n$ is a solution of the ODE $\dot{x} = f(x), x(t_1) = 0$. Let $i \notin Z_0(v \wedge w)$. Since $\dot{x}(t_1) = c$ and

$c_i \neq 0$, and due to continuity of $x(\cdot)$, there is also an open interval J_i containing t_1 such that $\forall t \in J_i : [\dot{x}_i(t)] = (v \wedge w)_i = w_i$.

Such an interval also exists for $i \in Z_0(v \wedge w)$:
Since $\forall t \in J_0 : \ddot{x}(t) = A\dot{x}(t)$ and $(v \wedge w)_{k(i)} \neq 0$,

$$\begin{aligned}
\ddot{x}_i(t_1) &= \sum_{j=1,\ldots,n} a_{i,j}\dot{x}_j(t_1) \\
&= a_{i,k(i)}c_{k(i)} + \sum_{\substack{j \notin Z_0(v\wedge w),\\ j\neq k(i)}} a_{i,j}c_j \\
&= n \cdot w_i \cdot (v \wedge w)_{k(i)} \cdot 1 \cdot (v \wedge w)_{k(i)} + \sum_{\substack{j \notin Z_0(v\wedge w),\\ j\neq k(i)}} a_{i,j}c_j.
\end{aligned}$$

Observe that

$$n \cdot w_i \cdot (v \wedge w)_{k(i)} \cdot 1 \cdot (v \wedge w)_{k(i)} = n \cdot w_i,$$

and

$$|\sum_{\substack{j \notin Z_0(v\wedge w),\\ j\neq k(i)}} a_{i,j}c_j| = |\sum_{\substack{j \notin Z_0(v\wedge w),\\ j\neq k(i)}} 1 \cdot w_i \cdot (v \wedge w)_j \cdot 1 \cdot (v \wedge w)_j| \leq n - 1.$$

Consequently $[\ddot{x}_i(t_1)] = w_i$, and, due to continuity, there is an open interval J_i containing t_1 such that $\forall t \in J_i : [\ddot{x}_i(t)] = w_i$. Now choose $t_0 < t_1 < t_2$ such that $t_0, t_2 \in \bigcap_{i=0,\ldots,n} J_i$ and define x' component-wise by

$$x'_i := x_i(t_1) - \int_{t_0}^{t_1} \dot{x}_i(\tau)d\tau,$$

such that $x(\cdot)$ is a solution of $\dot{x} = f(x), x(t_0) = x'$. Then, for all $t \in [t_0, t_1), i = 1, \ldots, n$:

$$[\dot{x}_i(t)] = \left[\dot{x}_i(t_1) - \int_t^{t_1} \ddot{x}_i(\tau)d\tau\right] = -w_i = v_i,$$

because $\dot{x}_i(t_1) = 0$ and $\forall \tau \in [t, t_1] : [\ddot{x}_i(\tau)] = w_i$.
By analogy for all $t \in (t_1, t_2) : [\dot{x}(t)] = w$. □

This proposition shows that every relevant edge actually corresponds to a solution of the monotonic ensemble. We now provide a simple example of a QDE and its state-transition graph.

EXAMPLE 5: Suppose we have a sign matrix

$$\Sigma = \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix},$$

and take $X = \mathbb{R}^2$ as state space. The monotonic ensemble $\mathcal{M}(\Sigma)$ contains all continuously differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\forall x \in \mathbb{R}^2 : D_2f_1 > 0, D_1f_2 < 0, D_1f_1 =$

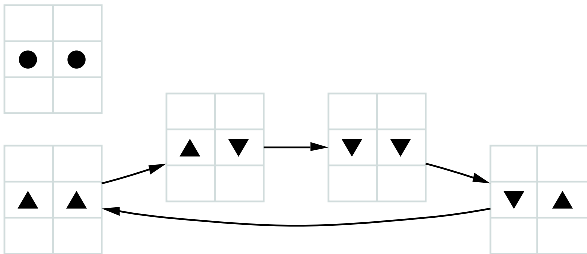

Figure 2.1: State-transition graph of the QDE of $\mathcal{M}(\Sigma)$ (computer-generated output, produced by the QSIM algorithm explained below in section 2.2.3, p. 32). The vertices represent qualitative states. The columns in each vertex correspond to the variables x_1, x_2. The triangles indicate whether $\dot{x}_i$ is positive, negative or vanishes.

$D_2 f_2 = 0$, and the QDE contains all initial value problems Eq. (2.3) with $f \in \mathcal{M}(\Sigma)$ and $x_0 \in \mathbb{R}^2$. The solution operator $\mathcal{S}_{\mathcal{M}(\Sigma)}(x_0)$ contains all reasonable solutions of these initial value problems which start in x_0, and $\tilde{\mathcal{S}}_{\mathcal{M}(\Sigma)}$ contains all abstractions, i.e. sequences of “trends” $[\dot{x}(\cdot)]$ of these trajectories which can be traced as paths in the state-transition graph G of the monotonic ensemble.

We infer the structure of G. As remarked above, $\{[-],[+]\}^2 \subseteq V(G)$. It follows from PROP. 1 (p. 24) that there is no other vertex in G except equilibrium: If $v = (0 \; v_2)^t$, and consequently $Z_0(v) = \{1\}$, both necessary conditions are violated – there are no two different indices $j, k \notin Z_0(v), j \neq k$. Only $1 \in Z_0(v)$ and $2 \notin Z_0 = (v)$, but $\sigma_{1,2} = [+] \neq 0$. The argument for $v = (v_1 \; 0)^t$ is analogous.

In this situation all edges can be found by using PROP. 2. For $v = ([+] \; [+])^t$ and $w = ([+] \; [-])^t$, it holds that $(v \wedge w) = ([+] \; 0)^t$ and $Z_0(v \wedge w) = \{2\}$. We have $w_2 \cdot (v \wedge w)_1 = [-] = \sigma_{2,1}$, such that $(v, w) \in E(G)$. But since $v_2 \cdot (v \wedge w)_1 = [+] \neq [-] = \sigma_{2,1}$, there is no edge from w to v. The arguments are analogous for all pairs of vertices (see Fig. 2.1 for the result). $\square$

2.2.2 Qualitative Differential Equations with Landmarks

This section extends the basic definitions, e.g. to prescribe zeros of the models of a monotonic ensemble and algebraic constraints. By introducing so called landmarks into the state space, the latter can be investigated more closely. Basically, landmarks are introduced (i) to exclude models from a monotonic ensemble where main isoclines do not pass through qualitatively prescribed points of the state space, and (ii) to allow regions of phase space with different sign matrices $[\mathcal{J}(f)(x)]$ which are separated by landmarks. If they are only used in the first way, landmarks provide a restriction of monotonic ensembles as defined in section 2.1 (p. 17). Again, the following definitions are different from the original work of Kuipers (1994): all landmarks are introduced by the modeller and not generated during qualitative simulation. Hence, the following theory is related to what Kuipers terms the “envisionment representation”.

Although the situation is almost analogue to DEF. 1, 2 and 3, we need new concepts, mainly to prepare more elaborate model ensembles and abstractions of reasonable functions. For every variable x_i of the model, we specify an ordered set of symbolic **landmarks** $\langle \lambda_{i,1}, \ldots, \lambda_{i,k_i} \rangle$. For every $i = 1, \ldots, n$ there is an ordered set

$$Q_i := \langle \lambda_{i,1}, \{\lambda_{i,1}, \lambda_{i,2}\}, \lambda_{i,2}, \ldots, \{\lambda_{i,k_i-1}, \lambda_{i,k_i}\}, \lambda_{i,k_i} \rangle,$$

where single landmarks and pairs of consecutive landmarks alternate. The cross product $Q := \times_{i=1,\ldots,n} Q_i$ is called the **quantity space**, and $S := \times_{i=1,\ldots,n}(Q_i \times \mathcal{A})$ is the **qualitative state space** of the model. To extract specific components of a **qualitative state** $v \in S$ by projection $\pi_i : \times_{i=1,\ldots,n} X_i \to X_i$, the standard notations in qualitative reasoning are:

$$\mathrm{qval}_i(v) = \pi_i(v) \in Q_i \times \mathcal{A}, \tag{2.6}$$

$$\mathrm{qmag}_i(v) = \pi_1(\mathrm{qval}_i(v)) \in Q_i, \tag{2.7}$$

$$\mathrm{qmag}(v) = \times_{i=1,\ldots,n} \mathrm{qmag}_i(v) \in Q, \tag{2.8}$$

$$\mathrm{qdir}_i(v) = \pi_2(\mathrm{qval}_i(v)) \in \mathcal{A}, \tag{2.9}$$

$$\mathrm{qdir}(v) = \times_{i=1,\ldots,n} \mathrm{qdir}_i(v) \in \mathcal{A}^n. \tag{2.10}$$

They are called **qualitative value** (Eq. 2.6), **qualitative magnitude** (Eqs. 2.7 and 2.8) or **qualitative direction** (Eqs. 2.9 and 2.10), respectively. Symbolic landmarks are interpreted as numbers in $\mathbb{R}$ and the pairs of landmarks as open intervals, such that a qualitative state can be assigned to each point of the state space X: A vector

$$\begin{aligned} \Lambda = (&\lambda_{1,1} \ \ldots \ \lambda_{1,k_1} \\ &\vdots \\ &\lambda_{n,1} \ \ldots \ \lambda_{n,k_n})^t, \end{aligned}$$

such that all $\lambda_{i,j} \in \mathbb{R}$, is called **landmark vector** if $\forall i = 1, \ldots, n : \lambda_{i,1} < \lambda_{i,2} < \ldots < \lambda_{i,k_i}$. For a given landmark vector, the **state abstraction** of a joint state and velocity vector $(x\,\dot{x})^t \in X \times \mathbb{R}^n$ is obtained by the mapping $a_\Lambda : X \times \mathbb{R}^n \to S$ such that

$$\begin{aligned} \mathrm{qdir}_i\big(a_\Lambda(x, \dot{x})\big) &= [\dot{x}_i], \\ \mathrm{qmag}_i\big(a_\Lambda(x, \dot{x})\big) &= \begin{cases} \lambda_{i,j} & \text{if } x_i = \lambda_{i,j} \\ \{\lambda_{i,j}, \lambda_{i,j+1}\} & \text{if } x_i \in (\lambda_{i,j}, \lambda_{i,j+1}). \end{cases} \end{aligned}$$

However, we will not consider a specific landmark vector Λ, but only assume the existence of landmark vectors with prescribed qualitative properties. This is why landmarks are introduced as *symbolic* values. One exception is the landmark "`0`", which is always associated with $0 \in \mathbb{R}$. This is necessary to extract the sign of a qualitative magnitude: If $0 \in Q_i$ and $q \in Q_i$, we define $[q] = [+]$ for $q > 0$, $[q] = [-]$ for $q < 0$, and $[q] = 0$ for $q = 0$; more generally

$$[q]_\lambda := \begin{cases} [+] & \text{if } q > \lambda, \\ 0 & \text{if } q = \lambda, \\ [-] & \text{if } q < \lambda, \end{cases}$$

where $\lambda \in Q_i$ is a landmark. We are now ready to proceed with further definitions extending the concepts from QDEs without landmarks.

DEFINITION 4: *For a state space X, a quantity space Q and the associated state space S, let $\mu : Q \to \mathcal{A}_*^{n\times n}$ be a mapping which assigns a matrix of (extended) signs $\mu(q)$ to each element q of the quantity space. Additionally, we take a family of relations $C := \{C_1, \ldots, C_m\}$ on the state space, $C_j \subseteq S, j = 1, \ldots, m$, called* ***constraints****, as given. Then, we obtain a* ***monotonic landmark ensemble***

$$\begin{aligned}\mathcal{M}(\mu, C) := \{f \in & C^1(X, \mathbb{R}^n) \mid \\ & \exists \textit{ landmark vector } \Lambda\, \forall x \in X : \\ & \big[\mathcal{J}(f)(x)\big] \approx \mu\big(\mathrm{qmag}(a_\Lambda(x, f(x)))\big) \\ & \textit{ and} \\ & \forall j = 1, \ldots, m : a_\Lambda(x, f(x)) \in C_j\}.\end{aligned}$$

The systems of a monotonic landmark ensemble $\mathcal{M}(\mu, C)$ are called a ***QDE****.*

It must be noted that μ and C can be chosen such that $\mathcal{M}(\mu, C) = \varnothing$ in a non-trivial way. In section 2.2.3 (p. 32) there are additional remarks on the consistency of μ and C. Taking the reasonable functions with values in X as space of admissible trajectories $\mathcal{E}$, we obtain a set-valued solution operator $\mathcal{S}_{\mathcal{M}(\mu,C)}(\cdot) : X \to \mathcal{P}(\mathcal{E})$. As before, not the solutions of the monotonic landmark ensemble themselves will be investigated, but their abstraction:

DEFINITION 5: *For a given reasonable function $x(\cdot)$ on $[0, T]$, possibly $T = \infty$, and a landmark vector Λ, there is an ordered sequence of jump points (t_j), with $t_0 = 0$ and subsequently containing all boundary points of the closures of all sets $\{t \in [0, T] | a_\Lambda(x(t), \dot{x}(t)) = v\}$ with $v \in S$ such that $\forall i = 1, \ldots, n : \mathrm{qdir}_i(v) \neq 0$ and $\mathrm{qmag}_i(v)$ is a pair of consecutive landmarks. The* ***landmark abstraction*** *of $x(\cdot)$ with respect to Λ is the sequence of qualitative states $\tilde{x} = (\tilde{x}_j) := \big(a_\Lambda(x(\tau_j), \dot{x}(\tau_j))\big)$ with arbitrarily chosen $\tau_j \in (t_j, t_{j+1})$. If (t_j) is finite with m elements, we choose $\tau_m \in (t_m, T)$.*

Again, the landmark abstraction does not depend on actual selection of $\tau_j, j \in \mathbb{N}$. By denoting $v(t) := a_\Lambda(x(t), \dot{x}(t))$, the definition implies that if t is a jump point, there exists an $i = \{1, \ldots, n\}$ such that the following three properties hold:

$$\left.\begin{array}{rl} (i) & \mathrm{qdir}_i(v(t)) = 0 \text{ or } \exists l \in \{1, \ldots, k_i\} : \mathrm{qmag}_i(v(t)) = \lambda_{i,l} \\ (ii) & \exists \epsilon\, \forall t^- \in [t - \epsilon, t) : \ \mathrm{qval}_i(v(t^-)) \neq \mathrm{qval}_i(v(t)) \\ & \qquad \text{or } \forall t^+ \in (t, t + \epsilon] : \ \mathrm{qval}_i(v(t^+)) \neq \mathrm{qval}_i(v(t)), \\ (iii) & \exists \epsilon\, \forall t^- \in [t - \epsilon, t),\ t^+ \in (t, t + \epsilon] : \mathrm{qval}_i(s(t^-)) \neq \mathrm{qval}_i(s(t^+)). \end{array}\right\} \tag{2.11}$$

The landmark abstraction of the solutions of a monotonic landmark ensemble are traced as paths of a directed graph as in the previous section.

DEFINITION 6: *Let $\mathcal{M}(\mu, C)$ be a monotonic landmark ensemble, $\mathcal{E}$ the space of reasonable trajectories and $\mathcal{S}_{\mathcal{M}(\mu,C)}(\cdot)$ the corresponding solution operator. We denote the set of the landmark abstractions of the solutions by*

$$\tilde{\mathcal{S}}_{\mathcal{M}(\mu,C)} := \{\tilde{x} \mid \exists x(\cdot) \in \mathcal{S}_{\mathcal{M}(\mu,C)}(x_0), x_0 \in X : \tilde{x} \textit{ is the abstraction of } x(\cdot)\}.$$

*Then, the directed **state-transition graph** G of the monotonic landmark ensemble is defined by the vertices*

$$V(G) := \{v \in S \mid \exists\, \tilde{x} \in \tilde{\mathcal{S}}_{\mathcal{M}(\mu,C)}, j \in \mathbb{N}:\ \tilde{x}_j = v\},$$

*called **qualitative states**, and the edges*

$$E(G) := \{(v,w) \mid\ \exists\, \tilde{x} \in \tilde{\mathcal{S}}_{\mathcal{M}(\mu,C)}, j \in \mathbb{N}:\ \tilde{x}_j = v \textit{ and } \tilde{x}_{j+1} = w\},$$

*called **qualitative transitions**.*

Again, G is always loop free. Although criteria for the existence of an edge in a state-transition graph are, in principle, a simple extension of PROP. 2 (p. 25), the situation becomes more complex since a larger number of cases has to be considered. Only a sketch of the various cases is provided here because they offer limited new insight. For this discussion (and later in Chapter 3), the extension of the notion of the intermediate state $v \wedge w \in S$ is helpful, which is defined component wise by

$$\operatorname{qdir}_i(v \wedge w) := \begin{cases} \operatorname{qdir}_i(v) & \text{if } \operatorname{qdir}_i(v) = \operatorname{qdir}_i(w), \\ 0 & \text{if } \operatorname{qdir}_i(v) \neq \operatorname{qdir}_i(w), \end{cases}$$

$$\operatorname{qmag}_i(v \wedge w) := \begin{cases} \operatorname{qmag}_i(v) & \text{if } \operatorname{qmag}_i(v) = \operatorname{qmag}_i(w), \\ \lambda_{i,j} & \text{if } \operatorname{qmag}_i(v) \neq \operatorname{qmag}_i(w) \\ & \text{and } \lambda_{i,j} \in \operatorname{qmag}_i(v) \cap \operatorname{qmag}_i(w) \neq \varnothing. \end{cases} \tag{2.12}$$

It should be noted that this operation is not defined for all pairs of states. In this case, $(v,w) \notin E(G)$, since their qualitative magnitudes are not adjacent. Due to continuity, it is not possible that a trajectory $x(\cdot)$ jumps from $x_i(t_1) < \lambda_{i,j}$ to $x_i(t_2) > \lambda_{i,j+1}$, $t_1 < t_2$, if there is not $t \in (t_1, t_2)$ such that $\lambda_{i,j} < x_i(t) < \lambda_{i,j+1}$.

An obvious necessary criterion for the existence of an edge (v,w) is that the intermediate state has to satisfy the relations in C, i.e. $\forall j = 1, \ldots, m : (v \wedge w) \in C_j$. For another criterion suppose that (i) $v \wedge w$ is defined, (ii) $Z_\lambda(v \wedge w) := \{i = 1, \ldots, n | \operatorname{qmag}_i(v) \neq \operatorname{qmag}_i(w)\} \neq \varnothing$ (i.e. some qualitative magnitudes change), (iii) for all $i \in Z_\lambda(v \wedge w)$ the qualitative magnitudes $\operatorname{qmag}_i(v)$ and $\operatorname{qmag}_i(w)$ are pairs of landmarks, and (iv) $Z_0(v \wedge w) = \varnothing$ (i.e. all qualitative directions remain unchanged). Then, if there is an $i \in Z_\lambda(v \wedge w)$: $\operatorname{qmag}_i(w) > \operatorname{qmag}_i(v)$, there can only be an edge (v,w) if $\operatorname{qdir}_i(v \wedge w) = \operatorname{qdir}_i(v) = [+]$. Otherwise, if $\operatorname{qmag}_i(w) < \operatorname{qmag}_i(v)$, the ith component has to decrease, $\operatorname{qdir}_i(v) = [-]$.

It can be seen that such criteria become very complicated because there are situations where only qualitative directions change (as investigated in PROP. 2, p. 25), where one or more qualitative magnitudes change, where some qualitative values remain constant on a landmark, or where multiple of such events occur in parallel. All these cases are incorporated into the QSIM algorithm which is presented in the next section. However, some of them are of limited practical relevance. Exclusion of solutions of the monotonic landmark ensemble where one component is constant is discussed in section 2.2.4 (p. 36). In section 3.2 (p. 62) I develop a method to exclude cases where several qualitative magnitudes and several qualitative directions change at the same time and which have limited relevance: both techniques restrict the set of admissible trajectories to a subset of the reasonable functions.

EXAMPLE 6: Suppose we have a system defined by an ODE of the form $\dot{x} = f(x)$ on the state space $X = \mathbb{R}_+$ with $f \in C^1(\mathbb{R}, \mathbb{R})$, but we only the following properties of f: There exist three real numbers $0 < \lambda_1 < \lambda_2 < \lambda_3$ such that λ_1, λ_3 are the only zeros of f, i.e.

$$f(\lambda_1) = f(\lambda_3) = 0.$$

Moreover, the function is increasing below λ_2 and decreasing above:

$$\begin{aligned} \forall x < \lambda_2 &: D_x f(x) > 0, \\ \forall x > \lambda_2 &: D_x f(x) < 0, \\ & D_x f(\lambda_2) = 0. \end{aligned}$$

Quantitative values are not known for $\lambda_1, \lambda_2, \lambda_3$. We want to describe this situation by a monotonic landmark ensemble to find all trajectories which may be produced by such a system. At first, the qualitative state space has to be constructed on the basis of adequate landmarks:

$$Q = Q_1 := \big\langle 0, \{0, \lambda_1\}, \lambda_1, \{\lambda_1, \lambda_2\}, \lambda_2, \{\lambda_2, \lambda_3\}, \lambda_3, \{\lambda_3, \infty\}, \infty \big\rangle,$$

yielding $S = Q \times \mathcal{A}$ and a state abstraction $a_\Lambda : \mathbb{R}_+ \times \mathbb{R} \to S$ for every landmark vector $\Lambda = (\lambda_0 \;\, \lambda_1 \;\, \lambda_2 \;\, \lambda_3 \;\, \lambda_4)^t$ with $\lambda_0 = 0, \lambda_4 = \infty$, which describes a possible quantitative configuration of the the landmarks. Secondly, we need the map μ and the set of constraints C. We define

$$\mu : Q \to \mathcal{A}_*, q \mapsto \begin{cases} [+] & \text{if } x < \lambda_2, \\ 0 & \text{if } x = \lambda_2, \\ [-] & \text{if } x > \lambda_2, \end{cases}$$

to express the monotonicity properties of f and $C := \{C_1, C_2\}, C_1, C_2 \subseteq S$ with

$$\begin{aligned} C_1 &:= \{v \in S \mid \operatorname{qmag}_1(v) \leq \lambda_2 \Rightarrow \operatorname{qdir}_1(v) = [\operatorname{qmag}_1(v)]_{\lambda_1}\}, \\ C_2 &:= \{v \in S \mid \operatorname{qmag}_1(v) \geq \lambda_2 \Rightarrow \operatorname{qdir}_1(v) = -[\operatorname{qmag}_1(v)]_{\lambda_3}\}, \end{aligned}$$

to express the zeros of f. All together, we obtain the monotonic landmark ensemble

$$\begin{aligned} \mathcal{M}(\mu, C) := \{f \in & C^1(\mathbb{R}_+, \mathbb{R}) \mid \\ & \exists \text{ landmark vector } \Lambda \, \forall x \in \mathbb{R}_+ : \\ & \big[D_x f(x)\big] = \mu\big(\operatorname{qmag}(a_\Lambda(x, f(x)))\big) \\ & \text{ and} \\ & \forall j = 1, \dots, m : a_\Lambda(x, f(x)) \in C_j\}, \end{aligned}$$

and a QDE consisting of all initial value problems given by functions $f \in \mathcal{M}(\mu, C)$ and initial values $x_0 \in \mathbb{R}_+$. This defines a set-valued solution operator $\mathcal{S}_{\mathcal{M}(\mu,C)}(\cdot) : \mathbb{R}_+ \to \mathcal{P}(\mathcal{E})$ with the reasonable functions $x(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ as space of admissible trajectories $\mathcal{E}$. It assigns all reasonable solutions to an initial value.

Based on the state abstraction, every solution $x(\cdot) \in \mathcal{S}_{\mathcal{M}(\mu,C)}(x_0)$ can be discretised by landmark abstraction, yielding a sequence $(\tilde{x}_j), \tilde{x}_j \in S$. All these sequences are paths in the state-transition graph, which can be computed with the QSIM algorithm (see next subsection). The result is presented in Fig. 2.2. Note that only qualitative states which persist over time intervals are represented as vertices in the graph. There is, e.g. no qualitative state v with $\mathrm{qmag}(v) = 0$ and $\mathrm{qdir}(v) = [-]$. □

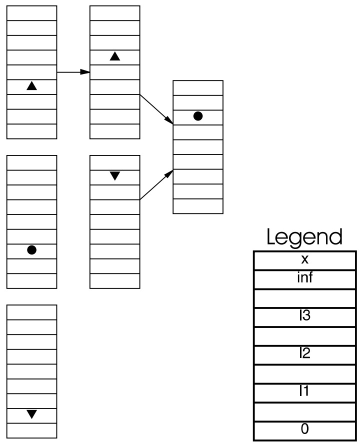

Figure 2.2: State-transition graph of the QDE defined in EX. 6 (computer-generated output). Each vertex represents a qualitative state, each edge a qualitative transition. The small boxes inside each vertex v represent all possible qualitative magnitudes where singular landmarks and intervals defined by pairs of consecutive landmarks alternate (see legend). The box which contains a symbol denotes $\mathrm{qmag}(v)$, at the same time indicating whether $\mathrm{qdir}(v) = [+]$, $\mathrm{qdir}(v) = 0$ or $\mathrm{qdir}(v) = [-]$.

Several larger QDEs will be presented and analysed in Chapter 4. QDEs with landmarks can describe much more complex settings than the causal loop diagrams introduced in Chapter 1. In addition to negative and positive influences on *change rates of variables*, there can be *direct* influences on the variables, represented by the constraints C. Moreover, whether an influence is negative or positive can depend on the (qualitative) value of another (or the same) variables via the mapping μ.

2.2.3 The QSIM Algorithm

Basic Ideas

The QSIM algorithm is a procedure to compute the state-transition graph of a QDE. This subsection outlines its basic ideas and the script language needed in Chapter 4 and the Appendix. For the complete formal specification we refer to the original work of Kuipers (1994). The core of the QSIM algorithm is a specialised solver for a so called constraint satisfaction problem for which various methods exist (Mackworth 1977; Mackworth 1987; Tsang 1993).

DEFINITION 7: *A **constraint satisfaction problem** is given by a set of n variables $V = \{x_1, \ldots, x_n\}$ and a set of n associated finite domains $D = \{D_1, \ldots, D_n\}$, where the variables take their values. Furthermore, a set of m **constraints** $C = \{C_1, \ldots, C_m\}$, which are* *subsets of the state space* $S = \times_{i=1,\dots,n} D_i$*, has to be given. The* ***solution*** *of a constraint satisfaction problem is the maximal subset* $L \subseteq S$ *such that* $\forall i = 1, \dots, m : L \subseteq C_i$*.*

Computing the solution can be complicated if $C_1, \dots, C_m$ are represented in a way such that $\cap_{i=1,\dots,m} C_i$ cannot be compute directly. For a monotonic landmark ensemble $\mathcal{M}(\mu, C)$, the basic idea is to take $D_i = Q_i \times \mathcal{A}$, $i = 1, \dots, n$, and the constraints C. Additional constraints are implied by the sign information given by μ. The solution L includes all qualitative states which are consistent with the model assumptions. This is a superset of the vertices of the state-transition-graph G, because it also contains states which cannot be obtained for a time interval (which is required by DEF. 2, p. 23 or DEF. 5, p. 29). Excluded are qualitative states v where for an $i \in \{1, \dots, n\}$: $\mathrm{qdir}_i(v) \neq 0$ and $\exists j \in \{1, \dots, k_i\}$: $\mathrm{qmag}_i(v) = \lambda_{i,j}$, i.e. where a component of a solution $x_i(\cdot)$ of a system of $\mathcal{M}(\mu, C)$ is at a landmark and $\dot{x}_i \neq 0$ – this is only possible for a *point* in time, i.e. for intermediate states $(v \wedge w)$, and not on a time interval. However, these intermediate states must also satisfy the constraints C. It is possible that the solution L does not contain every combination of qualitative directions (contrary to section 2.2.1, p. 21). The edges $(v, w) \in E(G)$ are found by considering the sign matrices $\mu(v), \mu(w)$, because they indicate under which conditions the qualitative direction or the qualitative magnitude of a state can change (cf. PROP. 2, p. 25).

In practice, the algorithm takes account of several technical details, mainly to cover all special cases for edges in G as discussed above. An example for another technicality is that the state space is augmented by the velocity space to allow for a more flexible formulation of constraints: The qualitative state space S is supplied with the components $i' = n+1, \dots, 2n$, $Q_{i'} := \big\langle \{-\infty, 0\}, 0, \{0, +\infty\} \big\rangle$ with $\dot{x}_i = x_{i'}$. This makes the state abstractions of vectors $(x\ \dot{x}\ \ddot{x})^t$ available via

$$\mathrm{qmag}_{i'}(a_\Lambda(x, \dot{x})) = \begin{cases} \{-\infty, 0\} & \text{if } \dot{x}_{i'} < 0, \\ 0 & \text{if } \dot{x}_{i'} = 0, \\ \{0, +\infty\} & \text{if } \dot{x}_{i'} > 0, \end{cases}$$

and $\mathrm{qdir}_{i'}(a_\Lambda(x, \dot{x})) = [\ddot{x}_i]$. The link of $\dot{x}_i$ to $x_{i'}$ is expressed by a constraint of the form

$$\{v \in S \mid \forall i = 1, \dots, n : \mathrm{qdir}_i(v) = [\mathrm{qmag}_{n+i}(v)]\}. \tag{2.13}$$

Altogether, the constraints describing $\mathcal{M}(\mu, C)$ can be specified using a script language as outlined below. Then, the QSIM algorithm applies a constructive filtering technique. It begins with one user-specified qualitative state v which is an element of the solution L of the constraint satisfaction problem. Then, all *potential* successor states of v are generated, i.e. states where qualitative magnitudes change in accordance with current qualitative directions, or where qualitative directions change in accordance with the signs of the Jacobian. This is as an application of PROP. 2 (p. 25) and its full extension to monotonic landmark ensembles. Each potential successor is checked to establish whether it belongs to L: the procedure efficiently evaluates all constraints $C_1, \dots, C_m$ for each potential successor. If one constraint is violated, the state is “filtered out”. With this procedure, sequences of “surviving” *actual* states are generated. In these sequences state which can occur for a time interval and for time points (so called I-states and P-states) alternate. Only the former are relevant for the state-transition graph, but all special cases like extrema of solutions occurring exactly on a landmark or concurrent changes of qmag and qdir can be covered by considering P-states. Thus, if $v \wedge w$ is a P-state succeeding v, and $w \in \cap_{i=1,\ldots,m} C_i$ is a new I-state succeeding $v \wedge w$, then w is introduced as a vertex and (v, w) as an edge in G. After this, new potential successors of w are generated and "filtered" iteratively, until no new valid successor is found.

One more subtlety in the filtering procedure arises from the fact that the number of potential successors is usually too large for efficient computation. In practice, a more complicated algorithm is applied. In a first step only the possible successors of some components of a qualitative state are generated (so-called partial assignments) and discarded if they violate a single constraint $C_1, \ldots, C_m$. Then, all surviving values for one component are combined with all surviving values for other components. This has to be checked against all constraints iteratively until all complete, consistent states are constructed. The algorithm is designed such that the guaranteed coverage theorem holds (Kuipers 1994, p. 118, adapted to the terminology of this thesis):

PROPOSITION 3: *Let $x(\cdot)$ be a reasonable solution of the monotonic landmark ensemble $\mathcal{M}(\mu, C)$, and Λ a landmark vector. Then, the landmark abstraction $\tilde{x}$ of $x(\cdot)$ is a path in the graph computed by the QSIM algorithm.*

It is important to note that there are also paths in the graph which are not an abstraction of a solution of the monotonic landmark ensemble. Such a path is called **spurious behaviour**. The QSIM algorithm is complete, meaning that the abstractions of all solutions are represented, but not sound, i.e. there is spurious behaviour – a fact that can generally be proven (Say and Akin 2002). In PROP. 2 (p. 25) soundness was only shown for edges (v, w), describing the short but important paths v, w. Thus, the information contained in the state-transition graph can be seen as negative: If there is no edge from a vertex v to a vertex w, we can be sure that there is no solution of the monotonic landmark ensemble for which the landmark abstraction attains w directly after v. Therefore, QDEs are complementary to common numerical simulation. While the latter strives to use sound but incomplete algorithms (every computed trajectory approximates a solution, but it is not possible to compute an infinite ensemble quantitatively), the QSIM algorithm is unsound but complete.

The QSIM Script Language

There are several implementations of the QSIM algorithm, e.g. in Lisp (University of Texas at Austin 1991) or in C (Dvorak 1998). The latter is the base for an extended C++ version developed for this thesis (which was already used for the examples in the previous subsections). We now introduce the basic syntax and semantics of the script language to formulate a QDE in these implementations. The variables of the model together with their quantity spaces are defined by a block

```
(quantity-spaces
  <(var ( <lm> ) ''text'')>
)
```

where <lm> is the sequence of landmarks associated with variable var, i.e. a component of the (qualitative) state space. Angular brackets < > indicate that a specification can be repeated multiple times. The description text comments on the variable. The most basic constraint requires that the derivative of one variable x is determined by a variable y (cf. Eq. 2.13).

```
((d//dt x y))
```

If i is the index of variable x and j the index of y this constraint is valid for qualitative states v with $\mathrm{qdir}_i(v) = [\mathrm{qmag}_j(v)]$. Simple monotonic relationships between two variables are expressed by

```
((M+ x y) <(lx ly)>)
```

or

```
((M- x y) <(lx ly)>)
```

with a (possibly empty) sequence of m pairs (lx ly), called **corresponding values**, where lx represents a landmark of the variable x (with index i) and ly a landmark of variable y (with index j), denoting sequences of landmarks $\lambda_{i,1}, \ldots, \lambda_{i,m}$ and $\lambda_{j,1}, \ldots, \lambda_{j,m}$. A qualitative state v satisfies the M+ constraint if

$$\begin{aligned} &\mathrm{qdir}_i(v) = \mathrm{qdir}_j(v), \\ &\text{and } \forall k = 0, \ldots, m: \ [\mathrm{qmag}_i(v)]_{\lambda_{i,k}} = [\mathrm{qmag}_j(v)]_{\lambda_{j,k}}, \end{aligned}$$

while for the M- constraint

$$\begin{aligned} &\mathrm{qdir}_i(v) = -\mathrm{qdir}_j(v), \\ &\text{and } \forall k = 0, \ldots, m: \ [\mathrm{qmag}_i(v)]_{\lambda_{i,k}} = -[\mathrm{qmag}_j(v)]_{\lambda_{j,k}}. \end{aligned}$$

Rows in the sign matrix Σ (without a coefficient $[?]$) are expressed by multivariate monotonic function constraints

```
(((M <s>) <var>))
```

where <s> denotes a sequence of m signs $\sigma_1, \ldots, \sigma_m \in \mathcal{A}$ (which are different from 0), and <var> denotes $m+1$ variables with indices $i_1, \ldots, i_{m+1}$. Then, a qualitative state v respects this constraint if one of the following two conditions holds:

$$\begin{aligned} \mathrm{qdir}_{i_{m+1}}(v) = 0 \Rightarrow & \exists j, k \in \{1, \ldots, m\}, j \neq k: \\ & \sigma_j \cdot \mathrm{qdir}_{i_j}(v) = -\sigma_k \cdot \mathrm{qdir}_{i_k}(v) \neq 0, \\ \mathrm{qdir}_{i_{m+1}}(v) \neq 0 \Rightarrow & \exists j \in \{1, \ldots, m\}: \sigma_j \cdot \mathrm{qdir}_{i_j}(v) = \mathrm{qdir}_{i_{m+1}}(v). \end{aligned}$$

This constraint can also be extended to corresponding values

```
(((M <s>) <var>) <(tuple)> )
```

with `<(tuple)>` being a sequence of $(m+1)$-tuples of landmarks with length k, denoted by $\lambda_{i_1,1}, \ldots, \lambda_{i_1,k}$, $\lambda_{i_2,1}, \ldots, \lambda_{i_2,k}$, ..., $\lambda_{i_{m+1},1}, \ldots, \lambda_{i_{m+1},k}$, and additionally requiring for a qualitative state v and for all $l = 1, \ldots, k$ that

$$\big(\forall j = 1, \ldots, m : \sigma_j \cdot [\text{qmag}_{i_j}(v)]_{\lambda_{i_j,l}} = \sigma_{m+1}\big) \Rightarrow [\text{qmag}_{i_{m+1}}(v)]_{\lambda_{i_{m+1},l}} = \sigma_{m+1}.$$

Other important constraints are defined in a similar way, e.g. `U-` and `U+`, capturing the monotonicity properties of functions $f(x), x \in \mathbb{R}$ which increase on one side of a landmark and decrease on the other (cf. Ex. 6, p. 31), and `((mult x y z))`, called qualitative multiplication, generalising the monotonicity properties of an algebraic equation like $x_k(t) = x_i(t) \cdot x_j(t)$. For a full specification of this and further constraints I refer to Kuipers (1994).

2.2.4 Advanced Techniques

I open this subsection with a short discussion of the strengths and weaknesses of QDEs. The latter motivates various extensions of the basic approach. Here, established extensions are presented, while in Chapter 3 I introduce new abstraction and restriction techniques. Using the framework of model ensembles developed in section 2.1 (p. 17) and the graph theoretical description introduced in section 2.2 (p. 20), I can formulate and systematise most of these techniques in a new and consistent way. The concept of a restriction technique is directly related to the restriction of a model ensemble, and is the guiding principle for most parts of Chapter 3. The generic definition of abstraction techniques I provide subsumes various known abstraction techniques. It will be used again and complemented with viability theory in Chapter 3.

The main advantages of QDEs concern generality, uncertainty and non-quantitative knowledge. As discussed in Chapter 1, problems of sustainability science show such properties. Based on the framework of model ensembles, QDEs clearly account for the first two of them (cf. section 2.1, p. 17): Each QDE represents a broad variety of systems with some common (monotonicity) properties. This can subsume a typical pattern comprising many instances, or a set of potentially valid model formulations between which the modeller cannot discriminate. QDEs also account for non-quantitative knowledge, since the characteristics of the system are only expressed in terms of trends (signs), influences (signs) and thresholds (landmarks). Hence, it is not necessary to measure variables of a real-world system quantitatively, making it much more easy to introduce variables which are difficult to operationalise (e.g. profit expectations, well-being, political power).

On the other hand, the core of the approach is deterministic: a monotonic landmark ensemble contains only autonomous models. We cannot take variability of parameters into account. Secondly, as discussed in the previous subsection, we have the problem of spurious behaviour: Not every path in the state-transition graph corresponds to a solution of the monotonic landmark ensemble. From the practical perspective, a main challenge is posed by very large state-transition graphs resulting from QDEs with many variables. The number of potential qualitative states increases exponentially with the dimension of the qualitative state space, termed the state explosion problem (Valmari 1998). A deeper obstacle is the tendency of larger QDEs to produce weakly structured state-transition graphs, indicated by a typically increasing average degree of vertices. This makes it less "predictable" which successor state w will be observed when a system is in a given qualitative state v. This is not an artifact of the method, but simply a consequence of the generality (or uncertainty) of the model assumptions contained in an influence diagram: If many partial derivatives of a component function $f_i, f \in \mathcal{M}(\Sigma)$ do not vanish, it is more likely that there exists a $j \notin Z_0(v \wedge w)$ such that $w_i \cdot (v \wedge w)_j \approx \sigma_{i,j}$, resulting in multiple successors (cf. PROP. 2, p. 25). All these problems are obstacles to drawing conclusions for the management of an environmental system. This is because only very limited knowledge is expressed by a monotonic landmark ensemble.

Most techniques to handle a large or weakly structured state-transition graph G can be classified by the following scheme:

1. Abstraction: simplify the representation of G to foster meaningful analysis.
2. Restriction: reduce the number of vertices and edges of G by introducing additional types of assumptions about the system. This is mainly by restricting the state space, the monotonic landmark ensemble, or the space of admissible trajectories.
3. Analysis: consider G as a database which is not displayed but queried for interesting model properties.

Here, the concept of "abstraction" is formally different from section 2.2.1 (p. 21) and section 2.2.2 (p. 27), but the use of both meanings is standard in the qualitative reasoning literature.

Abstraction techniques

There are two types of abstraction procedures: they can either be guided solely by the structure of state-transition graph G itself (automated abstraction), or can depend on user-specified criteria (user-guided abstraction). Both are useful, and thus they are very often combined. All abstraction techniques perform some kind of clustering of the state-transition graph, resulting in an abstracted state-transition graph which disregards some features of (and therefore abstracts from) the original state-transition graph (Clancy and Kuipers 1993; Mallory et al. 1996). There is also related work on the abstraction of finite state machines (Oikonomou 1996). We propose the following generic definition of abstraction, which is also fundamental for section 3.1 (p. 52).

DEFINITION 8: *Let G be a state-transition graph and $G_i, i = 1, \ldots, m$ a partition of disjoined subgraphs of G with $\bigcup_{i=1,\ldots,m} V(G_i) = V(G)$. The **abstracted state-transition graph** G' is then given by*

$$V(G') := \bigcup_{i=1,\ldots,m} \{V(G_i)\}$$

and

$$E(G') := \{(v', w') \mid v' \neq w' \textit{ and } \exists v \in v', w \in w' : (v, w) \in E(G)\}.$$

The new loop-free graph G' contains a vertex for every subgraph G_i (defined by the union of the qualitative states in this subgraph). The edges of G' are "inherited" from the edges of G. Different abstraction procedures are distinguished by the chosen subgraphs.

One major breakthrough in (automated) abstraction was so called chatter-box abstraction (Clancy and Kuipers 1993; Clancy 1997), since it reduces both the number of states and the computation effort of the QSIM algorithm drastically. The idea is to cluster qualitative states between which the system can arbitrarily float back and forth (called chattering). Based on the new DEF. 8 this technique can be described in a simple and concise way. For a graph G, define the bi-directed subgraph $G^- := G \cap G^{-1}$, which contains all vertices but only the bi-directed edges of G.

DEFINITION 9: *A **chatter-box** is a subgraph of G which is induced by a strongly connected component of the bi-directed subgraph G^-.*

Chatter-boxes can be used for abstraction since

PROPOSITION 4: *Chatter-boxes are disjoined subgraphs of G.*

PROOF: Due to maximality, the strongly connected components of G^- are disjoined. Thus, also the subgraphs of G induced by these components are disjoined. □

DEFINITION 10: *Let $G_i, i = 1, \ldots, l$ be the family of all chatter-boxes of a state transition graph G, and $G_i, i = l+1, \ldots, m$ the trivial subgraphs of G which have only one vertex which is not a member of a chatter-box. The abstracted state-transition graph G' based on this family is called the **chatter-box abstraction** of G.*

The most common version of the chatter-box algorithm detects only chatter-boxes where all qualitative states in the subgraph have the same qualitative magnitudes. If a chatter-box contains different qualitative magnitudes, it is split to different disjoined subgraphs: this **simple chatter-box abstraction** ignores so called landmark chatter. The advantage is that this version can be easily integrated into the QSIM algorithm and no post processing is needed. Since this reduces the number of qualitative states which have to be considered by the filtering algorithm, its efficiency increases. We will use simple chatter-box abstraction for most of the applications in Chapter 4.

Another very powerful simplification procedure is projection, also called generation of a variable focus (Mallory et al. 1996; Clancy et al. 1997). For this user-guided procedure, a set of variables of interest has to be specified by the modeller. The idea is to observe only differences in these variables and to disregard the values of all other variables. For example, merely technical auxiliary variables can be ignored or hypotheses about a restricted set of state variables can be verified by this technique. The variables not to be ignored are described as a index set $I = \{j_1, \ldots, j_p\} \subseteq \{1, \ldots, n\}$ (denoting components of the qualitative state space). We consider two types of projection: simple and faithful. The first abstracts the state-transition graph taking as subgraph states which have identical qualitative values in all variables of interest. The second type of projection additionally splits these subgraphs into weakly connected components, thus maintaining connectivity properties.

By $\pi_I : V(G) \to \times_{j \in I}(Q_j \times \mathcal{A})$, we denote the usual projection $v \mapsto (v_{j_1} \ \ldots \ v_{j_p})^t$. Two vertices $v, w \in V(G)$ are called **simple projection equivalent** with respect to I if $\pi_I(v) = \pi_I(w)$. Define the undirected supergraph $G^+ := G \cup G^{-1}$ of G, which contains a bi-directed edge for every edge in G. Two vertices v, w are called **faithful projection equivalent** if, additionally, in G^+ there is a path $v_0, \ldots, v_k$, $v_0 = v, v_k = w$ such that $\pi_I(v_0) = \pi_I(v_1) = \ldots = \pi_I(v_k)$ or if $v = w$. Both relations are obviously equivalence relations, and we obtain equivalence classes on $V(G)$ which induce disjoined subgraphs of G.

DEFINITION 11: *Let $G_i, i = 1, \ldots, m$ be the family of subgraphs of G induced by a simple projection equivalence defined by I. The resulting abstracted state-transition graph $G' =: \pi_I(G)$ is called the* ***simple projection*** *of G with respect to I. If $G_i, i = 1, \ldots, m$ are given by a faithful projection equivalence defined by I, $G' =: \bar{\pi}_I(G)$ is called the* ***faithful projection*** *of G with respect to I.*

Besides producing a simplified graph, these projections have some useful properties. For a path $v_0, \ldots, v_k$ in G, we can construct a sequence of vertices $v'_0, \ldots, v'_l$, in $\pi_I(G)$, $l \leq k$ by taking the sequence $\pi_I(v_0), \ldots, \pi_I(v_k)$ and removing all elements which are identical to their predecessor. Then $l = 1$ or $v'_0, \ldots, v'_l$ is a path in $\pi_I(G)$. The same holds true for $\bar{\pi}_I$. As a consequence, structures are preserved under $\bar{\pi}_I$ and π_I. A subgraph H of G is weakly connected, if for all $v, w \in V(H)$, $v \neq w$, there is a path $v, \ldots, w$ in G^+. If a subgraph $H \subseteq G$ is weakly or strongly connected, the same applies to $\pi_I(H) \subseteq \pi_I(G)$ (unless $|\pi_I(H)| = 1$) and to $\bar{\pi}_I(H) \subseteq \bar{\pi}_I(G)$ (unless $|\bar{\pi}_I(H)| = 1$). For the inverse case, we only have a weaker property:

PROPOSITION 5: *(1) If $v'_1, \ldots, v'_k$ is a path in $\bar{\pi}_I(G)^+$, then for all $v, w \in V(G)$ such that $\bar{\pi}_I(v) = v'_1$ and $\bar{\pi}_I(w) = v'_k$ there exists a path $v, \ldots, w$ in G^+.*
(2) If $H \subseteq \bar{\pi}_I(G)$ is weakly connected, so is $\bar{\pi}_I^{-1}(H) \subseteq G$.

PROOF: (1) Since for all $v'_i, v'_{i+1}, i = 1, \ldots, k-1$, there is always an edge (v'_i, v'_{i+1}) in $\bar{\pi}_I(G)^+$, there must also be an edge $(v_i, v_{i+1}) \in E(G^+)$ with $\bar{\pi}(v_i) = v'_i$ and $\bar{\pi}(v_{i+1}) = v'_{i+1}$. Since $\bar{\pi}_I$ is faithful, all $u \in V(G)$ with $\bar{\pi}(u) = v'_i$ are strongly connected in G^+. Thus, there is a path from v to w in G^+.
(2) Take $v, w \in V(\bar{\pi}_I^{-1}(H))$ and choose $v', w' \in V(H)$ such that $\bar{\pi}(v) = v'$ and $\bar{\pi}(w) = w'$. Since H is weakly connected, there is a path $v', \ldots, w'$ in $\bar{\pi}_I(G)^+$. Consequently, part (1) of the proposition guarantees the existence of a path $v, \ldots, w$ in G^+, i.e. $\bar{\pi}_I^{-1}(H)$ is weakly connected. □

Simple projection will be necessary to display some results in Chapter 4.

A third abstraction technique is called state-based (Fouché and Kuipers 1991; Clancy and Kuipers 1993), and can be used if the quantity space contains a larger number of landmarks. The modeller chooses some landmarks which are of limited interest. All states with a qualitative magnitude adjacent to these landmarks are regarded as equivalent if they also have the same qualitative direction. By this equivalence relation, disjoined subgraphs are defined and an abstracted state-transition graph can be computed.

Restriction Techniques

One simple restriction method is the analytical function constraint (Kuipers 1994). In principle, it is possible that a component of a solution of a monotonic landmark ensemble is constant over a time interval. If we restrict the set of admissible trajectories to functions which are analytical (at least in some components), this is only possible if the component remains constant forever. From the modelling perspective this restriction is reasonable if a constant evolution of one variable over some time must be regarded as a very improbable marginal case in the real world. If an analytical function constraint with a prescribed index set $I \subseteq \{1, \ldots, n\}$ is applied to a state-transition graph, all vertices v are for which at least one $i \in I$ exists such that $\mathrm{qdir}_i(v) = 0$ are eliminated. The effect of this restriction technique can be increased if it is integrated into the QSIM algorithm to reduce the number of states which have to be checked at each filtering step. The analytical function constraint is used for some of the applications in Chapter 4.

Phase plane constraints are a path-dependent technique (Lee and Kuipers 1988). Since a QDE contains only autonomous systems, non-constant trajectories cannot intersect. For each tuple of landmarks from the quantity space Q and a landmark vector a specific state in X is given. The set of admissible trajectories is restricted to the reasonable functions which always pass through such states in the same direction. However, this is not very restrictive for high dimensional systems.

The integration of quantitative knowledge to QDEs, so called semi-qualitative reasoning, is still a challenge (Berleant and Kuipers 1992; Kuipers 1994; Berleant and Kuipers 1998; Kay 1998; Moldenhauer et al. 1999). If we define a qualitative model, we possibly disregard quantitative knowledge which may be (partially) available: Quantitative knowledge cannot be expressed by a monotonic landmark ensemble. If we had full quantitative knowledge, we could restrict the model to a single ODE. In the more interesting case, we come up with a "hybrid" model ensemble by restricting a monotonic landmark ensemble (partially) with quantitative constraints. For such a semi-qualitative reasoning three types of quantitative knowledge are considered: (i) quantitative landmark intervals, (ii) functional envelopes, and (iii) temporal envelopes. For the first type, $\mathcal{M}(\mu, C)$ is restricted by constraining the values a landmark vector can take, e.g. by prescribing intervals $J_{i,j} \subset \mathbb{R}$ with $\lambda_{i,j} \in J_{i,j}$. For the second type, we define two functions $\underline{f}, \bar{f} : X \to \mathbb{R}^n$ and restrict $\mathcal{M}(\mu, C)$ to the models f such that $\forall x \in X : \underline{f}(x) \leq f(x) \leq \bar{f}(x)$ (inequalities defined component wise). By introducing temporal envelopes we restrict the space of admissible trajectories $\mathcal{E}$ with two functions $\underline{x}, \bar{x} : \mathbb{R}_+ \to X$ by requiring that $\forall x(\cdot) \in \mathcal{E}, t \geq 0 : \underline{x}(t) \leq x(t) \leq \bar{x}(t)$.

Combinations of these restrictions can be used to infer estimates for the sets of quantitative states which are consistent with a qualitative state v, or for the length of time intervals where a solution of the monotonic landmark ensemble has v as state abstraction. Various partially successful procedures have been proposed, but an efficient solution for systems with more than three variables is still lacking. The basic problem are tight approximations of solutions of interval-valued differential inclusions with state constraints (see section 2.3, p. 42), e.g. of the form

$$
\begin{aligned}
&\dot{x} \in [\underline{f}(x), \bar{f}(x)] \\
\text{subject to}\quad & \\
&x(0) \in J, \\
&\forall t \in \mathbb{R}_+ : [\dot{x}(t)] = \mathrm{qdir}(v) \\
&\quad \text{and } x(t) \in [\underline{x}(t), \bar{x}(t)],
\end{aligned}
$$

where $J \in \mathbb{R}^n$ describes an interval of possible initial values and $v \in S$ a qualitative state under consideration. The new method I develop in section 3.4 (p. 77) is a step in this direction.

Analysis Techniques

When a state-transition graph cannot be restricted or abstracted further, it may still be too large to be displayed or analysed by hand. In this case, we can interpret the graph as a data base and can employ search algorithms on this data to find paths which satisfy prescribed properties (Brajnik and Clancy 1996; Clancy 1997; Shults and Kuipers 1997). Temporal logic together with modal and standard predicate logic is a rich language to formulate expressions like “a path where $[\dot{x}_i] = [\dot{x}_j] = [-]$ after $[\dot{x}_k] = [+]$ is impossible”. An introduction to temporal logic is beyond the scope of this text. However, it should become clear that effective algorithms to test whether a state-transition graph fulfils given expressions of this kind can contribute to the understanding and interpretation of large QDEs.

In Chapter 3, I develop one novel abstraction and three new restriction methods which all fit into the scheme proposed in this section. The abstraction technique draws from basic concepts of viability theory, which will be introduced in section 2.4 (p. 45). The other techniques (i) exclude trajectories which represent unlikely developments in some specified sense, (ii) restricts monotonic landmark ensembles to models which satisfy a prescribed order on the coefficients of the Jacobian, and (iii) restricts them to models with quantitative interval bounds on the Jacobian. Before going on to this, I first introduce differential inclusions.

2.3 Differential Inclusions

In this section, basic concepts of differential inclusions are reviewed from the literature. This is a prerequisite for viability theory, which will be indispensable in Chapter 3, and provides a further elaborate example for a model ensemble (cf. section 2.1, p. 17). This contributes to the ongoing discussion about the relation of differential inclusions and QDEs. Differential inclusions are a generalisation of ordinary differential equations. An ODE assigns a single velocity to points in the state space, and is thus a special case of a differential inclusion, where *multiple* velocities can be assigned. We map a state x to a set of possible velocities $F(x)$, and admit a trajectory $x(\cdot)$ as a solution, if $\dot{x}(t)$ is always an element of $F(x(t))$. As in the case of QDEs we cannot generally expect to obtain unique solutions in such a setting, yielding a set-valued solution operator. The first ideas to this approach arose in the 30s of the last century (Zaremba 1936; Marchaud 1934), were the existence of continuous solutions was investigated under the terms of “contingent” or “paratingent” equations. Later, absolutely continuous solutions were considered (Ważewski 1961a; Ważewski 1961b). Filippov (1959) put differential inclusions into the context of optimal control. A broad overview to the fundamentals and subsequent development of the theory is provided by Aubin and Cellina (1984). One basic motivation – similar to QDEs – is to consider uncertainties which cannot be expressed in a probabilistic way. We may have an ODE $\dot{x} = f(x, t; u)$, depending on a parameter or a control u. If we do not know u exactly but can restrict the value, say, to an interval J such that $u \in J$, we obtain a set of possible values $F(x, t) := \{f(x, t; u) \mid u \in J\}$. Since the analysis of differential inclusions is more complicated than for ODEs, open research problems include accurate and efficient numerical schemes for state spaces of higher dimension and for differential inclusions with weak regularity properties. Differential inclusions are applied to problems from, e.g. population dynamics (Křivan and Colombo 1998; Guo et al. 2003), physics (Maisse and Pousin 1997), climate change (Chahma 2003), nonsmooth analysis (Clarke 1983), control theory (Leonov 2000; Lorenz 2005) and differential games (Chodun 1989; Ivanov and Polovinkin 1995).

We now present some basic definitions. Let X and Y be sets. A **set-valued map** $F : X \to \mathcal{P}(Y)$ is a map assigning to any $x \in X$ an element $F(x)$ of the power set of Y; we also write $F : X \rightsquigarrow Y, x \rightsquigarrow F(x)$. We denote the **domain** of F by $\mathrm{Dom}(F) := \{x \in X \mid F(x) \neq \varnothing\}$. A set-valued map F is called nontrivial if $\mathrm{Dom}(F) \neq \varnothing$. The **graph** of F is $\mathrm{Graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$. A standard example for set-valued maps are parameterised maps. We consider the sets X, Y and U and a (single-valued) map $f : X \times U \to Y$. Then $F(x) := \{f(x, u) \mid u \in U\}$ is called a set-valued map parameterised by U. There are several regularity concepts for set-valued maps which are helpful in practice, e.g. the following three. For a metric space X and $K \subseteq X$, the distance from $x \in X$ to K is defined by $d(x, K) := \inf_{y \in K} d(x, y)$, $d(x, \varnothing) = +\infty$, and $B(K, r) := \{x \in X \mid d(x, K) \leq r\}$ denotes the ball around K with radius $r > 0$.

DEFINITION 12: *Let $F : X \rightsquigarrow Y$ be a nontrivial set-valued map from a metric space X to a metric space Y. It is called*

1. ***Lipschitz*** *with constant $L \geq 0$ if*

$$\forall x, x' \in X : F(x) \subseteq B\big(F(x'), L d(x, x')\big).$$

2. ***Upper semicontinuous*** *at $x \in X$ if for any open neighbourhood $N \supseteq F(x)$ there exists a neighbourhood $M \supseteq x$ such that $F(M) \subseteq N$. It is called upper semicontinuous, if it is upper semicontinuous at any $x \in X$.*
3. ***Marchaud*** *if it is upper semicontinuous, has compact convex images and has linear growth, i.e. there exists a positive constant c such that*

$$\forall x \in \mathrm{Dom}(F) : \|F(x)\| \leq c(\|x\| + 1),$$

where $\|F(x)\| := \sup_{y \in F(x)} \|y\|$.

Set-valued maps which are parameterised by a closed set U and a (single-valued) continuous map f with linear growth such that $F(x)$ has convex values are always Marchaud (Aubin 1991, p. 203). It can be shown that Marchaud maps are characterised by nonempty and closed $\mathrm{Dom}(F)$, nonempty and closed $\mathrm{Graph}(F)$, convex values and linear growth (Aubin 2001).

For a given set-valued map $F : X \rightsquigarrow Y$ an “equation” of the form

$$\begin{aligned} \dot{x} &\in F(x), \\ x(0) &= x_0, \end{aligned} \tag{2.14}$$

is called a **differential inclusion**. What we define as solution of a differential inclusion depends on the chosen space of admissible trajectories. The situation is more complicated than for ODEs which have differentiable solutions if the right-hand side is continuous. As a standard case, we call an absolutely continuous function $x(\cdot) : J \to X$ on an interval $J = [0, T]$, possibly $T = \infty$ a **solution** of Eq. (2.14) if $x(0) = x_0$ and $\dot{x}(t) \in F(x(t))$ almost everywhere on J. For sake of simplicity we assume in the following that $X, Y \subseteq \mathbb{R}^n$, although results also hold for more general cases. There are various theorems on the existence of solutions to a differential inclusion (see e.g. Aubin 1991, p. 172).

PROPOSITION 6: *Let X be an open set, and $F : X \rightsquigarrow \mathbb{R}^n$ a set-valued map which is Lipschitz on X. Then, for any $x_0 \in X$ and $v_0 \in F(x_0)$, there exists a $T > 0$ and a solution $x(\cdot)$ to the differential inclusion Eq. (2.14) on $[0, T]$ such that $x(0) = x_0$ and $\dot{x}(0) = v_0$.*

How are differential inclusions related to model ensembles and causal loop diagrams? A Lipschitz map $F : X \rightsquigarrow \mathbb{R}^n$ and the resulting differential inclusion defines a model ensemble (cf. section 2.1, p. 17) by

$$\begin{aligned} \mathcal{M} := \{f : X \times \mathbb{R}_+ \to \mathbb{R}^n \mid & f(x,t) \text{ measurable with respect to } t \\ & \text{and } \forall t \in \mathbb{R}_+ : f(x,t) \in F(x)\}. \end{aligned}$$

Taking the set of absolutely continuous functions on intervals $J = [0, T]$ as space of admissible trajectories $\mathcal{E}$, we obtain a set-valued solution operator $\mathcal{S}_F(\cdot) : X \rightsquigarrow \mathcal{E}$,

$$\begin{aligned} \mathcal{S}_F(x_0) := \{x(\cdot) \in \mathcal{E} \mid & x(0) = x_0, \exists f \in \mathcal{M} : \\ & \dot{x}(t) = f(x(t), t) \text{ almost everywhere } \}, \end{aligned}$$

which assigns to an initial value $x_0 \in X$ the set of solutions of Eq. (2.14).

To find all possible trajectories which can be brought about by a causal loop diagram, one could start with a model ensemble $\mathcal{M}$ which contains all (autonomous) models consistent with the causal loop diagram (according to a concise interpretation of the diagram). We could then define a set-valued map by $F(x) := \{f(x) \mid f \in \mathcal{M}\}$ such that the solutions of the differential inclusion Eq. (2.14) describe all trajectories. However, if a causal loop diagram is specified by a sign matrix $\Sigma = (\sigma_{i,j}) \in \mathcal{A}_*^{n\times n}$ and a monotonic ensemble $\mathcal{M}(\Sigma)$, we run into trouble, as the following shows:

Suppose that $f \in \mathcal{M}(\Sigma)$. Since it follows from $\dot{x} = f(x)$ that $\ddot{x} = \mathcal{J}(f)(x) \cdot \dot{x}$, we obtain a second order differential inclusion in the joint state and velocity space:

$$\begin{aligned} \ddot{x} &\in F(\dot{x}, x), \\ F : (\dot{x}, x) &\rightsquigarrow \{\mathcal{J}(f)(x) \cdot \dot{x} \mid f \in \mathcal{M}(\Sigma)\}. \end{aligned}$$

This can be simplified to

$$\ddot{x} \in \hat{F}(\dot{x}) := \{A\dot{x} \mid [A] \approx \Sigma\},$$

where A denotes $n \times n$ matrices over the real numbers. We observe that the components $i = 1, \ldots, n$ of $\hat{F}(\dot{x})$ evaluate to

$$\hat{F}_i(\dot{x}) = \begin{cases} 0 & \text{if } \forall j = 1, \ldots, n : \dot{x}_j \cdot \sigma_{i,j} = 0, \\ \mathbb{R}_+ \setminus \{0\} & \text{else if } \forall j = 1, \ldots, n : [\dot{x}_j] = \sigma_{i,j} \neq 0 \text{ or } \dot{x}_j \cdot \sigma_{i,j} = 0, \\ \mathbb{R}_- \setminus \{0\} & \text{else if } \forall j = 1, \ldots, n : -[\dot{x}_j] = \sigma_{i,j} \neq 0 \text{ or } \dot{x}_j \cdot \sigma_{i,j} = 0, \\ \mathbb{R} & \text{otherwise .} \end{cases}$$

Except the trivial case, this unbounded set-valued map is not Lipschitz and not Marchaud. Also other well-known regularity concepts do not apply to $\hat{F}$. Thus, it cannot be expected that this simple approach provides valuable results. In section 3.4 (p. 77) I develop an alternative with better smoothness properties.

2.4 Viability Theory

Viability theory provides an elegant mathematical framework to consider normative settings in a model-driven analysis in a transparent way. Here, the basic definitions and results are recalled from the literature. They are the foundation for a new abstraction technique in section 3.1 (p. 52). I also outline the viability kernel algorithm, which is needed in section 3.4 (p. 77).

Viability theory considers whether prescribed state constraints can be satisfied by a set of trajectories. A viability constraint is a closed subset K of the state space X. It is assumed that in a state outside K the system is no longer viable. If a trajectory $x(\cdot)$ is given as a solution of an ODE with initial value x_0, and it remains in K forever, we call it viable. If we deal with a model ensemble, admitting multiple solutions starting from an initial state, the situation becomes more interesting: it may be that all solutions are viable, or at least one solution, or none. We can also formulate the control problem of how to steer a system such that it produces a viable solution. It turns out that these problems are also closely related to the reachability of a target set $C \subseteq X$, i.e. whether a viable trajectory reaches C in finite time. For ODEs, the first characterisation of viability by Nagumo (1942) has been forgotten and rediscovered several times. Later it was extended to differential inclusions (Yorke 1969; Gautier 1976; Aubin et al. 1977; Haddad 1981). A very comprehensive introduction is provided by Aubin (1991). The strength of viability theory is that it provides a consistent framework to conceptualise and analyse non-deterministic or uncertain dynamics under state constraints. By concentrating on initial states which admit viable solutions, questions about dynamics are reduced to geometrical considerations. This is in particular valuable to investigate complex control problems. However, as for differential inclusions, numerical schemes for viability theory are still difficult in state spaces of higher dimensions. Fields of applications include economics (e.g. Aubin et al. 2001; Aubin et al. 2005), engineering (e.g. Seube et al. 2002), population dynamics (e.g. Bonneuil and Müllers 1997; Bonneuil and Saint-Pierre 2005), and sustainability science (Petschel-Held et al. 1999; Bene et al. 2001; Eisenack et al. 2006; Cury et al. 2005).

Normative settings, which are important for sustainability issues, can be formalised by defining viability constraints as sets which subsume preferable or problematic states of the system under investigation. For example, it may be relevant to know whether there are trajectories reaching a problematic set from an initial value in finite time. In the “hopeless” case, all trajectories will become non-viable (and thus the system has to be altered structurally to make it sustainable). In the “foolproof” case, all trajectories will remain viable forever. Then, the system is on the safe side although we have to face the uncertainty or the generality expressed by the model. A third possibility is the “critical” case with both viable and non-viable solutions – the “fate” of the system depends on decisions (for a control problem) or on unknown system properties. This approach differs from usual criteria like optimality, which force the modeller to specify unique solutions. Traditionally, viability theory is based on differential inclusions (and difference inclusions for numerical schemes). In Chapter 3 we will see how qualitative reasoning profits from this approach, since we can apply similar concepts to the state-transition graph of a QDE.

2.4.1 The Viability Theorem

To define the basic objects of this section, the viability and invariance kernel, let $K \subseteq X$ be a subset of the the state space called a **constrained set**. For simplicity we assume that $X \subseteq \mathbb{R}^n$, although most results hold for more general cases. A trajectory $x(\cdot) : J \to X$, $J = [0, T]$ (possibly with $T = \infty$) which remains in K, i.e. $\forall t \in J : x(t) \in K$, is called **viable** in K on J. If multiple trajectories start from a given initial value $x_0 \in K$, we ask whether *all* trajectories with $x(0) = x_0$, or whether *at least one* trajectory has this property. These questions are further elaborated by considering a target $C \subseteq K$, which we prefer to reach with a viable trajectory in finite time.

DEFINITION 13: *Let F be a set-valued map on the state space X, defining a differential inclusion and the set-valued solution operator $\mathcal{S}_F(\cdot)$. For $C \subseteq K \subseteq X$ the*

1. ***Viability kernel** of K with target C, denoted by $\mathrm{Viab}_F(K, C)$, is the set of all $x_0 \in K$ such that*
$$\begin{aligned}&\exists\, x(\cdot) \in \mathcal{S}_F(x_0)\\ &\qquad \forall\, t \geq 0 : x(t) \in K\\ &\qquad \textit{or}\ \exists\, T > 0 : x(T) \in C\ \textit{and}\ \forall\, t \in [0, T] : x(t) \in K.\end{aligned}$$
2. ***Invariance kernel** of K with target C, denoted by $\mathrm{Inv}_F(K, C)$, is the set of all $x_0 \in K$ such that*
$$\begin{aligned}&\forall\, x(\cdot) \in \mathcal{S}_F(x_0)\\ &\qquad \forall\, t \geq 0 : x(t) \in K\\ &\qquad \textit{or}\ \exists\, T > 0 : x(T) \in C\ \textit{and}\ \forall\, t \in [0, T] : x(t) \in K.\end{aligned}$$
3. ***Capture basin** $\mathrm{Capt}_F(K, C)$ is the set of all $x_0 \in K$ such that $\exists\, x(\cdot) \in \mathcal{S}_F(x_0)$ for which $\exists\, T > 0 : x(T) \in C$ and $\forall\, t \in [0, T] : x(t) \in K$.*
4. ***Absorption basin** $\mathrm{Abs}_F(K, C)$ is the set of all $x_0 \in K$ such that $\forall\, x(\cdot) \in \mathcal{S}_F(x_0)$ there $\exists\, T > 0 : x(T) \in C$ and $\forall\, t \in [0, T] : x(t) \in K$.*

If the set-valued map F is clear from the context, it is usually omitted as a subscript. As important examples $\mathrm{Viab}_F(K, \varnothing)$ contains the initial states from which at least one viable solution starts, whereas $\mathrm{Inv}_F(K, \varnothing)$ contains the initial states for which all trajectories are viable. These important cases are simply called the viability kernel of K, $\mathrm{Viab}_F(K)$, or invariance kernel of K, $\mathrm{Inv}_F(K)$. We can thus define the following types of sets which we aim to characterise geometrically by the viability theorem.

DEFINITION 14: *As set K is **viable** if $K = \mathrm{Viab}_F(K)$, i.e.*

$$\forall\, x_0 \in K\, \exists\, x(\cdot) \in \mathcal{S}_F(x_0)\, \forall\, t \geq 0 : x(t) \in K.$$

*It is **invariant** if $K = \mathrm{Inv}_F(K)$, i.e.*

$$\forall\, x_0 \in K, x(\cdot) \in \mathcal{S}_F(x_0), t \geq 0 : x(t) \in K.$$

*It is **locally invariant** if*

$$\forall\, x_0 \in K, x(\cdot) \in \mathcal{S}_F(x_0)\, \exists\, T \geq 0\, \forall\, t \in [0, T] : x(t) \in K.$$

These concepts can also be translated to difference inclusions $x_{i+1} \in F(x_i), F : X \rightsquigarrow X$, where the solution operator $\mathcal{S}_F(x) := \{(x_i) \mid x_0 = x, \forall i \in \mathbb{N} : x_{i+1} \in F(x_i)\}$ contains all sequences of states which start from x and evolve according to the set-valued map F. This discrete version is needed for the viability kernel algorithm which is presented in the next subsection.

DEFINITION 15: *A sequence $(x_i) \in \mathcal{S}_F(x_0)$ is said to be **viable** in $K \subseteq X$ if $\forall i \in \mathbb{N} : x_i \in K$. The **discrete viability kernel** of K with target $C \subseteq K$, denoted by $\mathrm{Viab}_F(K, C)$, is the set of all x_0 such that there exists at least one sequence $(x_i) \in \mathcal{S}_F(x_0)$ which is viable in K or for which there is an $N \in \mathbb{N}$ with $x_N \in C$ and $\forall i \leq N : x_i \in K$. Again, we set $\mathrm{Viab}_F(K) := \mathrm{Viab}_F(K, \varnothing)$.*

We are interested in the structure of viable sets, and how it depends on F and the constrained set K. There are various such characterisations, and a simple one for discrete viability kernels:

PROPOSITION 7: *The discrete viability kernel $\mathrm{Viab}_F(K, C)$ is the largest closed set $D \subseteq K$ such that*

$$\forall x \in D \setminus C : F(x) \cap D \neq \varnothing.$$

In the viability and invariance theorem for differential inclusions, viable sets are characterised by tangential conditions. To state them, we need another definition:

DEFINITION 16: *Let $K \subseteq X$ be nonempty and $x \in K$. The **contingent cone** to K at x is the set*

$$T_K(x) := \Big\{ v \in \mathbb{R}^n \mid \liminf_{h \to 0+} \frac{d(x + hv, K)}{h} = 0 \Big\}.$$

The contingent cone contains the directions which point into K from a given point x in some sense. If x is on the boundary of a smooth manifold K, it is identical to the tangent space to K at x. For $x \in \mathrm{Int}(K)$ we have $T_K(x) = \mathbb{R}^n$, and if K is a singleton, $T_K(x) = 0$. If K is an open set, the contingent cone is always the whole space. The above definition also works for nonsmooth K. Sets with the following properties are viable or invariant, under certain regularity assumptions for F.

DEFINITION 17: *Let $F : X \rightsquigarrow X$ be a nontrivial set-valued map. We call a set $K \subseteq \mathrm{Dom}(F)$ a **viability domain** of F if*

$$\forall x \in K : F(x) \cap T_K(x) \neq \varnothing,$$

*and an **invariance domain** of F if*

$$\forall x \in K : F(x) \subseteq T_K(x).$$

Consequently, every open set is a viability and an invariance domain. The viability theorem states the following existence result (cf. Aubin 1991, p. 91):

PROPOSITION 8: *Let F be a Marchaud map. A closed set $K \subseteq \mathrm{Dom}(F)$ is viable iff K is a viability domain.*

The invariance theorem holds for Lipschitz maps (cf. Aubin 1991, p. 173).

PROPOSITION 9: *Let F be Lipschitz on* $\mathrm{Int}(\mathrm{Dom}(K))$ *with compact values. A closed set* $K \subseteq \mathrm{Dom}(F)$ *is locally invariant iff K is an invariance domain.*

Interestingly, viability and invariance kernels are maximal closed viability domains and invariance domains contained in the constrained set (Aubin 2001).

PROPOSITION 10: *If F is Marchaud, K and $C \subseteq K$ are closed, the viability kernel* $\mathrm{Viab}_F(K, C)$ *of K with target C is the largest closed subset D with $C \subseteq D \subseteq K$ satisfying*

$$\forall x \in D \setminus C : F(x) \cap T_D(x) \neq \varnothing.$$

PROPOSITION 11: *If F is Lipschitz, K and $C \subseteq K$ are closed, the invariance kernel* $\mathrm{Inv}_F(K, C)$ *of K with target C is the largest closed subset D with $C \subseteq D \subseteq K$ satisfying*

$$\forall x \in D \setminus C : F(x) \subseteq T_D(x).$$

2.4.2 The Viability Kernel Algorithm

Viability and invariance kernels as well as capture and absorption basins of a broad class of differential inclusions can be numerically approximated by the viability kernel algorithm (Frankowska and Quincampoix 1991; Cardaliaguet et al. 1994; Saint-Pierre 1994; Quincampoix and Saint-Pierre 1995; Cardaliaguet et al. 1999). It can be described in two steps. At first, time is discretised by replacing the differential inclusion by an appropriate difference inclusion (the semi-discrete scheme). Then, the fully discrete scheme introduces a grid for the state space and makes a further modification of the difference inclusion necessary. It can be shown that by refining the grid, the associated discrete viability kernel converges towards the viability kernel of the underlying differential inclusion. There are specialised numerical schemes for different types of differential inclusions and some methods to improve computation efficiency, e.g. the refinement principle which avoids re-computation of grid points in subsequent iterations. Some basic results from Cardaliaguet et al. (1999) are laid out in the following paragraphs. For a difference inclusion $x_{i+1} \in G(x_i)$ on the state space X, the discrete viability kernel (cf. DEF. 15) can be approximated as follows:

PROPOSITION 12: *Let $G : X \rightsquigarrow X$ be an upper semicontinuous map with compact, nonempty values, and $K \subseteq X$ be closed. Define a decreasing (with respect to inclusion) sequence of closed sets (K_i) by*

$$\begin{aligned} K_0 &:= K \\ K_{i+1} &:= \{x \in K_i \mid G(x) \cap K_i \neq \varnothing\}. \end{aligned}$$

Then, the discrete viability kernel is

$$\mathrm{Viab}_G(K) = \bigcap_{i=0}^{\infty} K_i.$$

This is not sufficient for practical computation, because determining K_i requires quantifying over an infinite set. Moreover, to compute the viability kernel of a *differential* inclusion, we have to relate it to a discretisation by an appropriate *difference* inclusion.

Let a differential inclusion be given by a Marchaud map $F : X \rightsquigarrow Y$ which is bounded by M, i.e. $\forall x \in X, y \in F(x) : \|y\| \leq M$. Choose a family of set-valued maps $F_\epsilon : X \rightsquigarrow Y$ such that the following regularity assumptions are satisfied:

(R1) Each F_ϵ is upper semicontinuous with compact, convex and non-empty values,

(R2) $\mathrm{Graph}(F_\epsilon) \subseteq B(\mathrm{Graph}(F), \phi(\epsilon))$ with $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\epsilon\to 0} \phi(\epsilon) = 0$,

(R3) $\forall x \in X : \bigcup_{\|y-x\|\leq M\epsilon} F(y) \subseteq F_\epsilon(x)$.

The discrete viability kernel with respect to a certain family of maps G_ϵ converges to the viability kernel with respect to F (see Cardaliaguet et al. 1999):

PROPOSITION 13: *Let $F : X \rightsquigarrow Y$ be a bounded Marchaud map, the family $F_\epsilon : X \rightsquigarrow Y$ satisfy (R0) – (R2), and define $G_\epsilon(x_i) := x_i + \epsilon F_\epsilon(x_i)$. Then,*

$$\operatorname*{Lim}_{\epsilon\to 0} \mathrm{Viab}_{G_\epsilon}(K) = \mathrm{Viab}_F(K).$$

Here, Lim denotes the Painléve-Kuratowski limit for sets which is defined as follows. If $A : \mathbb{R}_+ \rightsquigarrow X$ is a map assigning to each $\epsilon > 0$ a subset $A(\epsilon)$ of X, the upper limit

$$\operatorname*{Limsup}_{\epsilon\to 0} A(\epsilon) := \{x \in X \mid \liminf_{\epsilon\to 0} d(x, A(\epsilon)) = 0\},$$

and the lower limit

$$\operatorname*{Liminf}_{\epsilon\to 0} A(\epsilon) := \{x \in X \mid \lim_{\epsilon\to 0} d(x, A(\epsilon)) = 0\}.$$

A subset $K \subseteq X$ is the Painléve-Kuratowski limit $\operatorname*{Lim}_{\epsilon\to 0} A(\epsilon)$ if

$$K = \operatorname*{Liminf}_{\epsilon\to 0} A(\epsilon) = \operatorname*{Limsup}_{\epsilon\to 0} A(\epsilon).$$

EXAMPLE 7: If F is a Marchaud map bounded by M and Lipschitz with constant L,

$$F_\epsilon(x) := B(F(x), ML\epsilon)$$

satisfies the regularity assumptions (R1)–(R3) (Cardaliaguet et al. 1999). □

The next step is the discretisation of the state space. For $h \in \mathbb{R}_+$ we introduce a grid $X^h \subseteq X$ such that for any compact $K \subseteq X$, the intersection $K \cap X^h$ is finite and $\forall\, x \in X \,\exists\, x^h \in X^h : \|x - x^h\| \leq h$. For convenience, we define $K^h := B(K, h) \cap X^h$. On this grid, the discrete viability kernel of a difference inclusion $x_{i+1} \in H(x_i)$ can be computed in a *finite* number of steps (in contrast to PROP. 12, p. 48):

PROPOSITION 14: *Let $H : X^h \rightsquigarrow X^h$ be a set-valued map with finite, nonempty values, and K^h be finite. Define a decreasing sequence of sets (K_i^h) by*

$$\begin{aligned} K_0^h &:= K^h \\ K_{i+1}^h &:= \{x \in K_i^h \mid H(x) \cap K_i^h \neq \varnothing\}. \end{aligned}$$

Then there exists an $N \in \mathbb{N}$ such that the discrete viability kernel is

$$\mathrm{Viab}_H(K^h) = K_N^h.$$

Again, an appropriate family of maps $H_{\epsilon,h}$ can be constructed which approximates $\mathrm{Viab}_F(K)$ when the grid width h and the time step ϵ is refined. Let F be a bounded Marchaud map as before, F_ϵ a set-valued map satisfying (R1)–(R3), and $G_\epsilon(x_i) := x_i + \epsilon F_\epsilon(x_i)$. Then choose $H_{\epsilon,h}$ such that the following additional regularity assumptions hold:

(R4) $\mathrm{Graph}(H_{\epsilon,h}) \subseteq B(\mathrm{Graph}(G_\epsilon), \psi(\epsilon,h))$ with $\psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that $\frac{\psi(\epsilon,h)}{h} \to 0$ for $\epsilon \to 0$ and $\frac{h}{\epsilon} \to 0$.

(R5) $\forall x^h \in X^h : \bigcup_{||y-x^h|| \le h} B(G_\epsilon(y), h) \cap X^h \subseteq H_{\epsilon,h}(x^h)$.

EXAMPLE 8: When F is a Marchaud map, bounded by M and Lipschitz with constant L,

$$H_{\epsilon,h}(x^h) := B(x^h + \epsilon F_\epsilon(x^h), 2h + Lh\epsilon + ML\epsilon^2) \cap X_h$$

satisfies the regularity assumptions (R4) and (R5) (Cardaliaguet et al. 1999). □

We now end up with the fully discrete scheme, implicitly including an estimate for the approximation error (see Cardaliaguet et al. 1999).

PROPOSITION 15: *Let $F : X \rightsquigarrow Y$ be a bounded Marchaud map. If $H_{\epsilon,h}$ satisfy (R1) – (R5), then*

$$\mathrm{Viab}_F(K) \subseteq B(\mathrm{Viab}_{H_{\epsilon,h}}(K^h), h),$$

and

$$\operatorname*{Lim}_{\substack{\epsilon \to 0, \\ \frac{h}{\epsilon} \to 0}} \mathrm{Viab}_{H_{\epsilon,h}}(K^h) = \mathrm{Viab}_F(K).$$

Chapter 3

Abstraction and Restriction Techniques

The number of vertices of a state-transition graph of a larger QDE tends to explode as discussed in section 2.2.4 (p. 36). In this chapter I present several new methods to improve reasoning with such model ensembles. I define meaningful abstractions and subgraphs of the state-transition graph and restrict the model ensemble by including additional assumptions which cannot be expressed by a monotonic landmark ensemble. I develop algorithms to use these methods in practice.

For the first strategy the generic graph theoretical definition of abstraction techniques as supplied in section 2.2.4 is used. Graph theoretical versions of viable and invariant sets as introduced in section 2.4 (p. 45) are formulated to define subgraphs of interest. The so called no-return set appears as a new and useful concept which is associated with the notion of irreversibility. It turns out that there is a close relationship to strongly connected components, a well-known concept in graph theory. This is helpful to derive appropriate algorithms and to clarify the structure of a no-return abstraction. It can be tested in which regions of the qualitative state space the model respects certain specifications, even if the state-transition graph is too large to be visualised effectively. Then I analyse how the new abstraction technique can be combined with established ones.

For the second strategy, I posit restrictions of the space of admissible trajectories $\mathcal{E}$ and the model ensemble $\mathcal{M}$ in the sense defined in section 2.1 (p. 17). This idea is a guiding principle for the rest of the chapter. At first, $\mathcal{E}$ is restricted, which results in the elimination of edges representing unprobable marginal cases. Then, $\mathcal{M}$ is restricted by assumptions which cannot be expressed as a monotonic landmark ensemble, but still keep the model ensemble infinite, covering a broad set of functional relationships: in addition to requiring a sign matrix $\Sigma \approx [\mathcal{J}(f(x))]$, knowledge on the order of the coefficients of the Jacobian is used in the sense that for indices i, j, k, l assumptions hold like $D_i f_j(x) > D_k f_l(x)$. Finally, I further restrict $\mathcal{M}$ by prescribing quantitative intervals $u_{j,i}$ such that $D_i f_j(x) \in u_{j,i}$. Here, QDEs are combined with differential inclusions (cf. section 2.3, p. 42). The concept of an absorption basin and the viability algorithm (cf. section 2.4, p. 45) are central to determine the conditions under which one qualitative state is the successor of another one.

3.1 No-Return Abstraction

Abstraction techniques can be used to simplify a large state-transition graph. Recall that an abstracted state-transition graph G is determined from a family of disjoined subgraphs which cover G. Two important examples, chatter-box abstraction and projection, were introduced in section 2.2.4 (p. 36). The disjoined subgraphs can be displayed as clusters in G, or as a new graph H where each vertex corresponds to a subgraph, and edges are inherited from G.

In this section I introduce several new types of subgraphs which are meaningful in the context of sustainability science. Simply speaking, all of them are related to the notion of irreversibility in some sense: For example, if a region of the state or velocity space of a system is valued as problematic, the modeller wants to know whether the systems persists there once it is attained. Problems of this kind are closely related to the concepts from viability theory as introduced in section 2.4 (p. 45).

The basic ideas of this work appeared in Eisenack and Petschel-Held (2002), and are extended in this thesis. The main challenge is that the characterisation of the subgraphs exhibiting irreversible structures is not sufficient for their use as abstraction techniques – usually, they are not disjoined. Thus, further relations between these structures have to be investigated to develop an applicable method, the no-return abstraction. I show that it can be computed by combining well-known algorithms from graph theory related to reachability and connected components (e.g. Behzad et al. 1979). Finally, I explore how this new technique interferes with projection and chatter-box abstraction.

3.1.1 Characterisation of Subgraphs

Irreversibility is closely related to reachability: a system shifts from one region of the state space to another irreversibly if the former region cannot be reached from the latter. The system resides in a region forever if no state outside this region is reachable from inside it. In the following, G always denotes a state-transition graph. Reachability of a state $w \in V(G)$ from another state $v \in V(G)$ is expressed as the existence of a path $v, \dots, w$. Hence, the basic tool in this section is the set-valued successor map $\Gamma : V(G) \rightsquigarrow V(G), v \rightsquigarrow \Gamma(v)$. By $\Gamma^*(v)$ we denote the set of successors of vertex v in the transitive closure G^* of G. The vertex $w \in \Gamma^*(v)$ if and only if there is a path $v, \dots, w$. Recall that a state-transition graph is loop free (cf. section 2.2, p. 20), and therefore it is not generally true that $v \in \Gamma^*(v)$.

The relation between paths in G and solutions of a monotonic landmark ensemble $\mathcal{M}(\mu, C)$ should be kept in mind here. As shown in section 2.2 (p. 20), the existence of a path in G with length greater than two does not imply the existence of a solution of $\mathcal{M}(\mu, C)$ which has this path as landmark abstraction. On the other hand we know that if w is *not* reachable from v, i.e. $w \notin \Gamma^*(v)$, there is *no* corresponding solution.

We now introduce the basic types of sets $D \subseteq V(G)$ which will be investigated in this section. Let G be a state-transition graph of a monotonic landmark ensemble $\mathcal{M}(\mu, C)$.

DEFINITION 18: *A set $D \subseteq V(G)$ is*

1. ***Viable**, if for all $v_0 \in D$*

$$\exists \textit{ path } v_0, \ldots, v_i, \ldots \textit{ in } G\, \forall i \geq 0 : v_i \in D$$
$$\textit{or } \exists \textit{ path } v_0, \ldots, v_m \textit{ in } G : \Gamma(v_m) = \varnothing \textit{ and } \forall i = 0, \ldots, m : v_i \in D.$$

2. ***Invariant**, if for all $v_0 \in D$*

$$\forall \textit{ paths } v_0, \ldots, v_i, \ldots \textit{ in } G, i \geq 0 : v_i \in D.$$

3. ***No-return**, if for all $v_0 \in D$*

$$\forall \textit{ paths } v_0, \ldots, v_m, \ldots, v_i \textit{ in } G \textit{ with } v_m \in D^c, i \geq m : v_i \in D^c.$$

In a viable set a path starts from every vertex which remains in the set. Invariant sets correspond to regions in the phase space which cannot be left once they are entered, and no-return sets cannot be re-entered once they have been left. In the context of sustainability science, invariant sets correspond to robust facts under uncertainty or generality. Since there is no edge leaving an invariant set, no model of the ensemble $\mathcal{M}(\mu, C)$ has a solution leaving the associated region in the state and velocity space (i.e. every ODE with a Marchaud-map $f \in \mathcal{M}(\mu, C)$ as right-hand side has an invariance domain). In contrast, no-return sets correspond to a fragile configuration of states and velocities: Since there is no re-entering path, no solution of $\mathcal{M}(\mu, C)$ re-enters the region. A negative consequence holds for viable sets. If D is *not* viable, there are vertices in D where all successors are outside D, i.e. there is a region in the state and velocity space where any solution of $\mathcal{M}(\mu, C)$ (supposing it does not have a constant qualitative value) necessarily leaves this region – a problematic situation if such a region is valued as positive.

To find such sets in the state transition graph and to improve our understanding of these concepts, we provide further characterisation in the following propositions. They also make the connection to viability theory more clear and prepare for the development of efficient algorithms.

PROPOSITION 16: *A set $D \subseteq V(G)$ is viable iff the following criterion holds:*

$$\forall v \in D : \Gamma(v) \cap D \neq \varnothing \textit{ or } \Gamma(v) = \varnothing.$$

PROOF: First, choose $v \in D$. If $\Gamma(v) = \varnothing$, the criterion obviously holds. Otherwise, there is a path $v, w, \ldots$ in D, since D is viable. Thus, $w \in \Gamma(v) \cap D$.

Now, let D fulfil the criterion and choose $v_0 \in D$. If $\Gamma(v_0) = \varnothing$, it is obviously an element of a viable set. Otherwise, we can choose one $v_1 \in \Gamma(v_0)$ with $v_1 \in D$. This can be repeated infinitely or until some $\Gamma(v_m) = \varnothing$. Thus, D is a viable set. □

As discussed above, viable sets can only be used in a negative sense. In practice large state-transition graphs have viable sets comprising most vertices of G, making this structure not very distinctive. Viable sets were introduced here for sake of completeness and will not be considered further. This could be different if strong restriction techniques to eliminate edges are used before computing viable sets. In contrast, invariant sets, which are characterised now, will be more helpful.

PROPOSITION 17: *If $D \subseteq V(G)$ is an invariant set, this is equivalent to each of the following conditions:*

(i) $\forall v \in V(G) : \Gamma(v) \subseteq D$

(ii) $\phi(D) = D$*, where* $\phi : V(G) \rightsquigarrow V(G), v \rightsquigarrow \phi(v) := \Gamma^*(v) \cup \{v\}$.

Moreover, the set-valued map ϕ has the following properties:

1. *For all $D \subseteq V(G) : D \subseteq \phi(D)$, i.e. ϕ is* ***extensive****.*
2. *For all $C, D \subseteq V(G), C \subseteq D : \phi(C) \subseteq \phi(D)$, i.e. ϕ is* ***monotone****.*
3. *For all $D \subseteq V(G) : \phi(\phi(D)) = \phi(D)$, i.e. ϕ is* ***idempotent****.*

The properties of ϕ will be useful for various proofs in this section. As G is loop free, it holds for all $v \in V(G)$ that $v \notin \Gamma(v)$, but $v \in \phi(v)$. If $w \in \phi(v)$ then $w = v$ or there is a path from v to w in G.

PROOF: We begin with the properties of ϕ.
The map is extensive by definition.
If $C \subseteq D$, choose $w \in \Gamma^*(C)$. Then there is a path $v, \ldots, w$ with $v \in C$. Since also $v \in D$, we have $w \in \Gamma^*(D)$, and ϕ is monotone.
It holds for every set-valued map $F : X \rightsquigarrow X$ and $A, B \subseteq X$ that $F(A \cup B) = F(A) \cup F(B)$. For the successor map Γ^* it holds that $\Gamma^*(\Gamma^*(D)) \subseteq \Gamma^*(D)$, since it operates on the transitive closure of G. Thus,

$$\begin{aligned}\phi(\phi(D)) &= \phi(\Gamma^*(D) \cup D) = \Gamma^*\big(\Gamma^*(D) \cup D\big) \cup \big(\Gamma^*(D) \cup D\big) \\ &= \Gamma^*(\Gamma^*(D)) \cup \Gamma^*(D) \cup \Gamma^*(D) \cup D = \Gamma^*(D) \cup D = \phi(D),\end{aligned}$$

and ϕ is idempotent.

Equivalence to condition (i): Choose $v \in D$, D invariant. If $\Gamma(v) = \varnothing$, the condition is met. Otherwise, select an arbitrary $w \in \Gamma(v)$. Since D is invariant, it holds that $w \in D$, i.e. $\Gamma(v) \subseteq D$.
Let D fulfil the condition. If some $v_i \in D$, then all $v_{i+1} \in \Gamma(v_i) \subseteq D$. Thus, all paths starting from an $v_0 \in D$ remain in this set – it is invariant.

Equivalence to condition (ii): Let D be invariant. Since ϕ is expansive, only $\phi(D) \subseteq D$ has to be shown. If we choose $w \in \phi(D)$, there is a path $v, \ldots, w$ with $v \in D$. Due to invariance, also $w \in D$.
Let $D = \phi(D)$, take $v_0 \in D$ and an arbitrary path $v_0, \ldots, v_i, \ldots$ in G. Since $\forall i \geq 0 : v_i \in \phi(v_0) \subseteq \phi(D) = D$, the set is invariant. □

The notion of an invariant set can be used to further describe no-return sets (a further relation between both types of sets will be shown below):

PROPOSITION 18: *A set $D \subseteq V(G)$ is a no-return set iff $\phi(D) \cap D^c$ is invariant.*

PROOF: First, let D be a no-return set. Choose $v_m \in \phi(D) \cap D^c$. Then there is a path $v_0, \dots, v_m$ in G with $v_0 \in D$. Since $v_m \notin D$ and D is a no-return set, it holds for every continued path $v_0, \dots, v_m, \dots, v_i$ that $\forall i \geq m : v_i \in D^c$. Consequently, $v_i \in \phi(D) \cap D^c$, such that $\phi(D) \cap D^c$ is invariant.

Now, let $\phi(D) \cap D^c$ be invariant. Choose $v_0 \in D$. If there is no path leaving D, it is already a no-return set. Otherwise, there is some $v_m \in \phi(D) \cap D^c$ with $v_0, \dots, v_m$ a path in G. Then, due to invariance, it holds for all continued paths $v_0, \dots, v_m, \dots, v_i, \dots$ that $\forall i \geq m : v_i \in \phi(D) \cap D^c \subseteq D^c$, and D is a no-return set. □

3.1.2 Computing Invariant Sets and the No-Return Abstraction

Invariant sets need to be computed and displayed efficiently if they are to be exploited in applications from sustainability science. Moreover, we would like to integrate them into the generic definition of abstraction techniques as introduced in section 2.2.4 (p. 36) – requiring a disjoined family of subgraphs. In this section we will see important obstacles to this task – although the family of all invariant sets of a state-transition graph has a very regular structure. I show that no-return sets and connected components play the decisive role to overcome them.

At first we investigate the structure of the family of all invariant sets of a state-transition graph. Recall that for an arbitrary set X, a family of sets $\mathcal{L} \subseteq \mathcal{P}(X)$, ordered by inclusion $\subseteq$, is a **set lattice** if for all $A, B \in \mathcal{L}$ the supremum $A \cup B \in \mathcal{L}$ and the infimum $A \cap B \in \mathcal{L}$ (see, e.g. Davey and Priestley 1990).

PROPOSITION 19: *The family $\mathcal{L}$ of all invariant sets of G is a finite set lattice.*

PROOF: Let $C, D \in \mathcal{L}$. Since C and D are invariant, PROP. 17 yields

$$\phi(C \cup D) = \phi(C) \cup \phi(D) = C \cup D,$$

such that also $C \cup D$ is invariant by PROP. 17.
It holds for every set-valued map $F : X \rightsquigarrow X$ and $A, B \subseteq X$ that $F(A \cap B) \subseteq F(A) \cap F(B)$. Since C and D are invariant and ϕ is extensive, we can use PROP. 17 (p. 54) again to see that

$$C \cap D \subseteq \phi(C \cap D) \subseteq \phi(C) \cap \phi(D) = C \cap D,$$

and $C \cap D$ is invariant. □

Furthermore, this lattice can be constructed from simple invariant sets of the form $\phi(v)$ which are the atoms of $\mathcal{L}$:

PROPOSITION 20: *For all $v \in V(G)$, $\phi(v)$ is an invariant set.*
For all invariant sets D of G,

$$D = \bigcup_{v \in D} \phi(v).$$

PROOF: The set $\phi(v)$ is invariant by PROP. 17 (p. 54) since ϕ is idempotent.
By definition, $v \in \phi(v)$, and thus $D \subseteq \bigcup_{v \in D} \phi(v)$.
Now choose $w \in \bigcup_{v \in D} \phi(v)$. Then there is a $v \in D : w \in \phi(v)$, i.e. $w = v$ or there is a path $v, \ldots, w$. Since D is invariant, also $w \in D$, and $\bigcup_{v \in D} \phi(v) \subseteq D$. □

This situation seems comfortable at first, as ϕ can be easily obtained from the transitive closure of G for which various efficient algorithms are well-known. However, the sets $\phi(v)$ have a nested structure:

PROPOSITION 21: *Let $v, w \in V(G)$. Then $\phi(w) \subseteq \phi(v)$ iff $v = w$ or there is a path $v, \ldots, w$ in G.*

PROOF: If $w \in \phi(w) \subseteq \phi(v)$, then $w = v$ or there is a path $v, \ldots, w$.
Suppose there is a path $v, \ldots, w$, then $w \in \phi(v)$. Since ϕ is monotone and idempotent, we conclude that $\phi(w) \subseteq \phi(\phi(v)) = \phi(v)$. □

This makes it impossible to compute an abstracted state-transition graph (cf. DEF. 8, p. 37) directly from the invariant sets, since this needs disjoined subgraphs. To display all invariant sets of a state-transition graph, we would lose the overview due to the variety of nested clusters. Moreover, invariant sets need not be connected, e.g. if there is no path between $v, w \in V(G)$ and $D = \phi(v) \cup \phi(w)$. The basic idea for the solution of this problem lies in the following proposition which draws a new connection between invariant sets and no-return sets. I will go on to show that there is a "basis" of disjoined no-return sets from which all invariant sets can be constructed in a unique way.

PROPOSITION 22: *Let $D \in \mathcal{L}$ be an invariant set, and $D_1, \ldots, D_n \in \mathcal{L}$ the family of all invariant sets D_j with $D_j \subsetneqq D$. Then,*

$$B_D := D \setminus \bigcup_{j=1,\ldots,n} D_j$$

is a no-return set.

PROOF: Choose $v_0 \in B_D$ and a path $v_0, \ldots, v_m, \ldots, v_i, \ldots$ with $v_m \notin B_D$ (if there is no such path, B_D is a no-return set obviously). Since also $v_0 \in D$, due to invariance $v_m \in D$, and therefore $v_m \in \bigcup_{j=1,\ldots,n} D_j$, i.e. there is one j such that $v_m \in D_j$. Since D_j is invariant, $\forall i \geq m : v_i \in D_j \nsubseteq B_D$. Therefore, B_D is a no-return set. □

In other words, we obtain a no-return set by taking an invariant set and eliminating all included invariant sets. When this proposition is applied to the basic invariant sets $\phi(v)$, we obtain a family of no-return sets with stronger properties as building blocks for the no-return abstraction:

PROPOSITION 23: *For $v \in V(G)$ define*

$$B_v := \phi(v) \setminus \bigcup_{\substack{D_j \in \mathcal{L} \\ D_j \subsetneqq \phi(v)}} D_j.$$

Then, $v \in B_v$ and B_v is a single vertex or is strongly connected.

PROOF: Suppose, that $v \notin B_v$. Since $v \in \phi(v)$, there would be a $D_j \in \mathcal{L}$ with $D_j \subsetneqq \phi(v)$ and $v \in D_j$. Hence, since ϕ is monotone and D_j is invariant, this would imply that

$$\phi(v) \subseteq \phi(D_j) = D_j \subsetneqq \phi(v),$$

which is a contradiction. Therefore, $v \in B_v$.

If $|B_v| \geq 2$, take $w \in B_v, w \neq v$. As w is also an element of $\phi(v)$, there is a path $v, \ldots, w$. Supposing there is no path $w, \ldots, v$, then from PROP. 21 (p. 56) we yield the contradiction that $\phi(v) \nsubseteq \phi(w) \subseteq \phi(v)$. Thus, B_v is strongly connected. □

Taken with PROP. 22, it follows that all B_v consisting of more than one vertex are strongly connected no-return sets. The following proposition guarantees that they can be computed by standard algorithms from graph theory to detect strongly connected components (e.g. van Leeuwen 1990, p. 571). It is also another characterisation of no-return sets.

PROPOSITION 24: *A set $D \subseteq V(G)$ of a graph G is a strongly connected no-return set if and only if it is a strongly connected component.*

PROOF: Assume that D is a strongly connected no-return set. If it were not a strongly connected component, there would be a path $v, \ldots, w$ with $v \in D, w \notin D$, and a path $w, \ldots, v$. The latter is a contradiction since D is a no-return set.

If D is a strongly connected component, we have to show that D is a no-return set. Choose $v_0 \in D$ and a path $v_0, \ldots, v_m, \ldots, v_i$ with $v_m \notin D$. Suppose D is a no-return set. Then there is some $v_i \in D$ with $i \geq m$. Consequently, there is a cycle $v_0, \ldots, v_m, \ldots, v_i, \ldots, v_0$ since D is strongly connected. Hence, $v_m \notin D$ belongs to the same connected component as v_0, which is a contradiction to maximality. Thus, D is a no-return set. □

Thus, if the strongly connected components are computed, a large part of the set

$$\mathcal{B} := \{B_v \mid v \in V(G)\}$$

is known. What remains are the single vertices which are not part of any component. Luckily, all these vertices are no-return sets:

PROPOSITION 25: *If $v \in V(G)$ is not an element of any strongly connected component of G, then it is a no-return set and $B_v = \{v\} \in \mathcal{B}$.*

PROOF: If $v_0 \in V(G)$ does not belong to any strongly connected component, there is no cycle $v_0, \ldots, v_i, \ldots, v_0$, and thus it holds for every path $v_0, \ldots, v_i$ with $v_i \neq v_0$ that $\forall i \geq 1 : v_i \notin \{v_0\}$, making v a no-return set.

Due to PROP. 23, $v \in B_v$. If there were another $w \in B_v$ distinct from v, it would be an element of the same strongly connected component. □

We conclude from these results that:

PROPOSITION 26: *The family $\mathcal{B}$ is a cover of $V(G)$, which contains only no-return sets.*

PROOF: Due to PROP. 23 (p. 56), $\bigcup_{v\in V(G)} B_v = V(G)$. It holds for all $B_v, B_w \in \mathcal{B}$ that $B_v = B_w$ or $B_v \cap B_w = \varnothing$, since strongly connected components and/or single states not being part of a strongly connected component are always disjoined.

The elements of $\mathcal{B}$ are no-return sets due to PROP. 24 and PROP. 25. □

Thus, we arrive at the first of our aims: $\mathcal{B}$ has the necessary properties to compute an abstracted state-transition graph (cf. DEF. 8, p. 37):

DEFINITION 19: *Let G be a state-transition graph and $\mathcal{B} = \{B_v \mid v \in V(G)\}$ the disjoined partition of $V(G)$ with B_v as defined in* PROP. *23 (p. 56). The resulting abstracted state-transition graph G' with $V(G') = \mathcal{B}$ is the* ***no-return abstraction*** *of G.*

The other result is that there is a one-to-one correspondence between the atoms of the lattice $\mathcal{L}$ of invariant sets, $\phi(v), v \in V$, and the "no-return basis" $\mathcal{B}$: together with PROP. 20 (p. 55), where each invariant set is constructed from appropriate $\phi(v)$, every invariant set can be constructed by a union of appropriately chosen elements of $\mathcal{B}$, and every union of appropriately chosen elements of $\mathcal{B}$ yields an invariant set.

PROPOSITION 27:

(i) For all $v \in V(G) : \phi(v) = \phi(B_v)$.

(ii) For all $v \in V(G)$:

$$\phi(v) = \bigcup_{\substack{B_w\in\mathcal{B} \\ B_w\subseteq\phi(B_v)}} B_w.$$

(iii) If $D \subseteq V(G)$ is an invariant set, then

$$D = \bigcup_{\substack{B_v\in\mathcal{B} \\ v\in D}} B_v.$$

PROOF: Part (i): By PROP. 23 (p. 56), $v \in B_v \subseteq \phi(v)$. Since ϕ is monotone and idempotent, $\phi(v) \subseteq \phi(B_v) \subseteq \phi(\phi(v)) = \phi(v)$.

Part (ii): Generally,

$$\bigcup_{\substack{B_w\in\mathcal{B} \\ B_w\subseteq\phi(B_v)}} B_w \subseteq \phi(B_v) \subseteq \phi(v).$$

For the inclusion in the other direction take $u \in \phi(v)$. Since $\phi(u) \subseteq \phi(v)$, it holds by PROP. 21 (p. 56) and (i) that $u \in B_u \subseteq \phi(u) \subseteq \phi(v) = \phi(B_v)$, i.e. $u \in \bigcup_{\substack{B_w\in\mathcal{B} \\ B_w\subseteq\phi(B)}} B_w$.

Part (iii): It holds that $D \subseteq \bigcup_{\substack{B_v\in\mathcal{B} \\ v\in D}} B_v$ since $v \in B_v$ (cf. PROP. 23, p. 56).
Now, take a $u \in \bigcup_{\substack{B_v\in\mathcal{B} \\ v\in D}} B_v$. Then, by (i) and PROP. 20 (p. 55),

$$u \in \bigcup_{\substack{B_v\in\mathcal{B} \\ v\in D}} \phi(B_v) = \bigcup_{v\in D} \phi(v) = D.$$

□

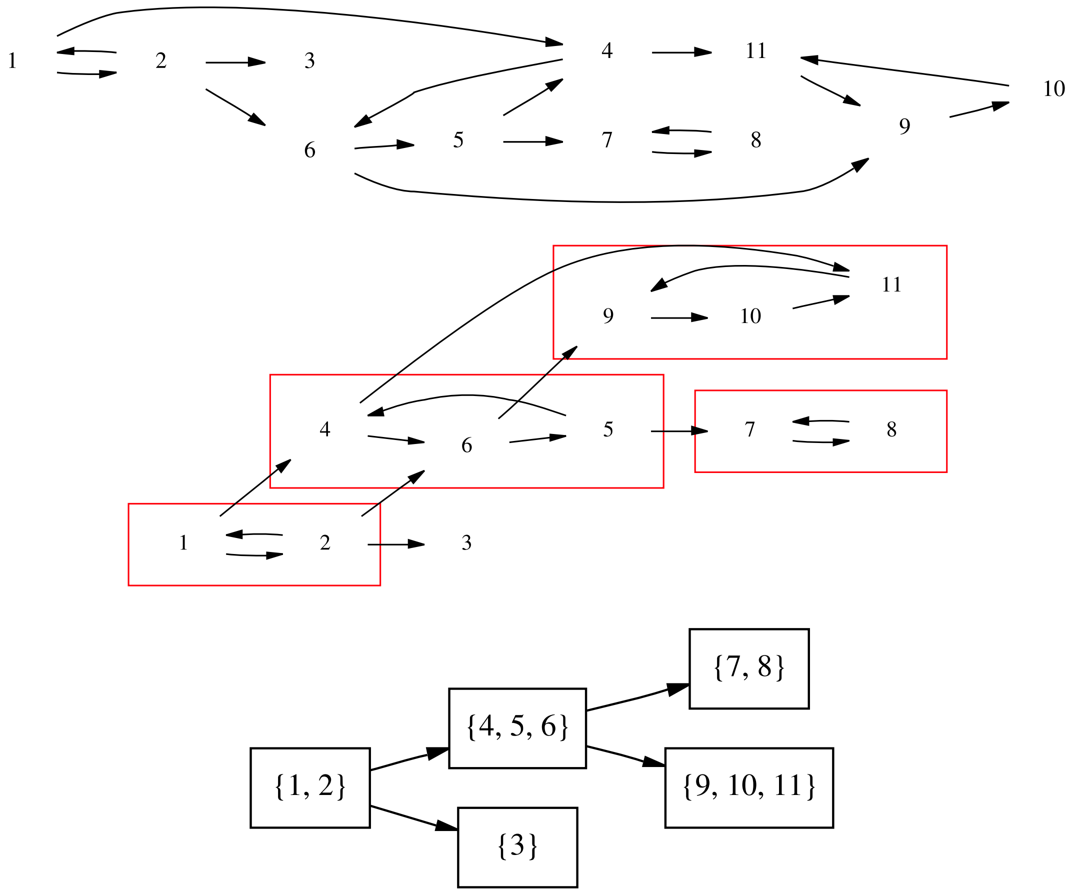

Figure 3.1: Stylised example for a no-return abstraction of a state-transition graph (top: a graph G, middle: the family $\mathcal{B}$ displayed as clusters, bottom: no-return abstraction of G).

In practice, it is sufficient to compute and display only the basic no-return sets $\mathcal{B}$ to obtain the no-return abstraction of G. It is quite simple for the end-user to find all relevant invariant sets from such a presentation: It is well-known in graph theory that directed graphs can be decomposed as an acyclic graph of strongly connected components (van Leeuwen 1990): If G' is the no-return abstraction of G, for all $B_0 \in V(G') = \mathcal{B}$ and for all paths $B_0, \ldots, B_i$ in G' it holds that $B_i \neq B_0$ for $i > 0$. Otherwise, there would be a path $v_0, \ldots, v_i, \ldots, v_0$ with $v_0 \in B_0$ and $v_i \notin B_0$, which is impossible since B_0 is a no-return set. Thus, by picking a no-return set $B \in V(G')$, one can trace "downstream" to all other vertices in G' which are attainable from B – which is straightforward in an acyclic graph – to obtain an invariant set. Of course, there are other invariant sets, but the most likely question when performing a no-return abstraction is: Where do irreversible changes of the qualitative state occur? This happens exactly for all edges entering a $B \in \mathcal{B}$, because $\phi(B)$ is an invariant set (see Fig. 3.1 for a stylised example).

3.1.3 Consistency of Projection, No-Return and Chatter-Box Abstraction

For the analysis of large state-transition graphs we would like to combine no-return abstraction with other methods, e.g. projection and chatter-box abstraction (cf. section 2.2.4 (p. 36), DEF. 9 and DEF. 11). Since all abstraction methods take an input graph and produce a new one which is not larger than the input graph, the idea is to apply the abstraction methods subsequently to substantially improve the clarity of the model result. The result is a sequence of graphs $G_0, \dots, G_n$, where G_{i+1} is an abstraction of G_i, determined by the abstraction methods used.

For a chosen set of abstraction methods, does the result G_n depend on the sequence in which they are computed? In other words, if the methods are regarded as operators, do they commute? As I will show below, they do not, so the appropriate sequence has to be determined. Since the objective of this exercise is to reveal properties of the state-transition graph which would not be easy accessible otherwise, there is a pragmatic criterion for appropriateness: If we observe a certain type of structure in G_n, we want to be sure that the structure also exists in G_0. In the following, I infer the sequence in which projection, chatter-box, and no-return abstraction should be applied. Start with a simple observation.

PROPOSITION 28: *Every chatter-box is contained in a no-return set.*

PROOF: If D be a chatter-box of G, it is strongly connected in the subgraph $G^- = G \cap G^{-1}$ by definition (cf. DEF. 9, p. 38). Since $E(G^-) \subseteq E(G)$, D is part of a strongly connected component of G, and thus also part of a no-return set due to PROP. 24 (p. 57). □

The consequence of this proposition is that we do not "split" chatter-boxes when we compute a no-return abstraction as second step after chatter box abstraction. On the other hand, the chatter-boxes are not visible any more in the no-return abstraction – we may only note the sets $B \in \mathcal{B}$ which contain a chatter-box. However, it makes no sense to apply the abstraction procedures the other way round, since there are no chatter-boxes in the no-return abstraction due to its acyclic structure. Now consider the case where projection is combined with no-return abstraction.

PROPOSITION 29: *For G and an index set I, denote $G' := \pi_I(G)$. If D' is a no-return set in G', then $D := \pi_I^{-1}(D')$ is a no-return set in G.*

PROOF: Suppose that D is *not* a no-return set. Then there is a path $v_0, \dots, v_m, \dots, v_n$ in G with $v_0, v_n \in D$ and $v_m \notin D$. By taking $\pi_I(v_0), \dots, \pi_I(v_m), \dots, \pi_I(v_n)$ and eliminating all elements of this sequence which are identical to their predecessor, we obtain a path $v'_0, \dots, v'_k, \dots, v'_l$ in G' with $v'_0 = \pi_I(v_0) \in \pi_I(D)$, $v'_l = \pi_I(v_n) \in \pi_I(D)$, and $v'_k = \pi_I(v_m)$. Therefore, since D is the inverse image of D', it holds that $v'_k \notin \pi_I(D)$. Thus, $\pi_I(D)$ is not a no-return set. Since surjectivity of π_I guarantees that $\pi_I(D) = \pi_I(\pi_I^{-1}(D')) = D'$, this is a contradiction to D' being a no-return set. □

This means that if there is a no-return set in a projection, it corresponds to a no-return set in the original state-transition graph. Conversely, the property of being a no-return set is not

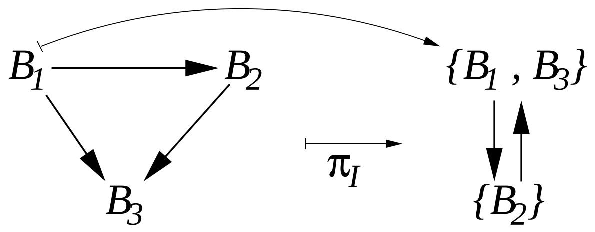

Figure 3.2: Example for a projection not preserving no-return sets. The graph to the left is the no-return abstraction G' of some state-transition graph with $V(G') = \mathcal{B} = \{B_1, B_2, B_3\}$. We consider a projection π_I and assume that $\forall v, w \in B_1 \cup B_3 : \pi_I(v) = \pi_I(w)$, and that $\forall v, w \in B_2 : \pi_I(v) = \pi_I(w)$, i.e. B_1 and B_3 are projection equivalent. The resulting projection $H = \pi_I(G')$ is displayed to the right. Although the vertex B_1 is a no-return set in G', it's image $\pi_I(B_1) = \{B_1, B_3\}$ is obviously not a no-return set in H.

generally preserved under projection, as the example in Fig. 3.2 shows. Thus, a no-return abstraction should be performed after a projection. To be complete, we consider the third possible combination of abstractions: projection and chatter-box abstraction.

PROPOSITION 30: *Let G be a state-transition graph and $H := \pi_I(G)$ its projection. If D is a chatter-box in G, then $D' := \pi_I(D) \subseteq V(H)$ is a single vertex or induces a chatter-box in H.*

PROOF: The subgraph induced by D is strongly connected in $G^- = G \cap G^{-1}$ by definition. It follows that $D' = \pi_I(D)$ is a single vertex in H or it is strongly connected in $\pi_I(G^-)$ (cf. p. 39). We show that $\pi_I(G^-) = \pi_I(G)^- = H^-$, making D' a chatter-box in H.
Since $V(G^-) = V(G)$, we have that $V(\pi_I(G)^-) = V(\pi_I(G)) = V(\pi_I(G^-))$. For the edges,

$$
\begin{aligned}
E(\pi_I(G)^-) &= E(\pi_I(G)) \cap E(\pi_I(G^{-1})) = \\
&= \{(\pi_I(v), \pi_I(w)) \mid (v, w) \in E(G)\} \cap \\
&\qquad \{(\pi_I(v), \pi_I(w)) \mid (w, v) \in E(G)\} = \\
&= \{(\pi_I(v), \pi_I(w)) \mid (v, w) \in E(G) \wedge (w, v) \in E(G)\} = \\
&= E(\pi_I(G^-))
\end{aligned}
$$

holds. $\square$

Simply speaking, chatter-boxes remain chatter-boxes under projection. Again, the converse is not generally true: If D' is a chatter-box in $\pi_I(G)$, it cannot be concluded that $\pi_I^{-1}(D')$ is a chatter-box in G (see example in Fig. 3.3). Hence, one has to be careful when applying chatter-box abstraction after projection. To sum up, for a large state-transition graph the different abstraction methods should be applied in the following sequence:

1. Chatter-box abstraction

2. Projection

3. No-return abstraction

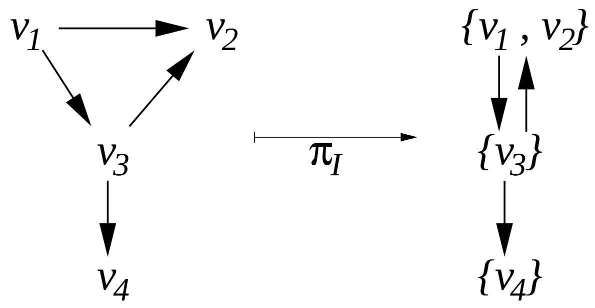

Figure 3.3: Example for a projection producing "artifical" chatter-boxes. The diagram to the left represents a state-transition graph G with $V(G) = \{v_1, v_2, v_3, v_4\}$. We consider a projection π_I and assume that $\pi_I(v_1) = \pi_I(v_2)$, but $\pi_I(v_1) \neq \pi_I(v_3) \neq \pi_I(v_4) \neq \pi_I(v_1)$, i.e. only v_1 and v_2 are projection equivalent. The resulting projection $H = \pi_I(G)$ is displayed to the right. The set $\{\{v_1, v_2\}, \{v_3\}\} \subseteq V(H)$ induces a chatter-box in H, while its inverse image $\{v_1, v_2, v_3\} \subseteq V(G)$ does not so in G.

3.2 Marginal Edges

In this section I present two techniques to simplify the state-transition graph of a QDE by eliminating edges which are of little relevance in most applications. They also preserve connectedness properties of the graph, thus being a special type of transitive reduction. Main ideas appeared in Eisenack and Petschel-Held (2002) and are formulated here more rigorously. There is related work of Bouwer and Bredeweg (2002), who compute another type of transitive reduction of the state-transition graph. Along some edges two or more qualitative directions or qualitative magnitudes change at the same time. This implies the existence of a solution which passes through the intersection of two main isoclines or the intersection of a main isocline with a landmark. It can be shown that for many systems the set of trajectories with such features is of measure zero (Bernard and Gouze 2002). The completeness of the QSIM algorithm guarantees that also the abstractions of such solutions are represented in the state-transition graph. By restricting the space of admissible trajectories $\mathcal{E}$ to functions which do not attain values on such intersections of measure zero, we obtain a graph with fewer edges. The implied loss of information is acceptable, since no features of relevance are left out.

3.2.1 Characterising Marginal Edges

In the following let S be a qualitative state space with n quantity spaces, and G the state-transition graph of a monotonic landmark ensemble $\mathcal{M}(\mu, C)$. For each edge $(v, w) \in E(G)$, there is at least one component i such that $\mathrm{qdir}_i(v \wedge w) = 0$ or that $\mathrm{qmag}_i(v \wedge w) = \lambda$ with a landmark $\lambda \in Q_i$. If $\mathrm{qdir}_i(v) \neq \mathrm{qdir}_i(w)$, we say in the first case that the edge **transgresses the ith main isocline**, while in the second we say that it **transgresses the landmark** λ if $\mathrm{qmag}_i(v) \neq \mathrm{qmag}_i(w)$. This corresponds to the existence of a solution $x(\cdot)$ of a system of the monotonic landmark ensemble such that for one $t \in \mathbb{R}_+ : f_i(x(t)) = 0$ (if it transgresses a main isocline), or that $x_i(t) = \lambda$, where $\lambda \in \mathbb{R}$ is defined by the landmark vector Λ associated with the system (cf. PROP. 2, p. 25, DEF. 4, p. 29 and Eq. 2.12).

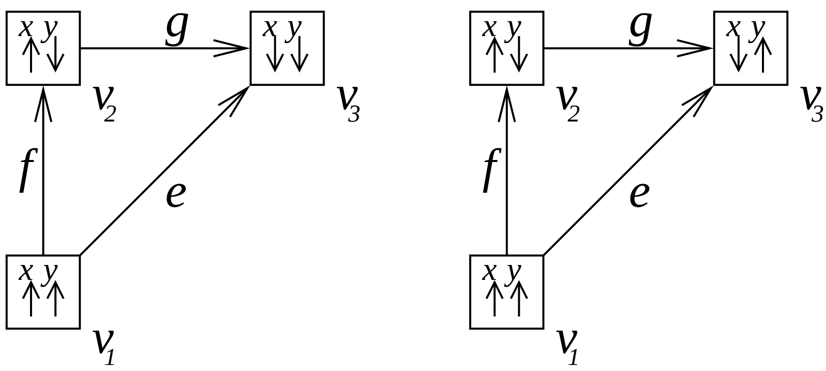

Figure 3.4: Different types of composed edges. Arrows in the boxes show qualitative directions of two variables.

Some edges $e = (v_1, v_3)$ in the state transition graph are composed in the sense that there are two other edges $f = (v_1, v_2)$ and $g = (v_2, v_3)$, in simple words, e is a "shortcut". If all landmarks or main isoclines which are transgressed along f and g are also transgressed along e, it is tempting to elemi nate e: Along this edge the transgressions occur simultaneously although they do not necessarily have to by the model assumptions (otherwise only the edge e would exist, but not the edges f and g). Eliminating all edges (v_1, v_m) for which a path $v_1, \ldots, v_m$ exists is conceptualised as transitive reduction in graph theory. However, it is well-known that – in contrast to transitive closure – this operation cannot be defined in a unique way (cf. van Leeuwen 1990). There is another subtlety: what happens along alternative paths from v_1 to v_3? Two cases have to be distinguished (see Fig. 3.4):

1. Along f some components change their qualitative direction or transgress a landmark, while along g other qualitative values change. Along the edge e the qualitative values change at the same time.

2. Along g some qualitative directions change back to the value they had in v_1 or transgress the same landmark in the other direction, while other components change to a new qualitative value. Along e only landmarks or main isoclines not transgressed along g are affected.

The first case is a marginal edge, since along e the landmarks or main isoclines are transgressed simultaneously "by accident". This interpretation is not valid for the second case, since some landmarks or main isoclines are crossed twice. Here, the edge e exhibits – compared to f and g – a notable new property which should not be ignored. Only composed edges e of the first type can be omitted, where transgressions coincide which do not necessarily have to. Usually, they have no special relevance: Nothing basically new happens, the result of both paths is the same (namely the system being in state v_3), and e is not likely to be observed in empirical studies.

Now, these considerations are formalised. Note that, since the quantity spaces $Q_i, i = 1, \ldots, n$ are ordered (cf. section 2.2.2, p. 27), expressions like "λ is between $\mathrm{qmag}_i(v)$ and $\mathrm{qmag}_i(w)$" and $\max(\mathrm{qmag}_i(v))$ are well-defined. If $\mathrm{qdir}_i(v) \neq \mathrm{qdir}_i(w)$, then $\mathrm{qdir}_i(v \wedge w) = 0$, and if $\mathrm{qmag}_i(v) \neq \mathrm{qmag}_i(w)$, then $\mathrm{qmag}_i(v \wedge w) = \lambda$ is a landmark (by Eq. 2.12).

Before defining a marginal edge, the following "qualitative intermediate value theorem" is formulated.

PROPOSITION 31: *Let $v_1, \dots, v_m$ be a path in G and $i \in \{1, \dots, n\}$. Then, for every landmark λ between $\mathrm{qmag}_i(v_1)$ and $\mathrm{qmag}_i(v_m)$ there is a $j \in \{1, \dots, m-1\}$ such that the edge (v_j, v_{j+1}) transgresses λ.*

PROOF: Since paths of length 1 in G correspond to abstractions of reasonable trajectories, an edge (v, w) can only transgress landmarks $\lambda \in \mathrm{qmag}_i(v) \cap \mathrm{qmag}_i(w)$ with $\lambda \in Q_i$ by continuity (cf. Eq. 2.12). Thus, along the path $v_1, \dots, v_m$ there must be an edge (v_j, v_{j+1}) with $\mathrm{qmag}_i(v_j \wedge v_{j+1}) = \lambda$. □

DEFINITION 20: *Let $v_1, \dots, v_m$ with $m \geq 3$ be a path in G. Then, an edge $(v_1, v_m) \in E(G)$ is called **marginal edge** if for all $i = 1, \dots, n$ there are not two edges $(v_j, v_{j+1}), j = 1, \dots, m-1$ which transgress the same landmark or the ith main isocline more than once.*

The concept is further clarified by the following equivalent characterisation, which is helpful for the elimination techniques. For its formulation we define for $(v, w) \in E(G)$ the **change set** $\mathrm{Ch}(v, w) := \{i \mid \mathrm{qmag}_i(v) \neq \mathrm{qmag}_i(w)\} \cup \{j = n + i \mid \mathrm{qdir}_i(v) \neq \mathrm{qdir}_i(w)\}$.

PROPOSITION 32: *An edge $(v_1, v_m) \in E(G)$ is marginal if and only if there is a path $v_1, \dots, v_m$ in G, $m \geq 3$, such that the change sets $\mathrm{Ch}(v_j, v_{j+1}), j = 1, \dots, m-1$, are pairwise disjoined.*

PROOF: If the change sets are pairwise disjoined, every $i = 1, \dots, 2n$ occurs no more than once in a change set. Since only changing components can transgress a landmark or main isoclines, (v_1, v_m) is a marginal edge.

Now assume that no landmark or main isocline is transgressed twice. Take an arbitrary $i = 1, \dots, 2n$.
For $i > n$, there is not more than one j such that $\mathrm{qdir}_{i-n}(v_j \wedge v_{j+1}) = 0$ and $\mathrm{qdir}_{i-n}(v_j) \neq \mathrm{qdir}_{i-n}(v_{j+1})$, and consequently not more than one j with $i \in \mathrm{Ch}(v_j, v_{j+1})$ – the intersections of the change sets with $\{n+1, \dots, 2n\}$ are pairwise disjoined.
For $i \leq n$ define $\lambda_0 := \mathrm{qmag}_i(v_1 \wedge v_m)$. Choose the smallest j such that $i \in \mathrm{Ch}(v_j, v_{j+1})$ and set $\lambda_j := \mathrm{qmag}_i(v_j \wedge v_{j+1})$. If $\lambda_j < \lambda_0$, then

$$\mathrm{qmag}_i(v_{j+1}) \leq \lambda_j \leq \mathrm{qmag}_i(v_j) = \mathrm{qmag}_i(v_1) \leq \lambda_0 \leq \mathrm{qmag}_i(v_m).$$

Due to PROP. 31, λ_j will be transgressed a second time on $v_{j+1}, \dots, v_m$, which is a contradiction. If $\lambda_0 < \lambda_j$, the analogue contradiction applies. Thus $\lambda_0 = \lambda_j$. The same argument applies for the next smallest $k > j$ with $i \in \mathrm{Ch}(v_k, v_{k+1})$. Therefore, it holds for all $l = 1, \dots, m-1$ with $i \in \mathrm{Ch}(v_l, v_{l+1})$ that $\mathrm{qmag}_i(v_l \wedge v_{l+1}) = \lambda_0$. Since λ_0 cannot be transgressed twice, also the intersection of the change sets with $\{1, \dots, n\}$ are pairwise disjoined. □

Later we will need the following property of the change sets to show the applicability of the restriction techniques:

PROPOSITION 33: *If* $(v_1, v_m) \in E(G)$ *and there is a path* $v_1, \ldots, v_m, m \geq 3$ *in* G *with pairwise disjoined change sets* $\mathrm{Ch}(v_j, v_{j+1}), j = 1, \ldots, m-1$*, then*

$$\bigcup_{j=1,\ldots,m-1} \mathrm{Ch}(v_j, v_{j+1}) = \mathrm{Ch}(v_1, v_m).$$

PROOF: Let $i \in \mathrm{Ch}(v_1, v_m)$. Since $\mathrm{qval}_i(v_1) \neq \mathrm{qval}_i(v_m)$ there must be at least one $j \in \{1, \ldots, m-1\}$ with $i \in \mathrm{Ch}(v_j, v_{j+1})$.

Now suppose that $i \notin \mathrm{Ch}(v_1, v_m)$, i.e. $\mathrm{qval}_i(v_1) = \mathrm{qval}_i(v_m)$. It is not possible that there is one and only one j such that $i \in \mathrm{Ch}(v_j, v_{j+1})$, since otherwise $\mathrm{qval}_i(v_1) = \mathrm{qval}_i(v_j) \neq \mathrm{qval}_i(v_{j+1}) = \mathrm{qval}_i(v_m)$. Because the change sets are disjoined, there also cannot be more than one such j, and therefore $i \notin \bigcup_{j=1,\ldots,m-1} \mathrm{Ch}(v_j, v_{j+1})$. □

3.2.2 Eliminating Marginal Edges

I present two algorithms to eliminate marginal edges. One uses a preprocessing approach, the other a postprocessing approach. Both can be combined. The preprocessing strategy requires the modeller to reason about marginal edges which are likely to occur. This can be based on earlier qualitative simulations, on algebraic reasoning or on knowledge about the application domain. Removing marginal edges means preventing simultaneous changes of qualitative values which can also occur subsequently. If two such changes are identified, associated edges can be filtered out by introducing a new constraint into the qualitative landmark system, called **correspondence-not**,

```
((cornot x y) <(lx ly)>)
```

with a sequence of m pairs `(lx ly)`, where `lx` represents a landmark of the variable `x` (associated with index i) and `ly` a landmark of variable `y` (associated with index j), denoting sequences of landmarks $\lambda_{i,1}, \ldots, \lambda_{i,m}$ and $\lambda_{j,1}, \ldots, \lambda_{j,m}$. An edge $(v, w) \in E(G)$ satisfies this constraint if

$$\begin{aligned} & i \notin \mathrm{Ch}(v, w) \\ \text{or } & j \notin \mathrm{Ch}(v, w) \\ \text{or } & \forall l = 1, \ldots, m : \mathrm{qmag}_i(v \wedge w) \neq \lambda_{i,l} \text{ or } \mathrm{qmag}_j(v \wedge w) \neq \lambda_{j,l}, \end{aligned}$$

i.e. if no pair of changing components transgresses the given landmarks at the same time. All marginal edges can be excluded if the qualitative directions of the state variables are included as additional qualitative magnitudes of auxiliary variables with indices $i = n+1, \ldots, 2n$, i.e. by augmenting the qualitative state space with the qualitative velocity space (as default in the QSIM algorithm, cf. section 2.2.3, p. 32).

PROPOSITION 34: *If the qualitative state space is augmented with the velocity space and* (v_1, v_m) *is a marginal edge, there is a correspondence-not constraint which contradicts the edge* (v_1, v_m)*, but all edges along the path* $v_1, \ldots, v_m$ *satisfy it.*

PROOF: At first recall that for the augmented qualitative state space $\forall v \in V(G), i = n+1, \ldots, 2n : [\mathrm{qmag}_i(v)] = \mathrm{qdir}_{i-n}(v)$, and consequently $\mathrm{Ch}(v, w) \cap \{1, \ldots, 2n\}$ in the augmented state space equals $\mathrm{Ch}(v, w)$ in the original state space.
Let (v_1, v_m) be a marginal edge. Choose $k, l \in \{1, \ldots, m-1\}, k \neq l$ and $i \in \mathrm{Ch}(v_k, v_{k+1}) \cap \{1, \ldots, 2n\}$, $j \in \mathrm{Ch}(v_l, v_{l+1}) \cap \{1, \ldots, 2n\}$, $i \neq j$, such that $\lambda_{i,1} = \mathrm{qmag}_i(v_k \wedge v_{k+1})$ is between $\mathrm{qmag}_i(v_1)$ and $\mathrm{qmag}_i(v_m)$, and also $\lambda_{j,1} = \mathrm{qmag}_j(v_l \wedge v_{l+1})$ is between $\mathrm{qmag}_j(v_1)$ and $\mathrm{qmag}_j(v_m)$. This is possible since for a marginal edge, by PROP. 32, the change sets are pairwise disjoined. The correspondence-not constraint given by the components i, j and the pair of landmarks $\lambda_{i,1}, \lambda_{j,1}$ has the required properties: Since $i \neq j$, all edges along $v_1, \ldots, v_m$ satisfy the constraint. However, (v_1, v_m) does not, because due to PROP. 33, $i, j \in \mathrm{Ch}(v_1, v_m)$, $\mathrm{qmag}_i(v_1 \wedge v_m) = \lambda_{i,1}$, and $\mathrm{qmag}_j(v_1 \wedge v_m) = \lambda_{j,1}$, because $\lambda_{i,1}$ is between $\mathrm{qmag}_i(v_1)$ and $\mathrm{qmag}_i(v_m)$, while $\lambda_{j,1}$ is between $\mathrm{qmag}_j(v_1)$ and $\mathrm{qmag}_j(v_m)$, and PROP. 31 (p. 64) holds. □

This strategy can substantially reduce computing costs, since fewer edges have to be generated during simulation.

The second method avoids the problem of the first – namely having to know potential marginal edges in advance – and is completely automatic. On the other hand, it requires the state-transition graph of the QDE to be determined first. Taking this as input, all marginal edges are determined and eliminated from the state transition graph by exploiting PROP. 32 (p. 64). In the basic version, the change set $\mathrm{Ch}(v, w)$ is assigned to every edge (v, w) of the state-transition graph. The method subsequently starts a depth first search (DFS) from every vertex $v_1 \in V(G)$, such that maximal paths $v_1, \ldots, v_m$ with disjoined change sets are traversed. Efficient algorithms are well-established for depth first search with a given criterion (cf. van Leeuwen 1990). If there is an edge (v_1, v_j), $j \in \{3, \ldots, m\}$, then it is marked as a marginal edge. After all the paths starting from all vertices in G are traversed, the marked edges are eliminated. Of course, there are various possibilities to increase the computational efficiency of the method, e.g. by not re-considering vertices which have already been visited. A problem would occur with this procedure if eliminating a marginal edge interrupts the path defining another marginal edge. Then, vertices which were connected in G would become unconnected. However, this can be excluded:

PROPOSITION 35: *Let G' be the graph obtained from the state-transition graph G by eliminating all edges which are marginal edges. There is a path $v_1, \ldots, v_m$ in G iff there is also a path $v_1, \ldots, v_m$ in G'.*

PROOF: Obviously, paths in G' are also paths in G, since $V(G) = V(G')$ and $E(G') \subseteq E(G)$.

Now, let $v_1, \ldots, v_m$ be a path in G. We have to clarify that there is still a path from v_1 to v_m in G' if some edges of G are marginal (and do not occur in G'). Suppose that $(w_1, w_k) \in E(G)$ is such a marginal edge with $w_1 = v_j$ and $w_k = v_{j+1}$, $j \in \{1, \ldots, m-1\}$. Then, there is a path $w_1, \ldots, w_k$ not containing the edge (w_1, w_k). If this path does not contain another marginal edge, there is a path $v_1, \ldots, w_1, \ldots, w_k, \ldots v_m$ which also exists in G', but there is no edge (v_j, v_{j+1}).

If there are multiple marginal edges in $v_1, \ldots, v_m$, or if one edge of $w_1, \ldots, w_k$ is marginal again, this argument can be repeated iteratively. The procedure stops when no further marginal edge is contained – with the consequence that there is still a path from v_1 to v_m in G'.
It would not stop if one of the paths from the source to the target of a marginal edge contains another marginal edge which was considered before during the iteration, yielding a cyclic procedure. To show that this can be excluded, consider a path $v_1, \ldots, w_1, \ldots, w_k, \ldots v_m$ in G and let $(v_1, v_m), (w_1, w_k)$ be marginal edges. Then, by PROP. 32 (p. 64) and PROP. 33, the change sets $\mathrm{Ch}(v_1, v_2), \ldots, \mathrm{Ch}(w_1, w_2), \ldots, \mathrm{Ch}(w_{k-1}, w_k), \ldots, \mathrm{Ch}(v_{m-1}, v_m)$ are pairwise disjoined. If the path $w_1, \ldots, w_k$ contains the marginal edge (v_1, v_m) again, we must conclude for the same reasons that the change sets

$$\begin{aligned}&\mathrm{Ch}(v_1, v_2), \ldots, \mathrm{Ch}(w_1, w_2), \ldots,\\ &\mathrm{Ch}(v_1, v_2), \ldots, \mathrm{Ch}(v_{m-1}, v_m), \ldots,\\ &\mathrm{Ch}(w_{k-1}, w_k), \ldots, \mathrm{Ch}(v_{m-1}, v_m)\end{aligned}$$

are also pairwise disjoined – a contradiction. □

To sum up, marginal edges can be defined so as to simplify a state-transition graph by restricting the space of admissible functions in a way that is not an obstacle from the application perspective. Two techniques with different advantages have been presented for this task.

3.3 Ordinal Assumptions

To define a monotonic landmark ensemble $\mathcal{M}(\mu, C)$, the signs of the components of the Jacobian $\mathcal{J}(f), f \in \mathcal{M}(\mu, C)$ are considered. Usually, this leads to a large state-transition graph G due to the generality of $\mathcal{M}(\mu, C)$. The no-return abstraction developed in section 3.1 (p. 52) helps to identify and display structural features in this case, and the elimination of marginal edges (section 3.2, p. 62) brings about more structure in G by restricting the space of admissible trajectories. However, practice shows that restriction techniques like the elimination of marginal edges together with eliminating non-analytical states (cf. section 2.2.4, p. 36) are not always sufficient to bring about enough interpretable subgraphs as no-return and invariant sets. In many cases, the graph consists of one connected component. If more edges can be eliminated, the value of abstraction techniques increases substantially. This is only possible if model ensembles are further restricted by including more assumptions than can be expressed by mere sign and landmark properties. We can use quantitative information (which will be investigated in section 3.4, p. 77), but when uncertainty or generality is high, this might not be available: we need to integrate additional, but not quantitative, information which is likely to be available when reasoning, for example, with causal loop diagrams.

3.3.1 The Effect of Ordinal Assumptions

Ordinal assumptions represent one such type of information. By these we mean statements that the value of some functions on the state space are always above or below the values of other functions. In this section I concentrate on ordinal assumptions for partial derivatives of models f of a monotonic ensemble $\mathcal{M}(\Sigma)$ of the form

$$\forall x \in X : D_k f_i(x) > D_l f_j(x),$$

with prescribed $i, j, k, l \in \{1, \ldots, n\}$, $\Sigma = (\sigma_{i,j})_{i,j=1,\ldots,n}$, $\sigma_{i,j} \in \mathcal{A}_*$ a matrix of (extended) signs and X the state space. In practice, ordinal assumptions are often supplied together with causal loop diagrams, when not only the positive or negative influences are stated, but also comparative propositions are made about their strengths. Such statements can be like "the influence of x_1 on x_3 is more important than the influence of x_2 on x_3, but it is not known how strong the influence of x_4 on x_3 is in comparison to the other two". This can be interpreted as a partial order of the partial derivatives of the models to be considered. In some cases we can deduce from such knowledge that a main isocline can be transgressed by trajectories only once. Before giving a general proposition to eliminate edges, the idea is illustrated with a simple qualitative model.

EXAMPLE 9: Define

$$\Sigma = \begin{pmatrix} + & + \\ - & - \end{pmatrix}.$$

The resulting state-transition graph is depicted in Fig. 3.5. It has one trivial no-return set, the graph itself, exhibiting a cycle through all four states. We perform a phase plane analysis. From Σ we derive – using the implicit function theorem – the monotonicity properties of both main isoclines $\dot{x}_1 = f_1(x) = 0, \dot{x}_2 = f_2(x) = 0$ (supposed they exist), which are denoted by

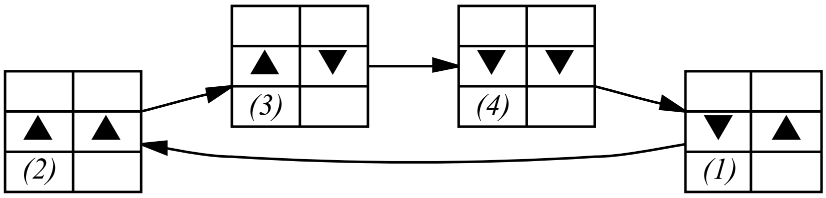

Figure 3.5: State-transition graph of the QDE of the monotonic system $M(\Sigma)$ (computer-generated output, one non-analytic state is eliminated).

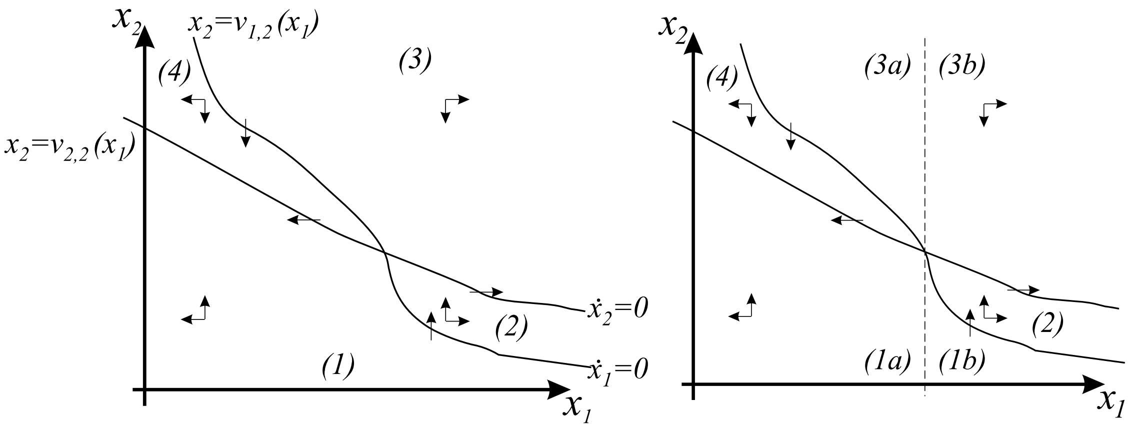

Figure 3.6: Phase plane analysis of an exemplary system $f \in \mathcal{M}(\Sigma)$. To the left: areas with constant direction of change, indicated by arrows and numbers corresponding to Fig. 3.5. To the right: area (1) and (3) are split to $(1a)$, $(1b)$ and $(3a)$, $(3b)$, respectively.

the functions $v_{2,2}, v_{1,2}$ (the indices will become clear in the next subsection). The functions $v_{2,2} : \mathbb{R} \to \mathbb{R}$ and $v_{1,2} : \mathbb{R} \to \mathbb{R}$ solve $f_1(x_1, v_{1,2}(x_1)) = 0$ and $f_2(x_1, v_{2,2}(x_1)) = 0$, yielding

$$[D_1 v_{1,2}] = [-\frac{D_1 f_1}{D_2 f_1}] = [-],$$
$$[D_1 v_{2,2}] = [-\frac{D_1 f_2}{D_2 f_2}] = [-],$$

i.e. both functions are decreasing. In Fig. 3.6, left, see the phase plane structure of an ODE with $f \in \mathcal{M}(\Sigma)$ where both isoclines exist and intersect exactly once. It can be verified that the state-transition graph is correct: From area (4), corresponding to the qualitative state (4), only area (1) can be reached. If the system is somewhere in area (1) only area (2) can be reached, etc. The result is a cycle in the state-transition graph. However (considering Fig. 3.6, right), it is possible to split areas (1) and (3), such that (2) and $(1b)$ cannot be reached from $(1a)$, while (4) and $(3a)$ cannot be reached from $(3b)$. It is tempting to draw a modified graph (Fig. 3.7), which exhibits a stronger invariance structure than the original state-transition graph. □

Why is it not possible to derive this structure directly from Σ? This stems from the fact that there are other systems $f \in \mathcal{M}(\Sigma)$ with a different phase plane structure. Suppose there is

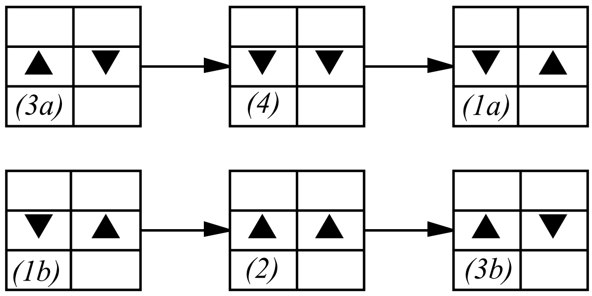

Figure 3.7: Modified state-transition graph derived from the example in Fig. 3.6.

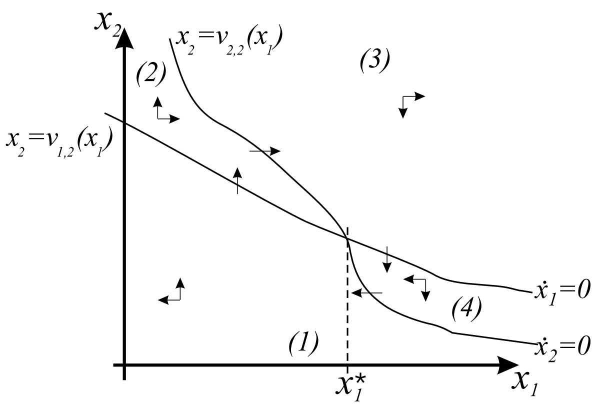

Figure 3.8: Phase plane analysis of a system with property Eq. (3.1).

an equilibrium for $x_1 = x_1^*$, i.e. $v_{1,2}(x_1^*) = v_{2,2}(x_1^*)$, and the system has the property that

$$\begin{aligned} &\forall x_1 < x_1^* : v_{1,2}(x_1^*) < v_{2,2}(x_1^*), \\ &\forall x_1 > x_1^* : v_{1,2}(x_1^*) > v_{2,2}(x_1^*). \end{aligned} \tag{3.1}$$

Then, the phase plane analysis may look like Fig. 3.8, showing that a cycle is possible under these conditions. This is also the case when there are multiple intersections of main isoclines. Since the state-transition graph covers all these cases, it is a strongly connected component. To exclude the cycle case, the model has to be supplied with additional information. If, for example, we impose the ordinal assumptions

$$\begin{aligned} \forall x \in X : \ & D_1 f_1(x) > D_2 f_1(x) (> 0), \\ & D_2 f_2(x) < D_1 f_2(x) (< 0), \end{aligned}$$

it follows that

$$\frac{D_1 f_1}{D_2 f_1} > \frac{D_1 f_2}{D_2 f_2},$$

and thus

$$\begin{aligned} 0 &< -\frac{D_1 f_2}{D_2 f_2} + \frac{D_1 f_1}{D_2 f_1} \\ &= D_1 \big(v_{2,2}(x_1) - v_{1,2}(x_1) \big), \end{aligned}$$

meaning that the distance between the main isoclines strictly increases with x_1, excluding multiple intersections and a situation as characterised by Eq. (3.1).

3.3.2 The ORDAS Algorithm

I now generalise this idea to state spaces with an arbitrary dimension. The result is a criterion to eleminate paths using ordinal assumptions and the knowledge that a certain main isocline was transgressed along a path before. We assume that $\Sigma \in \mathcal{A}^{n\times n}$. At first a simple proposition is shown, showing whether – for a given sign vector $[\dot{x}(t)]$ and Σ – a state $x(t)$ is above or below the ith isocline with respect to the direction of the kth unit vector. To avoid complications only models $f \in \mathcal{M}(\Sigma)$ are considered where for a given component i the implicit equation $f_i = 0$ **is soluble for** some $k \in \{1, \dots, n\}$, meaning that there is a unique solution $v_{i,k} : X \to \mathbb{R}$, independent from x_k, such that

$$\forall x \in X : f_i\big(x_1, \dots, x_{k-1}, v_{i,k}(x_1, \dots, x_n), x_{k+1}, \dots, x_n\big) = 0.$$

Uniqueness of the solution $v_{i,k}$ is guaranteed by the implicit function theorem since f is monotonic as specified by Σ, such that the *existence* on X is the assumption here.

PROPOSITION 36: *For a given Σ, let $f \in \mathcal{M}(\Sigma)$ be a function where f_i is soluble for k. Let $v_{i,k}$ be the solution, choose an arbitrary $x \in X$, and define $\sigma_i := [f_i(x)]$. If $\sigma_{i,k} \neq 0$ then*

$$\begin{aligned} \sigma_i\, \sigma_{i,k}\, x_k &> \sigma_i\, \sigma_{i,k}\, v_{i,k}(x) && \textit{if } \sigma_i \neq 0, \\ \textit{or } x_k &= v_{i,k}(x) && \textit{if } \sigma_i = 0. \end{aligned}$$

PROOF: The case $\sigma_i = 0$ is obvious.
If $\sigma_i > 0$ then

$$\begin{aligned} f_i(x_1, \dots, x_{k-1}, &x_k, x_{k+1}, \dots, x_n) > 0 = \\ = f_i\big(x_1, \dots, x_{k-1}, &v_{i,k}(x_1, \dots, x_n), x_{k+1}, \dots, x_n\big), \end{aligned}$$

and thus $\sigma_{i,k} x_k > \sigma_{i,k} v_{i,k}(x)$.
By analogy for $\sigma_i < 0$, it holds that $\sigma_{i,k} x_k < \sigma_{i,k} v_{i,k}(x)$. This and the former case together yield $\sigma_i \sigma_{i,k} x_k > \sigma_i \sigma_{i,k} v_{i,k}(x)$. □

Next, we introduce assumptions about terms of the form

$$d^{i,j}_{k,l} := D_l f_j \cdot D_k f_i - D_k f_j \cdot D_l f_i, \tag{3.2}$$

defined for a differentiable function $f : X \to \mathbb{R}^n$, and $i, j, k, l \in \{1, \dots, n\}$. It is assumed that the sign $\big[d^{i,j}_{k,l}\big]$ is constant on X. In some cases, this already follows from Σ, but in other cases it is a consequence of ordinal assumptions, as EX. 9 (p. 68) shows, where

$$\Sigma = \begin{pmatrix} + & + \\ - & - \end{pmatrix},$$

and $\big[d^{1,2}_{1,2}\big] = \big[D_1 f_1 \cdot D_2 f_2 - D_1 f_2 \cdot D_2 f_1\big]$ can be positive, negative or vanishing. If the modeller knows, e.g. that $D_2 f_1 > D_1 f_1$ and $D_2 f_2 > D_1 f_2$, then $\big[d^{1,2}_{1,2}\big] = [+]$. This kind of information is decisive for the elimination of edges:

PROPOSITION 37: *Let $\mathcal{M}(\Sigma)$ be a monotonic ensemble, G the resulting state-transition graph, v_0, v_1, v_2 a path of length 2 in G, and let the following criterion hold:*

(i) There exists an $i \in \{1, \ldots, n\}$ such that

$$-\mathrm{qdir}_i(v_0) = \mathrm{qdir}_i(v_1) = \mathrm{qdir}_i(v_2) \neq 0, \tag{3.3}$$

(ii) there is a $j \in \{1, \ldots, n\}$ such that

$$\mathrm{qdir}_j(v_0) = \mathrm{qdir}_j(v_1) = -\mathrm{qdir}_j(v_2) \neq 0, \tag{3.4}$$

(iii) for such i, j there exists a $k \in \{1, \ldots, n\}$ such that

$$\mathrm{qdir}_i(v_1)\, \sigma_{i,k} = \mathrm{qdir}_j(v_1)\, \sigma_{j,k} \neq 0, \tag{3.5}$$

(iv) for all $l \in \{1, \ldots, n\}, l \neq k$, $\left[d^{i,j}_{k,l}\right]$ is uniquely determined, fulfilling

$$\mathrm{qdir}_l(v_1)\, \mathrm{qdir}_i(v_1)\, \sigma_{j,k} \left[d^{i,j}_{k,l}\right] \geq 0. \tag{3.6}$$

Then there is no solution to an initial value problem $\dot{x} = f(x), [\dot{x}(0)] = v_0, f \in \mathcal{M}(\Sigma)$ with f_i soluble for k, which has the path v_0, v_1, v_2 as abstraction.

PROOF: Assume that there were a solution $x(\cdot)$ of the above type which has the path v_0, v_1, v_2 as abstraction. We demonstrate that this yields a contradiction.
For convenience, define $\sigma_i = \mathrm{qdir}_i(v_1)$ and $\sigma_j = \mathrm{qdir}_j(v_1)$. We know from Eq. (3.3) and Eq. (3.4) (using Eq. 2.2) that there are $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$ such that $\dot{x}_i(t_1) = 0 = \dot{x}_j(t_2)$ and $\dot{x}_i(t_2) \neq 0 \neq \dot{x}_j(t_1)$. Eq. (3.5) has the consequence that $\sigma_{i,k}$ and $\sigma_{j,k}$ do not vanish, and thus PROP. 36 can be applied to $x(t_1)$ and $x(t_2)$. Infer that

$$x_k(t_1) = v_{i,k}(x(t_1)), \tag{3.7}$$
$$\sigma_i\, \sigma_{i,k}\, x_k(t_2) > \sigma_i\, \sigma_{i,k}\, v_{i,k}(x(t_2)), \tag{3.8}$$
$$x_k(t_2) = v_{j,k}(x(t_2)), \tag{3.9}$$
$$\sigma_j\, \sigma_{j,k}\, x_k(t_1) > \sigma_j\, \sigma_{j,k}\, v_{j,k}(x(t_1)). \tag{3.10}$$

Now consider the function $\Delta(t) := v_{i,k}(x(t)) - v_{j,k}(x(t))$. It follows from Eq. (3.7) and Eq. (3.10) that

$$\sigma_j\, \sigma_{j,k}\, \Delta(t_1) = \sigma_j\, \sigma_{j,k}\, x_k(t_1) - \sigma_j\, \sigma_{j,k}\, v_{j,k}(x(t_1)) > 0,$$

and Eq. (3.5) implies that also $\sigma_i\, \sigma_{i,k}\, \Delta(t_1) > 0$. Compare $\Delta(t_2)$ with $\Delta(t_1)$: by differentiating and applying the implicit function theorem we obtain

$$D_t\Delta = \sum_{\substack{l=1,\ldots,n \\ l\neq k}} D_l v_{i,k} \cdot \dot{x}_l - \sum_{\substack{l=1,\ldots,n \\ l\neq k}} D_l v_{j,k} \cdot \dot{x}_l = \sum_{\substack{l=1,\ldots,n \\ l\neq k}} \dot{x}_l \Big(\frac{D_l f_j\, D_k f_i - D_k f_j\, D_l f_i}{D_k f_i\, D_k f_j} \Big).$$

The criterion guarantees that $\sigma_{i,k}\,\sigma_i\,D_t\Delta \geq 0$, since $\sigma_{i,k}, \sigma_{j,k}$ do not vanish and by Eq. (3.6) for all $l \neq k$

$$
\begin{aligned}
0 &\leq \sigma_l\,\sigma_i\,\sigma_{j,k}\,\left[d^{i,j}_{k,l}\right] = \\
&= \sigma_l\,(\sigma_{i,k}\,\sigma_{i,k})\,\sigma_i\,\sigma_{j,k}\,\left[d^{i,j}_{k,l}\right] = \\
&= \sigma_l\,\sigma_{i,k}\,\sigma_i\,\frac{\left[D_l f_j\,D_k f_i - D_k f_j\,D_l f_i\right]}{\sigma_{i,k}\,\sigma_{j,k}},
\end{aligned}
$$

where the last step exploits the multiplicative cancellation law of sign algebra (Eq. 2.1). Consequently, $\sigma_i\,\sigma_{i,k}\,\Delta(t_2) \geq \sigma_i\,\sigma_{i,k}\,\Delta(t_1) > 0$, and Eq. (3.9) implies

$$\sigma_i\,\sigma_{i,k}\,v_{i,k}(x(t_2)) > \sigma_i\,\sigma_{i,k}\,v_{j,k}(x(t_2)) = \sigma_i\,\sigma_{i,k}\,x_k(t_2),$$

which is a contradiction to Eq. (3.8).
We must conclude that there is no trajectory $x(\cdot)$ which has the path v_0, v_1, v_2 as abstraction.
$\square$

This proposition makes it possible to eliminate paths of length 2 if appropriate assumptions about $\left[d^{i,j}_{k,l}\right]$ are supplied by the modeller. Some of them are derived directly from Σ, while others represent a new structural property of the QDE not already entailed by the sign matrix. It should be noted that these assumptions cannot be chosen independently from each other since, by symmetry,

$$d^{i,j}_{k,l} = -d^{j,i}_{k,l} = -d^{i,j}_{l,k} = d^{j,i}_{l,k}. \tag{3.11}$$

In contrast to the elimination of marginal edges (section 3.2), the criterion cannot be applied to singular edges. In addition to the ordinal assumptions, some information about the “past” of a qualitative state v_1 is used to exclude a potential “future” v_2. It may be the case that the path v_0, v_1, v_2 is excluded, while a path v'_0, v_1, v_2 is still possible.

Thus, an elimination algorithm is not straightforward. In the following, I present the **ORD**inal **AS**sumptions algorithm. It exhausts PROP. 37 by “splitting” states as already indicated in EX. 9 (p. 68). This leads to a new graph containing multiple states with identical qualitative values. Each of them represents one case were PROP. 37 can be applied, having only the predecessors fulfilling Eq. (3.3) and Eq. (3.4), and the successors which cannot be eliminated under this condition.

DEFINITION 21: *The* ***ORDAS algorithm*** *inductively computes a sequence of graphs $H_0, \ldots, H_i$ by the following procedure, which takes as input a state transition graph G and a set of assumptions on the signs of expressions of the form Eq. (3.2).*

Define H_0 by

$$
\begin{aligned}
V(H_0) &:= \{v \times \{0\} \mid v \in V(G)\} \subseteq V(G) \times \mathbb{N}, \\
E(H_0) &:= \{(v \times \{0\}, w \times \{0\}) \mid (v, w) \in E(G)\}.
\end{aligned}
$$

For convenience, we write v_j for $v \times \{j\}$, and identify $\mathrm{qdir}(v_j)$ with $\mathrm{qdir}(v)$. Enumerate the vertices of H_0 as $v^1_0, \ldots, v^i_0, \ldots$ in an arbitrary order.

For the inductive definition of the sequence of graphs we consider the vertex v_0^i *at step* i *and define the set of* ***contradicting paths of a vertex*** v *in* H_i *as*

$$\mathrm{Cp}(v, H_i) := \{(u, v, w) \in V(H_i)^3 \mid (u, v) \in E(H_i), (v, w) \in E(H_i) \text{ and Prop. 37 (p. 72) can be used to eliminate the path } u, v, w\}.$$

This also defines the set of ***contradiction initiating predecessors***

$$\mathrm{Ci}(v, H_i) := \{u \in V(H_i) \mid (u, v, w) \in \mathrm{Cp}(v, H_i)\},$$

with its elements denoted by $u_1, \ldots, u_k$*,* $k := |\mathrm{Ci}(v, H_i)|$.

Then, H_{i+1} *is given by*

$$V(H_{i+1}) = V(H_i) \cup \{v_1^i, \ldots, v_k^i\},$$
$$E(H_{i+1}) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5,$$

where $v_1^i, \ldots, v_k^i$ *are introduced as new vertices which appear in the following edges:*

$$E_1 = \{(u, w) \mid (u, w) \in E(H_i) \text{ and } u \neq v_0^i \neq w\},$$
$$E_2 = \{(v_0^i, w) \mid (v_0^i, w) \in E(H_i)\},$$
$$E_3 = \{(u, v_0^i) \mid (u, v_0^i) \in E(H_i) \text{ and } u \notin \mathrm{Ci}(v_0^i, H_i)\},$$
$$E_4 = \{(u_j, v_j^i) \mid 1 \leq j \leq k \text{ and } u_j \in \mathrm{Ci}(v_0^i, H_i)\},$$
$$E_5 = \{(v_j^i, w) \mid (v_0^i, w) \in E(H_i), 1 \leq j \leq k \text{ and } (u_j, v_0^i, w) \notin \mathrm{Cp}(v_0^i, H_i)\}.$$

The set E_1 represents all edges which are not affected by considering v_0^i at induction step i and is consequently included in H_i as well as in H_{i+1}. Via E_2, all successors of v_0^i in H_i remain successors in H_{i+1}, since this vertex is made to represent all cases where PROP. 37 cannot be applied, and E_3 contains are all corresponding predecessors in H_i. In contrast, the new vertices $v_1^i, \ldots, v_k^i$ are introduced to represent a paths where the conditions of the proposition are met. The edges E_4 subsume the contradiction initiating predecessors corresponding to each of the new vertices, and E_5 subsumes all remaining successors of a v_j^i which cannot by excluded by PROP. 37. The next proposition shows that the result of the ORDAS algorithm is a graph H which contains no path of length 2 contradicting the ordinal assumptions, and maintains other important structural properties of G.

PROPOSITION 38: *The ORDAS algorithm computes a finite sequence of graphs* $H_0, \ldots, H_i, \ldots, H$ *such that* H *has the following properties:*

(i) For all vertices $v \in V(H)$ *the set of contradicting paths* $\mathrm{Cp}(v, H) = \varnothing$.

(ii) For all not-contradicting paths u, v, w *in* H_0*, i.e.* $(u, v, w) \notin \mathrm{Cp}(v, H_0)$*,* u, v, w *is a path in* H.

(iii) For all edges $(v', w') \in E(H)$ *there exists an edge* $(v, w) \in E(H_0)$ *with* $\mathrm{qdir}(v) = \mathrm{qdir}(v')$ *and* $\mathrm{qdir}(w) = \mathrm{qdir}(w')$.

The first property guarantees that all paths to which PROP. 37 applies are eliminated in H. The second property ensures that not too many paths are eliminated. The third safeguards that no essentially new edges are introduced (which would be artifacts of the procedure).

PROOF: Obviously, the algorithm terminates after a finite number of steps since the number of vertices in H_0 is finite.

To prove property (i), we show that at each induction step i it holds for all $v \in \{v_0^i, v_1^i, \ldots, v_k^i\}$ that

$$\mathrm{Cp}(v, H_{i+1}) = \varnothing,$$
$$\text{and } \forall l > i : \mathrm{Cp}(v, H_l) = \varnothing \Rightarrow \mathrm{Cp}(v, H_{l+1}) = \varnothing.$$

For a given $v \in \{v_0^i, v_1^i, \ldots, v_k^i\}$ consider an arbitrary path u, v, w in H_{i+1}.
If $v = v_0^i$, then by DEF. 21 $(u, v) \in E_3$ and $(v, w) \in E_2$. Hence, in H_i there is also an edge (v, w) and an edge (u, v) with $u \notin \mathrm{Ci}(v_0^i, H_i)$. This entails $(u, v_0^i, w) \notin \mathrm{Cp}(v_0^i, H_i)$, such that the path u, v, w cannot by excluded from H_i using PROP. 37. Likewise, it is not a contradicting path in H_{i+1}.
If $v = v_j^i$, then $(v, w) \in E_5$, implying $(u_j, v_0^i, w) \notin \mathrm{Cp}(v_0^i, H_i)$, i.e. there is a path u_j, v_0^i, w in H_i which cannot be excluded by PROP. 37. Since $\mathrm{qdir}(u_j) = \mathrm{qdir}(u)$ and $\mathrm{qdir}(v_0^i) = \mathrm{qdir}(v)$, the path u, v, w can also not be excluded from H_{i+1}.
Assume for the induction step that for a given vertex $v \in V(H_l)$ there is no contradicting path u, v, w in H_l. By DEF. 21, also $v \in V(H_{l+1})$. Since $l > i$, all paths of length 2 in H_{l+1} having v as second vertex are of the form $\hat{u}, v, \hat{w}$ with $\mathrm{qdir}(\hat{u}) = \mathrm{qdir}(u)$ and $\mathrm{qdir}(\hat{w}) = \mathrm{qdir}(w)$. Thus, none of the paths can be excluded in H_{l+1}.

For property (ii), let u, v, w be a path in H_i with $(u, v, w) \notin \mathrm{Cp}(v, H_i)$, and consequently $u \notin \mathrm{Ci}(v, H_i)$. Depending on whether u, v or w equals v_0^i we can determine if (u, v) and (v, w) are elements of E_1, E_2, E_3, E_4 or E_5, making u, v, w a path in H_{i+1}.
First consider that $v = v_0^i$. Then $(v, w) \in E_2$, and as $u \notin \mathrm{Ci}(v_0^i, H_i)$, also the edge $(u, v) \in E_3$.
Now let $v \neq v_0^i$. If $u \neq v_0^i$ then the edge $(u, v) \in E_1$, if $w \neq v_0^i$ then $(v, w) \in E_1$, and if $u = v_0^i$ then $(u, v) \in E_2$. For $w = v_0^i$ and $v \notin \mathrm{Ci}(v_0^i, H_i)$, it holds that $(v, w) \in E_3$. Otherwise, $v \in \mathrm{Ci}(v_0^i, H_i)$, and since $v \neq v_0^i$ there is one $j \geq 1$ such that $(v, v_j^i) \in E_4$. Note that $\mathrm{qdir}(v_j^i) = \mathrm{qdir}(w)$.
Thus, in any case there is a corresponding path in H_{i+1} which cannot be eliminated using PROP. 37 since it has the same qualitative directions as (u, v, w). By induction, the correspondence also holds from H_0 to H.

To show property (iii), consider an edge $e = (v, w) \in E(H_{i+1})$. Distinguish whether e is an element of E_1, E_2, E_3, E_4 or E_5. If $e \in E_1$, it is also an edge in $E(H_i)$. If $e \in E_2$, then $v = v_0^i$, and the edge (v_0^i, w) exists in H_i (and $\mathrm{qdir}(v_0^i) = \mathrm{qdir}(v)$). For $e \in E_3$, we have that $w = v_0^i$, making (v, v_0^i) an edge in H_i, and for $e \in E_5$ it is $v = v_j^i$, making (v_0^i, w) an edge in H_i. In the case that $e \in E_4$, it is $v = u_j$ and $w = v_j^i$ with $v \in \mathrm{Ci}(v_0^i, H_i)$, such that $(u_j, v_0^i) \in E(H_i)$ and $\mathrm{qdir}(v_j^i) = \mathrm{qdir}(v_0^i)$. Since there are no other types of edges in H_{i+1}, all of them correspond to an edge in H_i with identical qualitative directions. By induction, there is also a correspondence from H to H_0. □

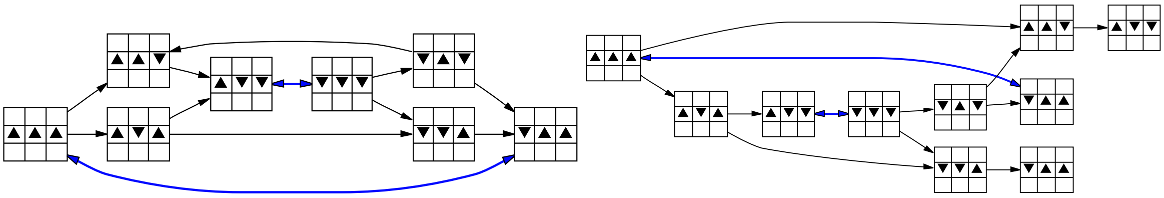

Figure 3.9: The state-transition graph of Ex. 10 (left) and the result of the ORDAS algorithm (right).

I present an example for the ORDAS algorithm.

EXAMPLE 10: Let a monotonic ensemble be given by

$$\Sigma = \begin{pmatrix} + & + & - \\ - & - & 0 \\ - & 0 & + \end{pmatrix}.$$

The resulting state-transition graph is displayed in Fig. 3.9 (left). If we impose the assumptions

$$\begin{aligned} -d_{1,2}^{1,2} = -d_{2,1}^{2,1} = d_{1,2}^{2,1} = d_{2,1}^{1,2} = [+], \\ -d_{1,3}^{3,1} = -d_{3,1}^{1,3} = d_{1,3}^{1,3} = d_{3,1}^{3,1} = [+], \end{aligned}$$

the ORDAS algorithm yields the result as also given in Fig. 3.9 (right). As the result H of the ORDAS algorithm is a graph where vertices represent qualitative states, abstractions techniques can also be applied to H. Performing a no-return abstraction yields a new result: The is a non-trivial no-return set D, i.e. there is no path in H which re-enters $D \subseteq V(H)$ after it is left: all solutions $x(\cdot)$ given by a model $f \in \mathcal{M}(\Sigma)$ which respects the ordinal assumptions and which is soluble for the appropriate components cannot re-enter D. If there is one $t_1 > 0$ such that $[f(x(t_1))] \in D$, and a $t_2 > t_1$ such that $[f(x(t_2))] \notin D$, then $\forall t > t_2 : [f(x(t))] \notin D$. Since there are paths in G which do not occur in H, no-return sets are more likely to occur in H (see Fig. 3.10 for the example). $\square$

Thus, there is a synergy between the new abstraction and reduction techniques presented in this chapter, making qualitative models more valuable for questions of sustainable system design. I illustrate this added value in more detail in Chapter 4. But first the sequence of restricted model ensembles is completed in the next section.

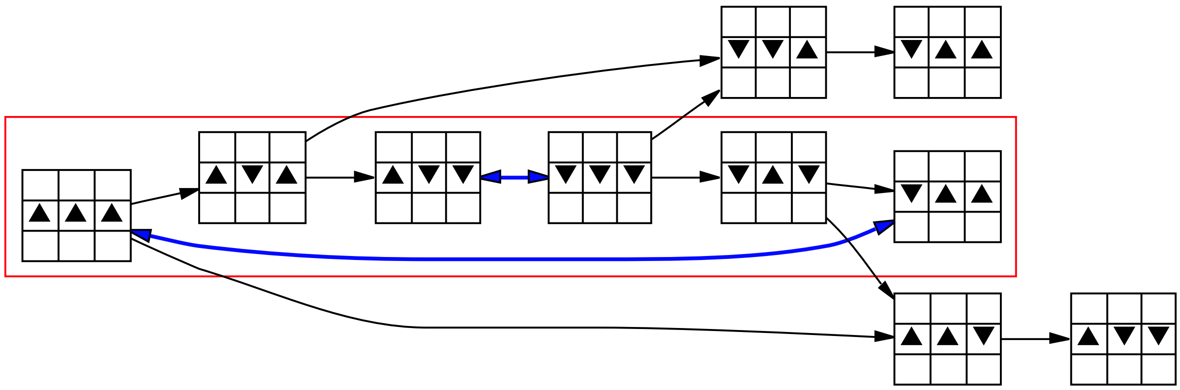

Figure 3.10: No-return abstraction of the result of the ORDAS algorithm for Ex. 10. The no-return set consisting of more than one vertex is displayed as a cluster.

3.4 Quantitative Bounds

In this section I introduce linear-interval differential inclusions, which restrict a monotonic ensemble $\mathcal{M}(\Sigma)$ to models for which interval constraints hold for the components of the Jacobian. If there are multiple successors of a single vertex in the state-transition graph, the question which of them will be attained can be formulated as a viability problem. After defining linear-interval maps and investigating their absorption basins, I show how this can be used to analyse the state-transition graph of a QDE.

3.4.1 Absorption Basins of Linear-Interval Differential Inclusions

Linear-interval differential inclusions are given by set-valued maps. Throughout this section we regard singletons as intervals and consider a state space $X \subseteq \mathbb{R}^n$.

DEFINITION 22: *Let U be a matrix of compact intervals $(u_{i,j})_{i,j=1,\dots,n}$, where each interval either vanishes or does not contain 0. A set-valued map $F : X \rightsquigarrow \mathbb{R}^n, F(x) := Ux$, where the latter denotes interval-valued multiplication, is called a* ***linear-interval map****.*

Interval-valued multiplication is defined in the usual way by $Ux := \{Mx \mid M \in U\}$, where a matrix $M = (m_{i,j})_{i,j=1,\dots,n} \in U$ if and only if $\forall i, j = 1, \dots, n : m_{i,j} \in u_{i,j}$. DEF. 22 guarantees that every coefficient of U has a prescribed sign (which will be related to Σ below). Note that a linear-interval map F defines a model ensemble (cf. section 2.3, p. 42) which includes nonlinear models $f : X \to \mathbb{R}^n$ such that $\forall x \in X : f(x) \in F(x)$. Before analysing absorption basins, the regularity properties of linear-interval maps are investigated. Based on a matrix norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$, we define the norm $\|F\| := \max_{M\in U} \|M\|$.

PROPOSITION 39: *A linear-interval map F has compact interval-valued images. It is Marchaud, Lipschitz (both with constant $\|F\|$) and homogeneous, i.e. $\forall \lambda \in \mathbb{R} : F(\lambda x) = \lambda F(x)$.*

PROOF: (i) F has interval-valued images: Each component $F_i, i = 1, \dots, n$ has the form $\sum_{j=1,\dots,n} u_{i,j} x_i$, where $u_{i,j}$ are compact intervals. The properties of interval arithmetic imply that this yields an interval. As it results from a continuous operation on a compact set, it is also compact.

(ii) F is Marchaud: $Dom(F) = \mathbb{R}^n$ is obviously closed. It has convex values, because they are interval-valued. It has linear growth, because $\|F(x)\| = \|Ux\| \leq \|U\|\|x\| = \|F\|\|x\| \leq \|F\|(\|x\| + 1)$. Its graph is closed, because F has compact values and the upper and lower bounds of the values depend continuously on x (cf. p. 43).
(iii) F is Lipschitz: Let $x, x' \in \mathbb{R}^n$. Since U is compact we can choose a matrix $M \in U$ such that $\|M(x' - x)\| = \|F\|\|x' - x\|$, and define

$$e := -\frac{M(x' - x)}{\|M(x' - x)\|} \in B(0, 1).$$

Therefore,

$$M(x' - x) + \|F\|\|x' - x\|e = M(x' - x) - \frac{\|F\|\|(x' - x)\|}{\|M(x' - x)\|}M(x' - x) = 0,$$

and $0 \in B\big(U(x' - x), \|F\|\|x' - x\|\big)$. Hence, $Ux \subseteq B\big(Ux', \|F\|\|x' - x\|\big)$, i.e. $F(x) \subseteq B\big(F(x'), \|F\|\|x' - x\|\big)$ (cf. p. 42).
(iv) F is homogeneous: Choose arbitrary $x \in \mathbb{R}^n, \lambda \in \mathbb{R}$. Due to the properties of interval arithmetic it holds for all $i = 1, \ldots, n$ that

$$\begin{aligned} F_i(\lambda x) &= \sum_{j=1,\ldots,n} u_{i,j}(\lambda x_i) = \sum_{j=1,\ldots,n} \lambda u_{i,j} x_i \\ &= \lambda \sum_{j=1,\ldots,n} u_{i,j} x_i = \lambda F_i(x). \end{aligned}$$

□

As discussed below, we need to compute absorption basins of linear-interval differential inclusions $\dot{x} \in F(x)$. In principle, this can be done with the viability kernel algorithm (cf. section 2.4, p. 45). But since it is designed for a bounded constrained set and target, and we will have to deal with (unbounded) cones, some remarks are necessary. We start with an observation resulting from the homogeneity of linear-interval maps.

PROPOSITION 40: *Let F be a linear-interval map, $x(\cdot) \in \mathcal{S}_F(x_0)$ a solution for $x_0 \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$. Then $y(\cdot) := \lambda x(\cdot) \in \mathcal{S}_F(x_0)$.*

PROOF: For almost every $t \in \mathbb{R}_+$ it holds that $\dot{y}(t) = \lambda\dot{x}(t) \in \lambda F(x(t))$. Due to PROP. 39, the last term equals $F(\lambda x(t)) = F(y(t))$, making $y(\cdot)$ a solution. □

As a consequence, the absorption basin of a cone is also a cone:

PROPOSITION 41: *Let $C, K \subseteq \mathbb{R}^n, C \subseteq K$ be cones, and $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ a linear-interval map. Then $D = \mathrm{Abs}_F(K, C)$ is a cone.*

PROOF: Choose $x_0 \in D$ and $\lambda > 0$. We show that $y_0 := \lambda x_0 \in D$.
By definition of the absorption basin, for all solutions $x(\cdot) \in \mathcal{S}_F(x_0)$ there exists a $T > 0$ such that $x(T) \in C$ and $\forall t \in [0, T] : x(t) \in K$ (see DEF. 13, p. 46). Define $y(\cdot) := \lambda x(\cdot)$, which is an element of $\mathcal{S}_F(y_0)$ by PROP. 40. Since C, K are cones, this proposition also implies that $y(T) = \lambda x(T) \in \lambda C = C$ and $\forall t \in [0, T] : y(T) = \lambda x(t) \in \lambda K = K$. Therefore, $y_0 \in D$. □

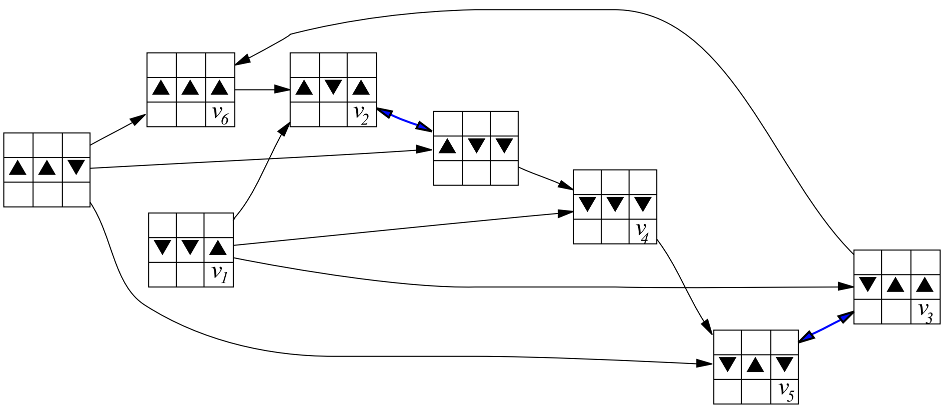

Figure 3.11: State-transition graph of the QDE defined in EX. 11 after removing marginal edges, non-analytical and equilibrium states. The vertices $v_1, \ldots, v_6$ are discussed in the text.

For cones C, K we can compute the bounded set $D_\lambda := \mathrm{Abs}(K \cap \lambda Q, C \cap \lambda Q)$, where $\lambda > 0$ and Q is an appropriately chosen bounded set.

PROPOSITION 42: *Let* $C, K \subseteq \mathbb{R}^n, C \subseteq K$ *be cones,* $Q \subseteq \mathbb{R}^n$ *and* $\lambda > 0$*. Then* $D_\lambda = \lambda D_1$.

PROOF: For $x_0 \in D_\lambda$, we show that $y_0 := \frac{1}{\lambda} x_0 \in D_1$: Choose $x(\cdot) \in \mathcal{S}_F(x_0)$ and set $y(\cdot) := \frac{1}{\lambda} x(\cdot)$, which is an element of $\mathcal{S}_F(\frac{1}{\lambda} x_0)$ due to PROP. 40. If $\exists T > 0 : x(T) \in C \cap \lambda Q = \lambda C \cap \lambda Q = \lambda(C \cap Q)$, then $y(T) \in C \cap Q$. If $\forall t \in [0, T] : x(t) \in K \cap \lambda Q = \lambda(K \cap Q)$, then $y(t) \in K \cap Q$. Thus $y_0 \in D_1$, i.e. $D_\lambda \subseteq \lambda D_1$.
By symmetry, $\lambda D_1 \subseteq D_\lambda$ also holds. □

Finally, the absorption basin $\mathrm{Abs}_F(K, C)$ can be recovered from D_λ by the following property:

PROPOSITION 43: *If* $0 \in \mathrm{Int}(Q)$ *then* $\mathrm{Abs}_F(K, C) = \bigcup_{\lambda > 0} D_\lambda$.

PROOF: If $x_0 \in \bigcup_{\lambda > 0} D_\lambda$ there exists one $\lambda > 0$ such that $x_0 \in D_\lambda$. Thus, for all $x(\cdot) \in \mathcal{S}_F(x_0) \exists T > 0 : x(T) \in C \cap \lambda Q \subset C$ and $\forall t \in [0, T] : x(t) \in K \cap \lambda Q \subset K$. Therefore, $x_0 \in \mathrm{Abs}(K, C)$.

Now choose $x_0 \in \mathrm{Abs}_F(K, C)$ and a solution $x(\cdot) \in \mathcal{S}_F(x_0)$. By DEF. 13 (p. 46), there is one $T > 0$ such that $x(T) \in C$ and $\forall t \in [0, T] : x(t) \in K$. Since $x(\cdot)$ is continuous, $x(T)$ finite and $0 \in \mathrm{Int}(Q)$, there is one $\lambda > 0$ such that $x(T) \in C \cap \lambda Q$ and $\forall t \in [0, T] : x(t) \in K \cap \lambda Q$. Therefore, $x_0 \in \bigcup_{\lambda > 0} D_\lambda$. □

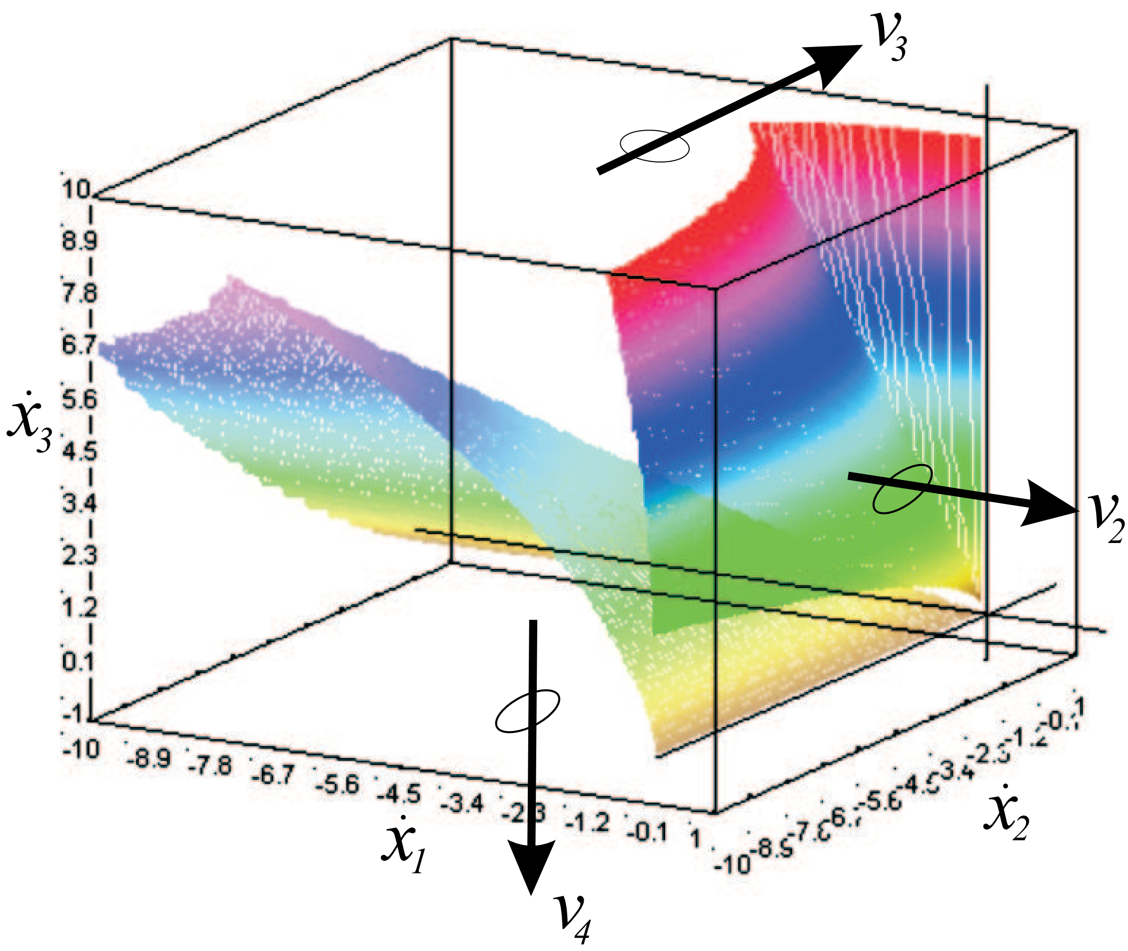

Figure 3.12: Boundaries of the absorption basins of v_1 with target v_2, v_3, v_4 restricted to a cube Q. Large arrows indicate the directions towards successors. Qualitative transitions to v_2 necessarily occur from $Abs_F\big(\bar{K}(v_1)) \cap Q, \bar{K}(v_1) \cap \bar{K}(v_2) \cap Q\big)$, the region between the surface to the right and the plane given by $\dot{x}_1 = 0$. A shift to v_4 happens from $Abs_F\big(\bar{K}(v_1) \cap Q), \bar{K}(v_1) \cap \bar{K}(v_4) \cap Q\big)$ between the lower surface and the plane given by $\dot{x}_3 = 0$. $Abs_F\big(\bar{K}(v_1) \cap Q, \bar{K}(v_1) \cap \bar{K}(v_3) \cap Q\big)$, which would lead to state v_3, is empty.

3.4.2 Analysing a State-Transition Graph with Linear-Interval Differential Inclusions

I now show how the absorption basins of linear-interval differential inclusions can be used infer system knowledge from a QDE when quantitative bounds are available. Starting with a sign matrix Σ, we set-up a monotonic ensemble and solve it with the QSIM algorithm. Then, quantitative bounds are considered by setting up a linear-interval differential inclusion where the signs of the intervals correspond to the signs of Σ. I give a precise definition of the differential inclusion and discuss its relation to the associated QDE. Recall that a monotonic ensemble $\mathcal{M}(\Sigma)$ defines a set of systems $\dot{x} = f(x)$ such that for all $x \in X : \big[\mathcal{J}(f(x))\big] \approx \Sigma$. Changing the perspective from the state space to the velocity space, we saw in section 2.3

(p. 42) that it is not possible to investigate a QDE by considering a differential inclusion

$$\ddot{x} \in \hat{F}(\dot{x}) := \{A\dot{x} \mid [A] \approx \Sigma\},$$

since the set-valued map $\hat{F}$ is unbounded. However, if intervals $u_{i,j}$ are known such that $\forall x \in X : D_j f_i(x) \in u_{i,j}$, the linear-interval differential inclusion

$$\ddot{x} \in F(\dot{x}) = U\dot{x},$$

can be set-up. It is very regular by PROP. 39 (p. 77), and "simulates" the monotonic ensemble $\mathcal{M}(\Sigma)$ in the following sense. Define the restricted model ensemble

$$\mathcal{M}'(\Sigma, U) := \{f \in M(\Sigma) \mid \forall x \in X : \mathcal{J}(f)(x) \in U\} \subseteq \mathcal{M}(\Sigma).$$

with the solution operator $\mathcal{S}_{\mathcal{M}'(\Sigma,U)}(\cdot)$. Then $\forall x_0 \in X, x(\cdot) \in \mathcal{S}_{\mathcal{M}'(\Sigma,U)}(x_0) : \dot{x}(\cdot) \in \mathcal{S}_F(\dot{x}(0))$. On the other hand, the differential inclusion also covers solutions of non-autonomous ODEs $\dot{x} = f(x,t)$ with $\mathcal{J}(f)(x,t) \in U$ for all $t \in \mathbb{R}_+$. Linear-interval differential inclusions are more general than QDEs in the sense that they also include non-autonomous models, and are more specific in the sense that they only include bounded models.

When the state-transition graph G of the QDE is computed and a linear-interval differential inclusion F is defined, I propose the following procedure. The modeller takes a close look at the state-transition graph of the QDE and identifies subgraphs of importance, i.e. vertices with multiple successors (some of which may be problematic or preferable by value judgement). We want to identify conditions for a given successor to be reached. If there is an edge (v, w) in G, we know from section 2.2.1 (p. 21) that there is an initial valued $x_0 \in X$ and a solution $x(\cdot) \in \mathcal{S}_{\mathcal{M}(\Sigma)}(x_0)$ such that $[\dot{x}(0)] = v$ and it $\exists T > 0 : [\dot{x}(T)] = v \wedge w$ and $\forall t \in [0, T) : \operatorname{sgn}(\dot{x}(t)) = v$.

To describe these reachability conditions we define – in the velocity space – the cones $K(v) := \{\dot{x} \in \mathbb{R}^n \mid [\dot{x}] = v\}$ for $v \in \mathcal{A}^n$. For the linear-interval differential inclusion $\ddot{x} \in F(\dot{x}) = U\dot{x}$, the absorption basin $\mathrm{Abs}_F(\bar{K}(v), \bar{K}(v) \cap \bar{K}(w))$ of the closure of such cones contains all initial velocities $\dot{x}_0$ such that for all solutions $\dot{x}(\cdot) \in \mathcal{S}_F(\dot{x}_0)$ with $[\dot{x}_0] = v$ there exists a $T > 0$ with $\dot{x}(T) \in \bar{K}(v \wedge w)$ and $\forall t \in [0, T] : \dot{x}(t) \in \bar{K}(v)$. The results from the previous subsection can be used to compute this absorption basin with the viability kernel algorithm as outlined in section 2.4 (p. 45). If $\mathrm{Abs}_F(\bar{K}(v), \bar{K}(v) \cap \bar{K}(w))$ is empty, the edge (v, w) can be eliminated. Otherwise, the algorithm provides insights about the velocities for which a qualitative state is reached from another one. This can be valuable in the context of sustainability science, as such conditions yield early warning indicators of the form "once the rates of change are in such and such a relation, the following trend will necessarily reverse at a later time".

EXAMPLE 11: We present an application of the method, where the monotonic ensemble is described by the sign matrix

$$\Sigma := \begin{pmatrix} 0 & 0 & + \\ - & 0 & 0 \\ + & + & 0 \end{pmatrix}.$$

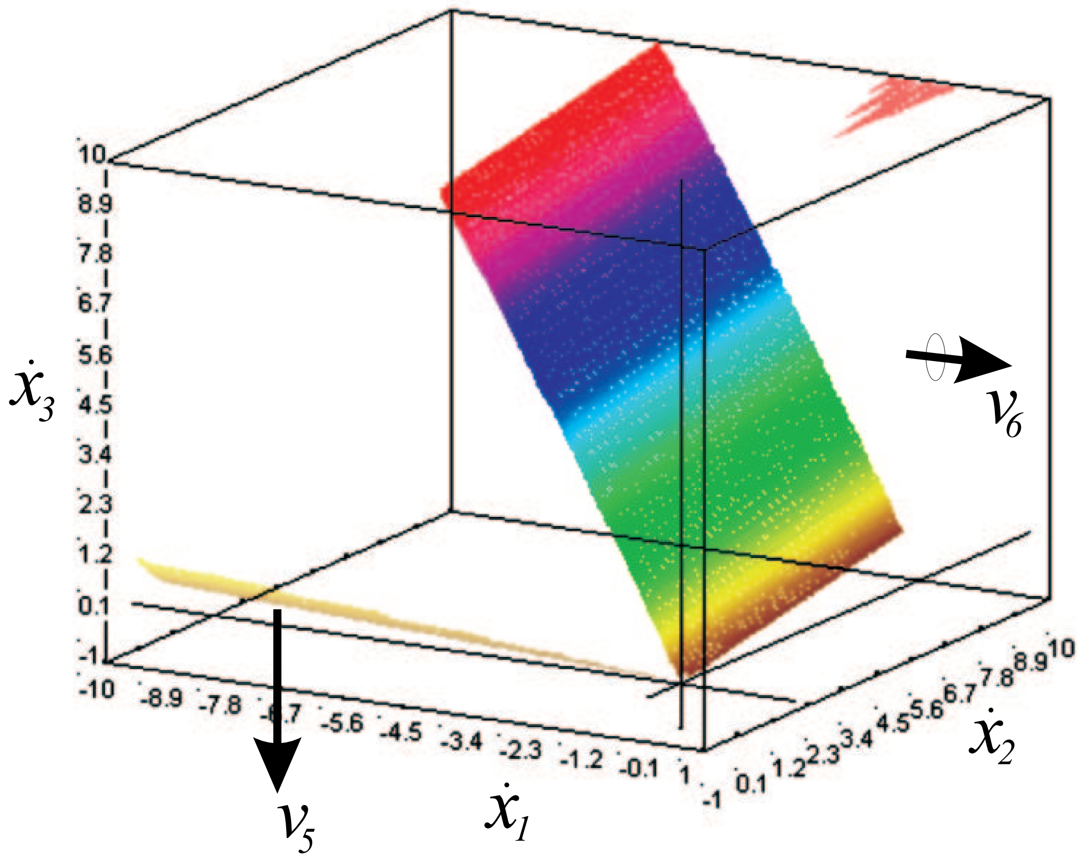

Figure 3.13: Boundaries of the absorption basins of v_3, with target v_5, v_6, restricted to a cube Q. $Abs_F\big(\bar{K}(v_3) \cap Q, \bar{K}(v_3) \cap \bar{K}(v_5) \cap Q\big)$ is a very small cone, whereas the boundary of $Abs_F\big(\bar{K}(v_3) \cap Q, \bar{K}(v_3) \cap \bar{K}(v_6) \cap Q\big)$ appears to be a plane separating a large part of the quadrant. The small part of the boundary to the upper right is an artifact resulting from restricting the absorption basin to Q, which can be eliminated due to PROP. 43 (p. 79).

The resulting graph after eliminating marginal edges (cf. section 3.2, p. 62) and non-analytical state (cf. section 2.2.4, p. 36) is shown in in Fig. 3.11. Supposing that quantitative information is available we set up an interval matrix

$$U := \begin{pmatrix} 0 & 0 & [0.7, 0.9] \\ [-0.7, -0.4] & 0 & 0 \\ [0.5, 3.0] & [0.5, 3.0] & 0 \end{pmatrix},$$

where the coefficients have signs corresponding to Σ. This defines the linear-interval differential inclusion $\ddot{x} \in U\dot{x}$. We now analyse two exemplary qualitative states $v_1 = ([-]\,[-]\,[+])^t$, $v_3 = ([-]\,[+]\,[+])^t$ in G where multiple successors occur (numbers in Fig. 3.11 correspond to indices). The computed absorption basins of state v_1 with targets v_2, v_3, v_4 are shown in Fig. 3.12. One absorption basin is empty. The boundaries of the other basins are smooth except along one ray from the origin. There are no combinations of velocities which safeguard that v_3 is reached, but there is a considerable likelihood that v_2 is a guaranteed successor. However, a large part of this quadrant necessarily leads to v_4. In state v_3, we have the successors v_5 and v_6. Since there is no edge (v_3, v_1) the corresponding absorption basin $Abs_F\big(\bar{K}(v_3)), \bar{K}(v_3) \cap \bar{K}(v_1)\big)$ (and even the respective capture basin) has to be empty. The size of the regions necessarily leading from v_3 to one of the outcomes is considerably different (Fig. 3.13). As both absorption basins of state v_3 intersect $K(v_1 \wedge v_3)$, in some cases the successor of v_3 can already be predicted at a time t, when $[\dot{x}(t)] = v_1$. If v_5 is reached, (which may be less likely as the absorption basin is significantly smaller than the other), the state-transition graph implies that the only possible subsequent qualitative transition is (v_5, v_3). $\square$

Chapter 4

Management of Natural Resources

In this chapter I apply the framework of model ensembles proposed Chapter 2, in particular qualitative differential equations (QDEs), differential inclusions and viability theory to develop novel conceptual models for three examples of natural resource problems, namely subsistence farming, marine capture fisheries and fresh water use. These models demonstrate the scope and limits of the abstraction and restriction techniques developed in Chapter 3 (which are intensively used) and show how the investigation of model ensembles yields robust results for resource management.

I first give a short outline of the resource problems. In many cases, natural resources are *common pool resources*, so it is often difficult to exclude users from the resource or limit extraction and pollution. While users benefit individually from resource utilisation, the costs of a degraded resource are likely to be shared by the community (Hardin 1968). Thus, to avoid degradation, management strategies and institutional arrangements are needed which guarantee sustainable use. Actually, some kind of management is implemented for most exploited natural resources, but not always in a sustainable way (Ostrom 1990).

Agricultural production increased substantially during the 20th century, primarily due to industrialised agriculture, intensification of cultivated systems, and expansion of cultivated areas (Millennium Ecosystem Assessement 2005). One key factor for agriculture is the quality of soil. Much land is still used for subsistence farming, which is important for food safety in developing countries. Subsistence farming can evolve along the so called impoverishment-degradation spiral: existential rural poverty forces farmers to intensify their land use. This leads to soil degradation, which reduces yield and thereby further exacerbates rural poverty (Leonhard 1989; Reenberg and Paarup-Laursen 1997). There is ongoing research on adequate practices to stop or prevent this problematic interaction of processes (e.g. Lüdeke et al. 1999; Reij et al. 2005).

Marine fish stocks are degrading worldwide. Due to globally decreasing catches, in many cases the fishing industry can only be sustained at an economic level by paying high subsidies, while at the same time increased capitalisation puts additional pressure on the stocks (Banks 1999; Munro 1999; Pauly et al. 2002). As a consequence of this intricate situation there is an ongoing debate on adequate control and management instruments. Recent years have seen a number of bio-economic models examining the effects of commercial fishery on marine resources.

Fresh water is threatened by eutrophication, triggered by high nutrient loads of urban runoff and excessive agricultural use of fertilisers. In this sense, inland waters are overused as a sink. Eutrophication is typically associated with algal blooms, declining fish populations, and loss of recreation opportunities (Lathrop et al. 1998).

Within the list proposed by the German Advisory Council on Global Change (WBGU 1996) the above land use problem can be classified as Sahel Syndrome (overcultivation of marginal land), the fishery problem as Overexploitation Syndrome (overexploitation of natural ecosystems), and the eutrophication problem as Waste Dumping Syndrome (environmental degradation due to controlled and uncontrolled waste disposal). As discussed in Chapter 1, several properties of these problems make it promising to use model ensembles and further methods I presented and developed in Chapters 2 and 3:

- They appear at different places in a similar way, making a generalised identification of patterns of global environmental change valuable.
- They are characterised by various uncertainties. For example, in the domain of fisheries we must live with the fact that the amount of fish and its growth properties as well as the functions describing changes in behaviour of fishing firms are not exactly known (Clark 1999; Whitmarsh et al. 2000; Charles 2001). The latter also holds for the behaviour of subsistence farmers. In the case of eutrophication, storages and flows of nutrients are not all easily measured (e.g. in the mud of a lake), and some highly variable processes are not completely understood in a quantitative sense (Lathrop et al. 1998).
- They involve normative considerations, since a *problematic* pattern has to be understood.
- In addition to addressing generality and uncertainty, we demonstrate in this chapter how QDEs can be used to advance from identifying dynamical patterns to designing management options.

4.1 Land-Use Changes in Developing Countries

A well-known QDE model from sustainability science is presented in this section to demonstrate the most basic ensemble methods. Its state-transition graph is computed, which has a simple no-return abstraction, and examples are given how the methods contribute to the understanding of the motivating real-world problem. Some management interventions can be analysed in a straightforward way within this framework.

The Sahel Syndrome Model

The model studies regional land-use changes due to subsistence farming in developing countries (for details see Petschel-Held et al. 1999; Petschel-Held and Lüdeke 2001; Eisenack and Petschel-Held 2002). State variables are the quality of the resource R, agricultural activities A, and the poverty level P. Obviously, high agricultural activities reduce R due to overuse, while low A has a positive effect on the quality of the resource. Poverty increases agricultural activity, being the constituting behavioural assumption in a context of subsistence farming. Poverty inversely depends on agricultural yield, which increases with activity and the quality of the resource. The first question is whether these mechanisms necessarily bring about the poverty-degradation spiral. We are interested in measures shifting the system in a favourable direction. The model is described by the following equations:

$$\begin{aligned} \dot{A} &= b(P), \\ \dot{R} &= r(A), \\ P &= y(A, R), \end{aligned}$$

with $A, R, P \in \mathbb{R}_+$. The behavioural function $b \in C^1(\mathbb{R}_+, \mathbb{R}), D_P b > 0$ assigns to a given poverty level the change of agricultural activity. The soil regeneration function $r \in C^1(\mathbb{R}_+, \mathbb{R})$ is strictly decreasing with respect to A. Poverty is reduced by yield via the function $y \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ with $D_A y, D_R y < 0$: economic production increases with effort A and resource quality R, thus reducing poverty.

By substituting y for P, poverty can be eliminated from the model, but as poverty is an important component of the Sahel Syndrome, we want to keep it as a state variable. By differentiation,

$$\begin{aligned} \dot{P} &= D_A y \cdot \dot{A} + D_R y \cdot \dot{R} \\ &= D_A y(A, R) \cdot b(P) + D_R y(A, R) \cdot r(A), \end{aligned}$$

yielding some sign ambiguities in the Jacobian since

$$\begin{aligned} \left[D_A \dot{P}\right] &= \left[D_{AA} y \cdot b + D_{RA} y \cdot r + D_R y \cdot D_A r\right] = [?], \\ \left[D_R \dot{P}\right] &= \left[D_{AR} y \cdot b + D_{RR} y \cdot r\right] = [?], \\ \left[D_P \dot{P}\right] &= \left[D_A y \cdot D_P b\right] = [-], \end{aligned}$$

unless we make appropriate assumptions on the second derivatives of the yield function. Thus, taking $(A\,R\,P)^t$ as state vector, the basic monotonicity properties of the model are captured by the sign matrix

$$\Sigma = \begin{pmatrix} 0 & 0 & [+] \\ [-] & 0 & 0 \\ [?] & [?] & [-] \end{pmatrix}.$$

The model is refined by introducing landmarks and setting up a monotonic landmark ensemble. The maximum sustainable agriculture is denoted by the landmark $\texttt{ms}$, and it is assumed that $r(\texttt{ms}) = 0$, i.e. below $\texttt{ms}$ the soil regenerates, while it degrades above. Similarly, we introduce $\texttt{ex}$ for poverty (existential poverty level) with $b(\texttt{ex}) = 0$, meaning that for a poverty level above $\texttt{ex}$ agricultural activities increase (to offset yield losses). Furthermore, landmarks for the upper bounds of the variables are set. As introduced in section 2.2.3 (p. 32), the state space is augmented with the velocity variables dA, dR and dP. We obtain the quantity spaces

$$\begin{aligned}
Q_A &:= (0, \{0, \texttt{ms}\}, \texttt{ms}, \{\texttt{ms}, \texttt{Amax}\}, \texttt{Amax}), \\
Q_R &:= (0, \{0, \texttt{Rmax}\}, \texttt{Rmax}), \\
Q_P &:= (0, \{0, \texttt{ex}\}, \texttt{ex}, \{\texttt{ex}, \texttt{Pmax}\}, \texttt{Pmax}), \\
Q_{dA} &:= (\{-\infty, 0\}, 0, \{0, \infty\}), \\
Q_{dR} &:= (\{-\infty, 0\}, 0, \{0, \infty\}), \\
Q_{dP} &:= (\{-\infty, 0\}, 0, \{0, \infty\}),
\end{aligned}$$

the resulting quantity space Q and the qualitative state space S. By default, three constraints link state and velocity variables:

$$\begin{aligned}
C_1 &:= \{v \in S \mid \mathrm{qdir}_A(v) = [\mathrm{qmag}_{dA}(v)]\}, \\
C_2 &:= \{v \in S \mid \mathrm{qdir}_R(v) = [\mathrm{qmag}_{dR}(v)]\}, \\
C_3 &:= \{v \in S \mid \mathrm{qdir}_P(v) = [\mathrm{qmag}_{dP}(v)]\}.
\end{aligned}$$

The basic properties of the yield function are expressed by the constraint

$$\begin{aligned}
C_4 := \{v \in S \mid \\
&[\mathrm{qmag}_A(v)][\mathrm{qmag}_R(v)] = -[\mathrm{qmag}_P(v)] \wedge \\
&[\mathrm{qmag}_A(v)]\mathrm{qdir}_R(v) + [\mathrm{qmag}_R(v)]\mathrm{qdir}_A(v) = -\mathrm{qdir}_P(v)\},
\end{aligned}$$

which accounts for the cases where one or more of the qualitative directions or magnitudes vanish. The zeros of r and m are expressed by

$$\begin{aligned}
C_5 &:= \{v \in S \mid [\mathrm{qmag}_P(v)]_{\texttt{ex}} = \mathrm{qdir}_A(v)\}, \\
C_6 &:= \{v \in S \mid [\mathrm{qmag}_A(v)]_{\texttt{ms}} = -\mathrm{qdir}_R(v)\}.
\end{aligned}$$

Defining $C := \{C_1, \ldots, C_6\}$ and the constant mapping $\mu : Q \to \mathcal{A}^{3\times 3}, q \mapsto \Sigma$, we obtain a monotonic landmark ensemble $\mathcal{M}(\mu, C)$. All solutions of ODEs with right-hand side $f \in \mathcal{M}(\mu, C)$ on the state space $X = \mathbb{R}^3_+$, i.e. $S_{\mathcal{M}(\mu,C)}(\mathbb{R}^3_+)$, are possible evolutions of agricultural systems as described by the Sahel Syndrome model. These can be computed using the QSIM algorithm with a model code as follows:

```
(quantity-spaces
 (A  (0 ms Amax)    "Activity")
 (dA (minf 0 inf)   "dA")
 (R  (0 Rmax)       "Resource")
 (dR (minf 0 inf)   "dR")
 (P  (0 ex Pmax)    "Poverty" ))

(constraints
 ((d//dt A dA))
 ((d//dt R dR))
 ((M+ P dA) (ex 0))
 ((M- A dR) (ms 0))
 (((M - -) A R P)))
```

The landmarks for the upper and lower bounds of the quantity spaces are not designed to appear in any constraint, but have another purpose. If a qualitative state v is considered by the QSIM algorithm where one $\mathrm{qmag}_i(v), i \in \{1, 2, 3\}$ attains one of these landmarks and $\mathrm{qdir}_i(v)$ is such that the landmark will be transgressed, the state v is regarded as a final state, i.e. no further successors of this state are generated. Thus, states are automatically detected where the soil totally degrades, where efforts come their limits, etc.

Results

The quantity space of the model consists of 2025 qualitative states. By applying the QSIM algorithm, we obtain a state-transition graph with 158 edges and 49 vertices. Some basic abstraction and restriction techniques further simplify the result (see section 2.2.4, p. 36). We end up with 20 edges and 20 vertices, of which 16 are final states where at least one variable attains its bound (see Fig. 4.1, Tab. 4.1). The no-return abstraction (see section 3.1, p. 52) of the graph is simple in this case because the graph contains no strongly connected components, i.e. every edge is irreversible.

The irreversibility of all edges expresses an important feature of the model – it brings the agricultural system to a situation which cannot be changed without an intervention. Since the qualitative model subsumes a set of ODEs defined by right-hand sides f in a monotonic landmark ensemble, interventions which change the quantitative state of the system without crossing a landmark or which replace f by another right-hand side $f' \in \mathcal{M}(\mu, C)$ have no substantial effect. Fig. 4.1 shows that not every final state is problematic: although there are cases where the resource quality is reduced to a minimum level or poverty comes to a maximum, there are also final outcomes with a recovered resource or a level of well-being above the existential level.

Value judgements enter the analysis at this stage. In Fig. 4.1 an example is provided for such a valuation of final states, based on the qualitative magnitudes and directions of P and R. A degrading resource and existential poverty are considered as problematic, while

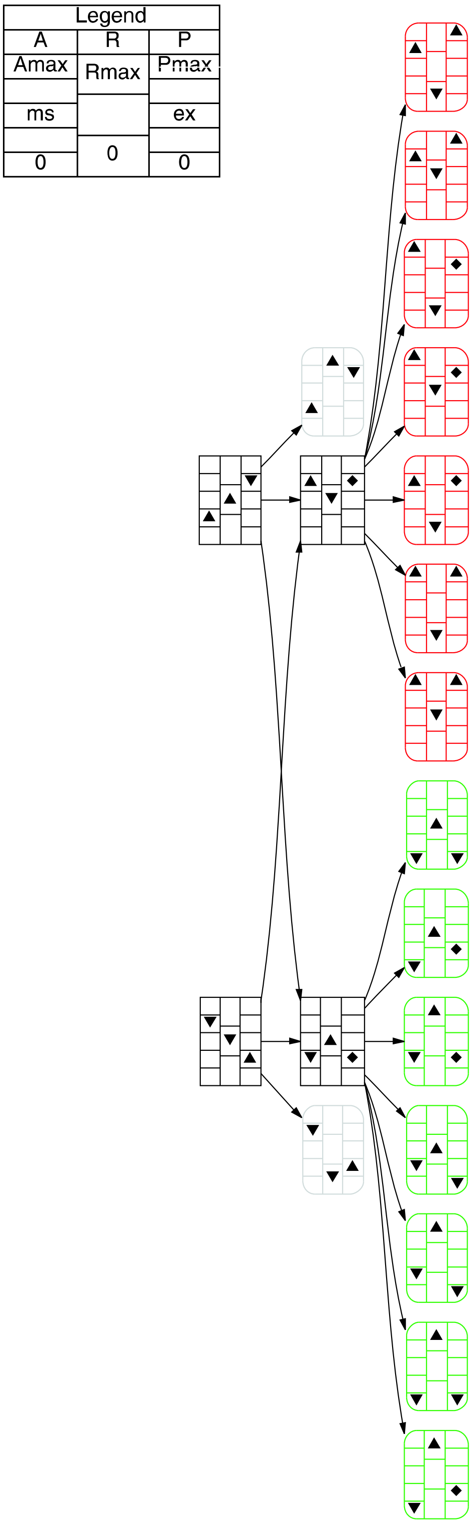

Figure 4.1: Abstracted and restricted state-transition graph of the core of the Sahel Syndrome (computer-generated output). The columns in the vertices represent qualitative values as given in the legend, where landmarks and intervals between landmarks alternate. Diamonds abstract multiple qualitative directions detected by chatter-box abstraction. Colours indicate value judgements as explained in the text.

Graph	Vertices	Edges
State-transition graph	49	158
After chatter-box abstraction	33	42
Removing marginal edges	33	36
Removing non-analytical states	20	20

Table 4.1: Number of vertices and edges resulting from different restriction techniques applied subsequently to the Sahel Syndrome model.

a recovering resource or low poverty is preferable. If a variable is in a preferable state and the other not a problematic one, it is coloured green. If one is problematic and the other not preferable, it is red. The ambiguous cases where one variable is in a problematic and the other in a preferable state are grey.

Both problematic and preferable outcomes are possible for an initial state with increasing agricultural activities below the maximum sustainable level and decreasing, but existential poverty. The same applies for decreasing activities which degrade soils, combined with increasing poverty below the existential level. But, once activity *and* poverty are above the critical landmarks, it is *inevitable* for every solution of the monotonic landmark ensemble $\mathcal{M}(\mu, C)$ that the resource totally degrades or poverty remains critical. Conversely, a positive development necessarily occurs if poverty and agricultural activity are low at the same time.

Management

Three types of potential interventions into systems without control variables can be distinguished (Eisenack and Petschel-Held 2002):

> **External interventions**: A manager is temporarily introduced who alters state variables to shift the system directly to another qualitative state. During the intervention, the mechanism of the QDE is postponed but becomes active afterwards again.
>
> **Structural management**: The social-ecological conditions are changed such that another model ensemble has to be chosen, e.g. by another sign matrix or the introduction of new landmarks and variables. This results in a different state-transition graph where, e.g. problematic invariant sets may be resolved.
>
> **Micro-management**: Management changes parameters such that the ODE describing the system is defined by a new right-hand side which is a member of the same model ensemble as before. The effect can be a change in the tendency of the system to shift to one or another successor state. As this does not change the state-transition graph, the evaluation of micro-management is beyond the scope of qualitative reasoning.

Petschel-Held et al. (1999) discuss three external interventions into the Sahel Syndrome dynamics. (i) A policy to combate poverty is initialised if P is existential, and results in poverty reduction to below the existential level. (ii) The agricultural impact on soils is mitigated when $\mathrm{qmag}_A(v) > \texttt{ms}$, with the effect that activity is below the critical level afterwards. (iii) Application of both policies at the same time. Interestingly, only the latter guarantees

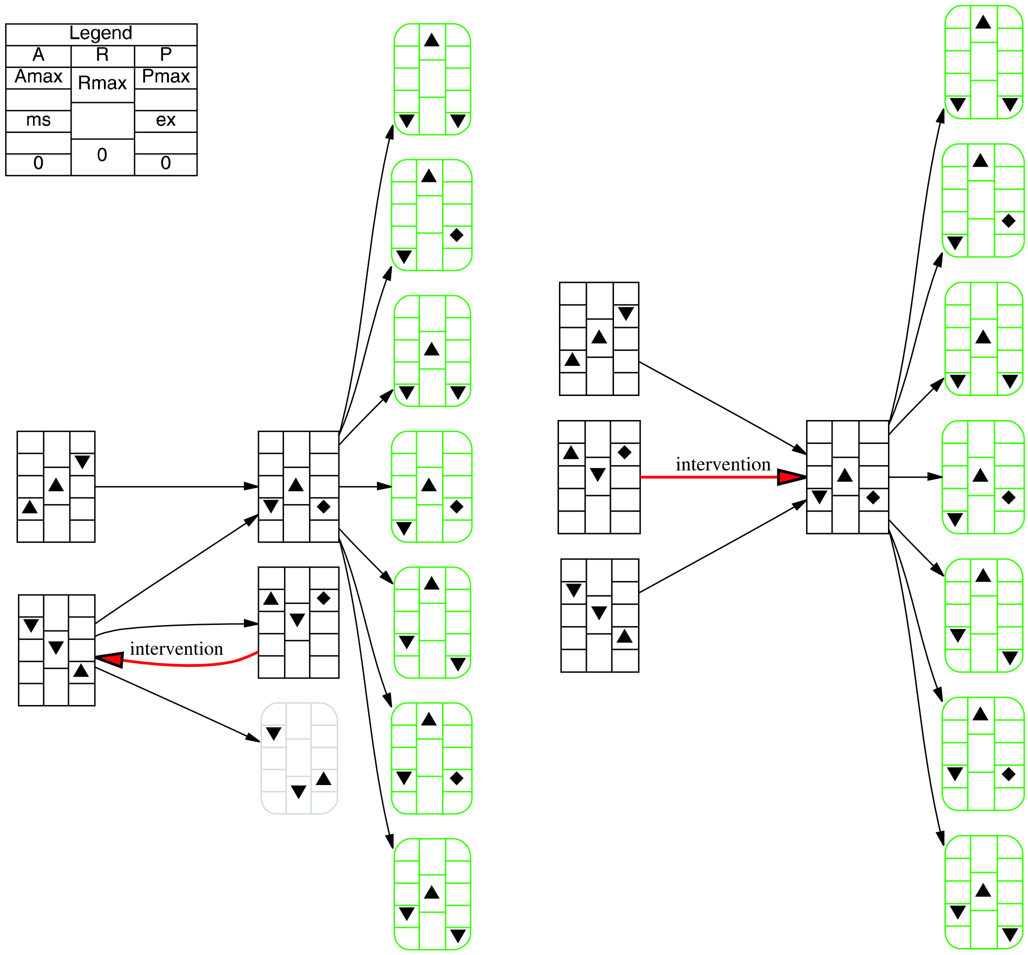

Figure 4.2: State-transition graph modified by external interventions (left: only combating poverty, right: combating poverty and mitigating agricultural impact). The columns represent the same qualitative values as in Fig. 4.1.

an improvement in every case (cf. Fig. 4.2). Combating poverty shifts the system back to a state where the resource may recover, but does not necessarily have to. It may also come back to the situation where the intervention has to be applied again. If only the agricultural impact is changed, the effect is symmetrical. This is avoided if both types of intervention are combined.

Now we analyse an example of structural management. In Eq. (4.1) we observed two sign ambiguities. Suppose some policy influences the agriculture such that these ambiguities are resolved so that $[D_A\dot{P}] = [+]$ and $[D_R\dot{P}] = [-]$. This means that a high resource quality always reduces poverty (if nothing else changes), while high activities have an adverse effect. This can be a consequence of introducing agricultural techniques more sensitive to the resource and measures so as to dampen the influence of income on poverty. Would such a policy be beneficial or not? By including the constraint

```
(((M + - -) A R P dP))
```

in the model description, i.e. defining the sign matrix

$$\Sigma' = \begin{pmatrix} 0 & 0 & [+] \\ [-] & 0 & 0 \\ [+] & [-] & [-] \end{pmatrix},$$

and the sign map $\mu' : Q \to \mathcal{A}_*^{3\times 3}, q \mapsto \Sigma'$, we obtain a new QDE defined by the monotonic landmark system $\mathcal{M}(\mu', C)$ on the same state space as $\mathcal{M}(\mu, C)$. The resulting state-transition graph can be computed and compared to that of $\mathcal{M}(\mu, C)$ (see Fig. 4.3). As expected, certainties increase slightly: In the state with low agricultural activity, recovering resource and low as well as decreasing poverty (a), it is already sure that the outcome will be positive. In the original model it is also possible that poverty begins to increase again (there, the edge (b) is bidirectional, making both states a chatter-box). The situation is symmetric for edge (c), making high and increasing poverty and high agricultural activity a safe predictor for a bad outcome: it would be dangerous to recommend the structural management proposed here as a panacea, since its success or failure depends on the actual situation of the system. This is emphasised by the fact that some parts of the state-transition graph cannot be reached from every initial state. This has the consequence that combining this structural management with intervention (i) from above (combating poverty) is sufficient for a good outcome. However, there is a caveat to investigating structural management in a way like here. Expanding the two assumptions about the effect of management, we obtain

$$\begin{aligned} D_{AA}y \cdot b + D_{RA}y \cdot r + D_R y \cdot D_A r &> 0, \\ D_{AR}y \cdot b + D_{RR}y \cdot r &< 0. \end{aligned}$$

Thus, the monotonic landmark ensemble $\mathcal{M}(\mu', C)$ contains all functions $f \in \mathcal{M}(\mu, C)$ for which both relations hold for every state in X. A restriction of this kind may make $\mathcal{M}(\mu', C)$ an empty set, although this is not always obvious. In this case the required relations can still hold on a restricted region of the state space $X' \subsetneq X$. If it can be justified that an investigated system stays in X', the state-transition graph remains meaningful. Otherwise, the solution can be determined independently for monotonic landmark ensembles on different regions of the state space and the solutions have to be combined appropriately.

Summing up, we have learned from the qualitative Sahel Syndrome model that the underlying mechanism does not always bring about the impoverishment-degradation spiral. On the other hand, there are qualitative states where the outcome is more predictable and management can avoid a critical development. Two management options were analysed using QDEs and the state-transition graph. It can be seen that simple interventions are not sufficient: combining different external interventions, or external interventions with structural management are more efficient.

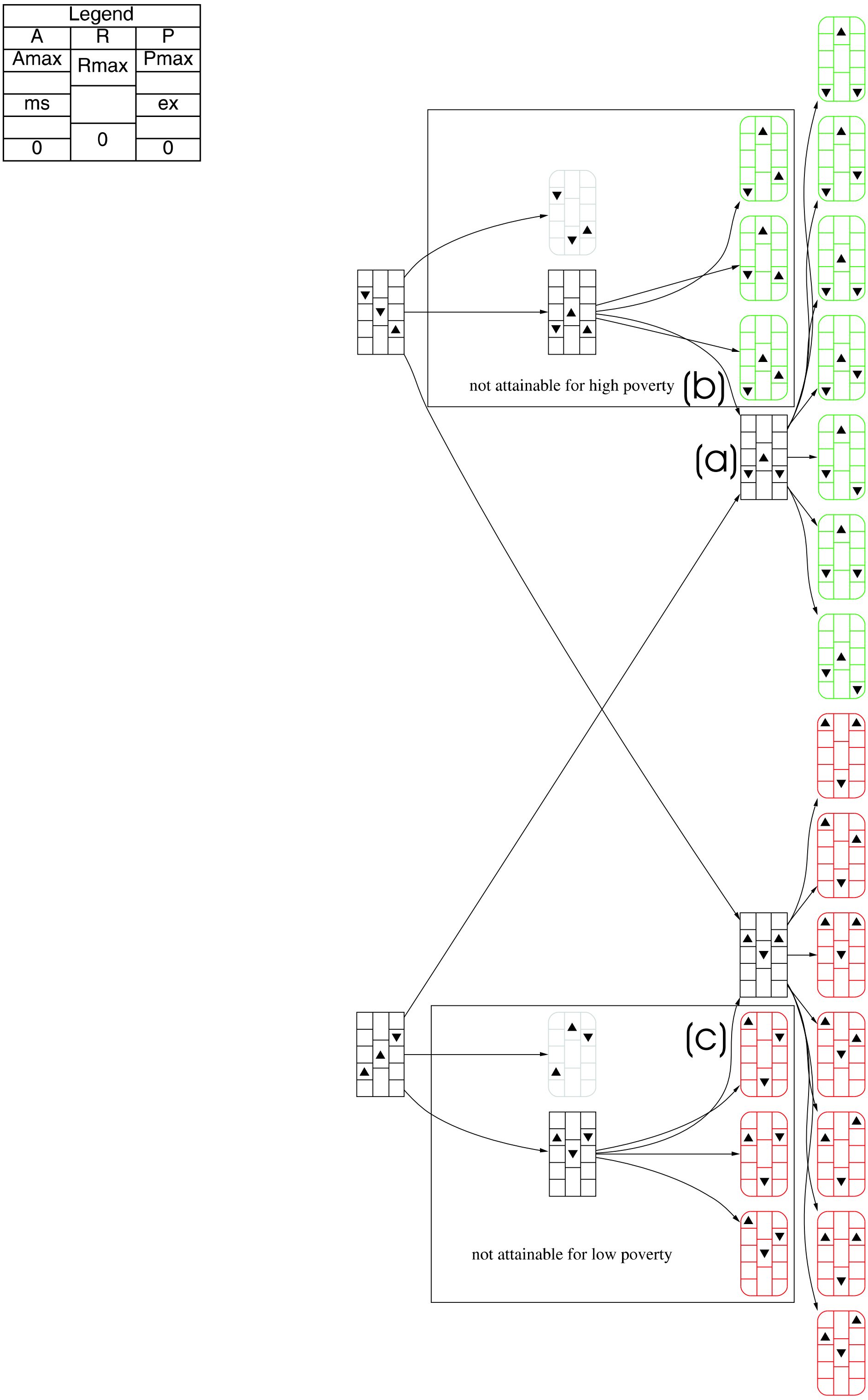

Figure 4.3: State-transition graph of the Sahel Syndrome with additional assumptions ($D_A\dot{P} > 0, D_R\dot{P} < 0$). The boxes indicate sets of states which are not attainable from a state with high or from a state with low poverty. The columns of the state representation are again the same as in Fig. 4.1. The vertex (a) and the edges (b), (c) are discussed in the text.

4.2 Capital Accumulation in Unregulated Fisheries

In this section I present a qualitative model about the problematic interaction of capital and biological stocks in marine fisheries (for details, I refer to Eisenack and Kropp 2001; Kropp and Eisenack 2001; Kropp, Zickfeld, and Eisenack 2002; Eisenack et al. 2006). I concentrate on the demonstration of the abstraction and restriction techniques developed and presented in this thesis: projection (section 2.2.4, p. 36), elimination of marginal edges (section 3.2, p. 62) and no-return abstraction (section 3.1, p. 52).

Capital accumulation has been a major issue in fishery economics over the last two decades; commercial fishery is portrayed as a system in which a biological stock and a capital stock interact dynamically (Clark et al. 1979; McKelvey 1985; Boyce 1995; Jørgensen and Kort 1997; Munro 1999; Pauly et al. 2002). The biological stock is the amount (number of fish or biomass) of the target species, whereas the capital stock consists of fishing gear (boats, nets, technical equipment etc.). As the capital stock is highly specialised and cannot readily be converted to other uses, investment decisions are irreversible. In many contributions this is understood as a major cause of over-fishing. If a fish stock is overexploited, making the fishery less profitable, there is no opportunity to sell the fishing gear. Consequently, more capital than efficient is allocated to the fishery, or equipment is transfered to other fisheries, putting other target stocks at risk – a pattern known as serial overfishing (Goñi 1998). With the model below I reveal one major cause of overfishing. It is shown that every solution of the monotonic landmark ensemble necessarily produces a period where excess capacities are built up, making the fishery less efficient and contributing to the risk of serial overfishing. Since this is driven by profit-oriented resource use, the model is qualified as a representative system for the Overexploitation Syndrome (cf. Cassel-Gintz and Petschel-Held 2000; Kropp, Eisenack, and Scheffran 2006).

First I set up an analytical model based on standard bio-economics. Then we use a monotonic landmark ensemble for its analysis for the following reasons:

- Due to tractability, previous efforts in this field have relied on a variety of simplifying assumptions, and many of them are restricted to equilibrium analysis. As QDEs extend the possibilities to handle the global dynamic properties of a system, the model can be substantially extended and some simplifications can be avoided.

- We are uncertain about exact functional relationships and parameters in marine fisheries (see p. 86).

The Capital Fisher Model

The capital fisher model investigates the dynamics of capital accumulation in an unregulated marine fishing industry with nonlinear investment costs and stock-dependent harvesting productivity. It describes a situation where N identical and profit maximising firms compete for an unregulated resource, i.e. a marine fish stock of size x. Assuming that any harvesting requires capital k (e.g. ships, fishing gear), and that the productivity of these inputs depends on the biological stock x, we can set up a variable cost function $v(h, x, k) : \mathbb{R}^3_+ \to \mathbb{R}_+$ which describes the harvesting costs at a given time for a given harvest h, fish stock x and capital stock k. We assume that this function has the following monotonicity properties:

$$D_h v > 0, D_x v, D_k v < 0,$$
$$D_{hh} v, D_{kk} v, D_{xx} v > 0,$$
$$D_{kx} v > 0, D_{hx} v, D_{hk} v < 0.$$

These inequalities describe consequences of economic standard properties of positive but decreasing marginal productivity, and that certain attributes of capital enhance the accessibility of the fish stock (improved fishing gear and technology, increased horsepower of boats, etc.). Additionally, we assume that the Hessian of v is positive definite, which is no contradiction to the above inequalities (see Eisenack et al. 2006 for details).

The regeneration of the resource is given by a recruitment function $R \in C^1(\mathbb{R}_+, \mathbb{R}_+), x \mapsto R(x)$ of logistic type. It attains a unique maximum sustainable yield (MSY) for $x = x_{MSY}$ (both parameters do not have to be known quantitatively). Furthermore, $R(0) = R(Q) = 0$, where $Q > x_{MSY}$ the carrying capacity of the biological system (which also does not have to be known qualitatively). For $x < x_{MSY}$, $D_x R > 0$, but $D_x R < 0$ if $x > x_{MSY}$. The fish stock changes according to

$$\dot{x} = R(x) - (h + h'), \tag{4.1}$$

where h denotes the harvest of a firm under consideration and h' that of all the others. The change of each firm's capital stock is described by

$$\dot{k} = I - \delta k, \tag{4.2}$$

where $I \geq 0$ represents the investment rate and δ a depreciation rate which is assumed to be constant. Investment costs are expressed by a strictly convex increasing function $c \in C^1(\mathbb{R}_+, \mathbb{R}_+), I \mapsto c(I)$. The convexity reflects inelastic supply of highly specialised equipment and rising adjustment costs for higher investment. The demand for fish is described by the downward sloping inverse demand function $p \in C^1(\mathbb{R}_+, \mathbb{R}_+), h + h' \mapsto p(h + h')$, assigning a market price to a given amount of harvested fish.

The decision of each firm about h and I now has to be determined. We assume that the harvest decision is myopic in contrast to the investment decision, i.e. fishing firms only take the current state of the system into account when deciding about h, while they take long-term effects into consideration when choosing the level of investment. The latter is justified by the long time scale of capital dynamics (ships are typically used for 10 to 50 years). The former is partly because of a lack of knowledge about the recruitment function, and partly because firms consider their own influence on the fish stock to be negligible. Moreover, they tend to assume that other firms behave in the same way. Thus, we suppose that the impact of harvesting on the biological stock are neglected by the individual firms in their short-term decision making. If each fishing company acts in an economically rational way, it chooses harvest to maximise profits $p(h+h')h - v(h, x, k)$ at each time. By using the implicit function theorem, some standard properties of the inverse demand function, the monotonicity and convexity properties of v, it can be guaranteed that the solution to this static problem is a harvest supply function $h : \mathbb{R}_+^2 \to \mathbb{R}_+, (x, k) \mapsto h(x, k)$ with $D_x h, D_k h > 0$. The investment plan is chosen such that it maximises the discounted profit given by

$$\Pi := \int_J e^{-rt} \Big(p(h + h')h - v(h, x, k) - c(I) \Big) dt,$$

subject to Eq. (4.1), Eq. (4.2) and $h = h(x,k)$. Here, r denotes a constant discount rate and $J = [0,T]$ a planning interval. Assuming that all firms are characterised by the same technology and behave in the same way (i.e. $h + h' = Nh$), the decision problem can be solved with a theorem on dynamical optimisation by Mangasarian (1966). The resulting analytical model can be written as:

$$\begin{aligned}
\dot{x} &= R(x) - Nh,\\
\dot{k} &= I - \delta k,\\
h &= h(x,k),\\
\dot{I} &= \frac{1}{D_{II}c(I)}\big((r+\delta)D_I c(I) + D_k v(h,x,k)\big).
\end{aligned}$$

Unfortunately, many signs of the Jacobian display ambiguities, and only some of these can be resolved by introducing landmarks. To distinguish the monotonically increasing part of R from the monotonically decreasing part, x is supplied with the landmark $\texttt{xmsy}$:

$$\begin{aligned}
D_x\dot{x} &= D_x R - N D_x h = \begin{cases} < 0 & \text{if } x > \texttt{xmsy},\\ \gtrless 0 & \text{otherwise}, \end{cases}\\
D_k\dot{x} &= -N D_k h < 0,\\
D_I\dot{x} &= D_x\dot{k} = 0,\\
D_k\dot{k} &= -\delta < 0,\\
D_I\dot{k} &= 1 > 0,\\
D_x\dot{I} &= \frac{1}{D_{II}c(I)}\big(D_{kx}v + D_{kh}v D_x h\big) \gtrless 0,\\
D_k\dot{I} &= \frac{1}{D_{II}c(I)}\big(D_{kk}v + D_{kh}v D_k h\big) \gtrless 0,\\
D_I\dot{I} &= (r+\delta)\big(1 - \frac{D_I c D_{III} c}{(D_{II}c(I))^2}\big).
\end{aligned}$$

Assuming $\frac{D_I c D_{III} c}{(D_{II}c(I))^2}$ to be small, for a qualitative state q with $\mathrm{qmag}_x(q) < \texttt{xmsy}$, we have the sign matrix

$$\mu(q) = \begin{pmatrix} [?] & [-] & 0\\ 0 & [-] & [+]\\ [?] & [?] & [+] \end{pmatrix},$$

while for $\mathrm{qmag}_x(q) < \texttt{xmsy}$

$$\mu(q) = \begin{pmatrix} [-] & [-] & 0\\ 0 & [-] & [+]\\ [?] & [?] & [+] \end{pmatrix}.$$

To formulate a set of constraints C we introduce the landmark $\texttt{MSY} := R(\texttt{xmsy})$ for R and h. Also, harvest increases monotonically with x and k. As in the previous section 4.1 (p. 87),

Graph	Vertices	Edges
State-transition graph		
after chatter-box abstraction	134	599
Removing marginal edges in runtime	103	330
Removing non-analytical states	59	103
Removing further marginal edges	59	93
Simple projection	30	47

Table 4.2: Number of vertices and edges for abstraction and restriction techniques subsequently applied to the Capital Fisher model.

upper and lower bounds are introduced into the quantity spaces to detect cases with extreme outcome, e.g. a diminishing fish stock or harvest rate. Several `cornot` constraints (cf. section 3.2, p. 62) are defined to eliminate marginal edges during computation (see Appendix for the model code).

Results

Without the `cornot` constraints (but using simple chatter-box abstraction, cf. section 2.2.4, p. 36), the state-transition graph has 134 vertices and 599 edges. Several restriction techniques are applied (chatter-box abstraction, projection and restriction to analytical functions, cf. section 2.2.4, p. 36, and elimination marginal edges, cf. section 3.2, p. 62; see Tab. 4.2). The result is presented in Fig. 4.4 in a manually improved form, where equilibria are omitted and all remaining final states are classified into two categories: (A) represents a catastrophic state where the fish stock is fully exploited ($x = 0$), while in (B) the stock recovers ($\dot{x} > 0$) but no harvest takes place ($h = 0$). The former is an environmental and economical disaster, while the latter is only an economic disaster. (A) can only directly be reached if $x < \texttt{xmsy}$ and $\dot{x} < 0$, while for type (B) $\dot{x} > 0$ and $\dot{h} < 0$ is a precondition.

The no-return abstraction (cf. section 3.1, p. 52) yields that the subgraph containing all except the final states is a strongly connected component, i.e. as long as no final state is reached, every vertex can *possibly* be re-entered. This is in contrast to established bioeconomic ODE models, where the system evolves monotonically towards equilibrium, or where equilibrium is reached after one turning point (e.g. Clark et al. 1979; McKelvey 1985; McKelvey 1986; Boyce 1995). However, the occurrence of boom-and-bust cycles is an empirical fact in many industrial fisheries (Hilborn and Walters 1992; Charles 2001): the state-transition graph can be used to reconstruct case studies which cannot be reconstructed by the older models (e.g. the collapse of the North Atlantic cod fishery or the historical development of the blue whale industry, cf. Eisenack et al. 2006). This shortcoming is mainly due to various linearity assumptions which are used to derive tractable solutions. QDEs allow for greater flexibility in this respect.

The non-linearities also bring about another strong feature of the state-transition graph: every fishery described by the model *necessarily* undergoes a phase of over-capitalisation, i.e. capital increases although catches are declining. In Fig. 4.4 this is the case in vertices #3, #6, #8 and #18. It is easy to see that every path in the abstracted state-transition graph which starts from vertex #1 and has at least length 3 reaches one of these vertices or results

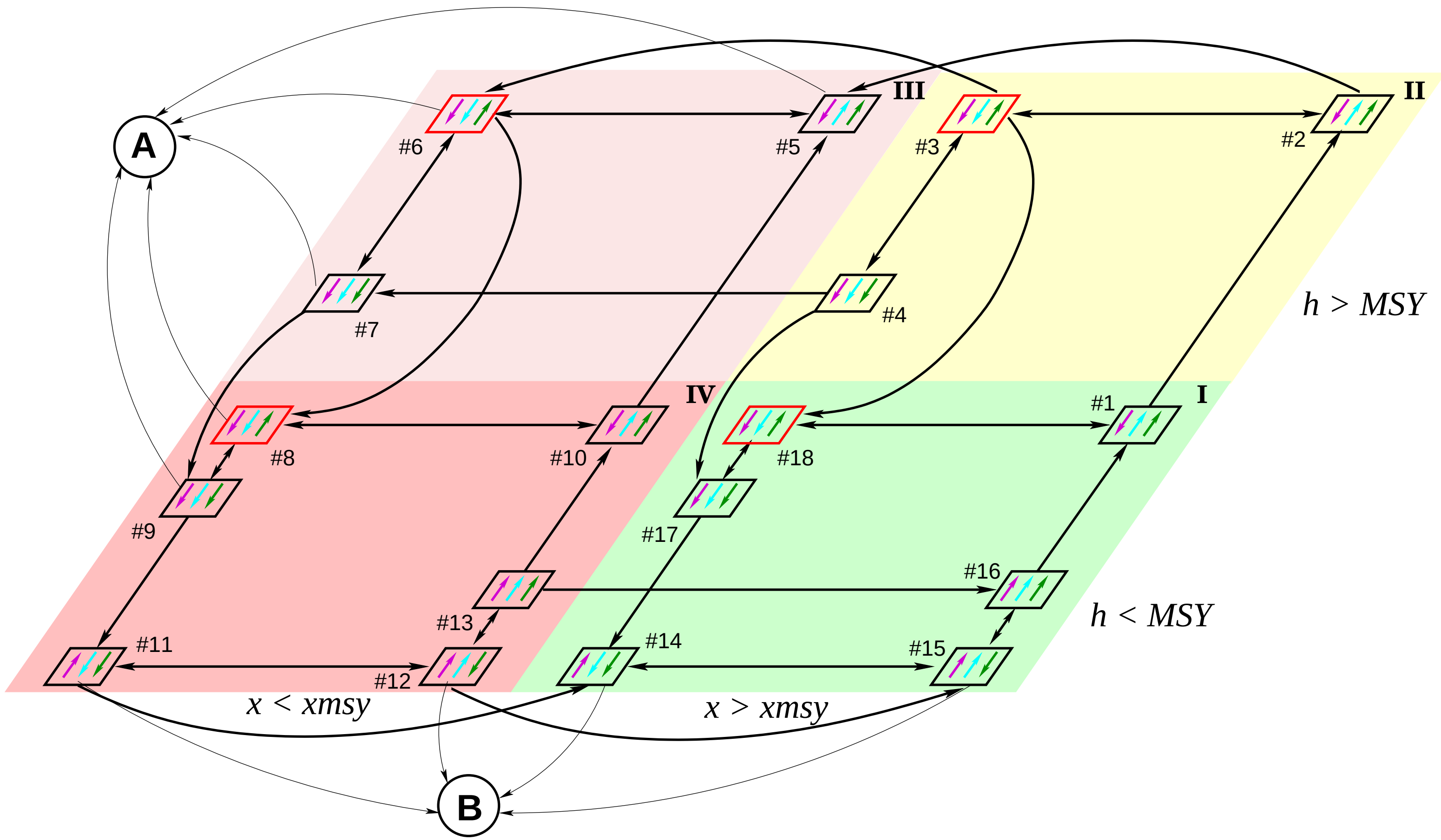

Figure 4.4: Projection of the state-transition graph of the capital fisher model with respect to x, h and k (after several restriction techniques are applied, cf. text). The arrows in the vertices (boxes) label the direction of the first derivative (from left to right $\dot{x}$, $\dot{h}$, $\dot{k}$), the red boxes indicate situations where overcapitalisation occurs; (A) and (B) are final states (adapted from Kropp, Zickfeld, and Eisenack 2002).

in a collapse. State #1 is a typical situation in a fishery where exploitation begins: Fish stocks are still high but declining, while investment increases the capital stock and harvests. But overcapitalisation also occurs for other initial conditions unless the system remains in a chatter-box forever or reaches a final state. This property is rooted in the fact that $v_{hx} < 0$ and $v_{hk} < 0$, i.e. that the harvest supply function h increases in x and k. As long as increased harvest is observed although the fish stock is reduced, net investment must be positive to compensate losses from increasing marginal costs. In other words, an increase in marginal costs due to a decreasing fish stock may trigger additional investment in an effort to keep marginal costs from rising excessively. Therefore, k cannot start to decrease before h.

The above model is less constrained than the Sahel Syndrome model, which becomes obvious from the no-return abstraction: there are no invariant sets except final states – it concludes that substantially more knowledge than considered by a monotonic landmark ensemble is needed to make crisper predictions. This is supported by experiments in which all ordinal assumptions consistent with the basic monotonicity properties were tested (cf. section 3.3, p. 68). Although some paths can be excluded such that several no-return sets consisting of single states emerge, they do not improve the overall situation. However, in spite of this ill-posed nature, diverse abstraction and restriction techniques substantially simplify the state-transition graph, and robust properties are revealed which are common to all systems given by the monotonic landmark ensemble.

4.3 Participatory Fishery Management

In this section I present and assess various management schemes of participatory resource management using viability theory and qualitative differential equations. The analysis is based on two models which are examined as dynamic control systems (cf. previous work in Eisenack 2003; Kropp, Eisenack, and Scheffran 2004; Eisenack et al. 2006). I show that serious problems may result if participatory management is purely resource-based. The analysis investigates whether a less risky management strategy can be implemented even with only limited data.

Viability criteria are imposed on both models of a management framework. We assess whether different management strategies comply with these requirements and how they change the structure of the resulting state transition graph. The first model is investigated solely by analytical techniques from viability theory (cf. section 2.4, p. 45). The second model describes a closely related but more complex setting. It uses QDEs to account for uncertainties more thoroughly and to design a qualitative closed-loop control in a systematic way. But would a management strategy which is promising in the setting of the first model remains robust in the second? The new methods developed in Chapter 3, elimination of marginal edges (section 3.2, p. 62) and no-return abstraction (section 3.1, p. 52), are important tools when addressing this question. The model also demonstrates another way in which viability theory can be fruitful for qualitative reasoning by restricting the qualitative state space to a region close to the boundary of a constrained set.

Recently, participatory strategies to fisheries management have been seen as a promising way. The basic idea is to include stakeholders – e.g. fishing firms, processing companies, scientific institutions and NGOs – in the decision-making process on catch restrictions. (Jentoft et al. 1998; Noble 2000; Charles 2001; Potter 2002). This is in contrast to the common type of top-down management where a government agency imposes restrictions on the fishery. When a fishery reaches a state of crisis, scientific institutions are criticised for putting too much emphasis on conservation objectives and neglecting the economic sustainability. If fishermen are involved in the decision-making process, it is assumed that economic objectives will complement conservation goals of government organisations and that compliance with regulations will be better (Pinkerton 1989; Mahon et al. 2003). In many cases, this type of management is exercised via a fishery council where the representatives of stakeholder groups negotiate, e.g. about the total allowable catch. This plan is executed by a management organisation which works in close collaboration with local fishermen.

One precondition for the success of both top-down and participatory frameworks is a proper information base, usually delivered by scientific institutions (e.g. ICES 2002). In this context it is sometimes argued that fishery management is focused too much on an ecological viewpoint – compared with efforts to examine the behaviour of the resource users, their economic settings, and aims – sometimes referred to as "ichthyocentrism" (Lane and Stephenson 2000; Davis and Gartside 2001). Thus, the following models are designed to discuss the potential benefits and risks of participatory and ichthyocentric management frameworks. We want to know if there is still a risk of overexploitation.

4.3.1 Viability Analysis of Management Frameworks

The quota negotiation model of a participatory management framework uses game theory and includes scientific catch recommendations as control variable (for details, I refer to Kropp, Eisenack, and Scheffran 2004; Eisenack, Scheffran, and Kropp 2006). The model is supplied with viability constraints to identify conditions for viable control. Different recommendation strategies are assessed against these conditions. One of them will also be assessed within the extended model in the next subsection.

The Quota Negotiation Model

As in section 4.2 (p. 95), the basic state variables is the biomass of a fish stock x, which is influenced by the total harvest h and the recruitment function R, yielding for the stock dynamics the ordinary differential equation

$$\dot{x} = R(x) - h.$$

Recall that $R(x_{MSY}) = MSY$, $R(0) = R(Q) = 0$ and for $x > x_{MSY}$ we have $D_x R(x) < 0$, while $D_x R(x) > 0$ for $x = x_{MSY}$. Due to the complexity of ecosystems we have only limited knowledge about the behaviour of a fish stock. Thus, no additional assumptions about R are made.

In a participatory framework the total harvest h is determined in a negotiation process about the allocation of catch quotas, written as vector $q \in \mathbb{R}_+^N$, to N groups of fishing firms. The resulting total harvest is $h = \sum_{i=1,\dots,N} q_i$. The negotiation process is modelled with the following assumptions: A scientific institution and representatives from the fishing industry bargain for the total harvest h and the individual quotas q_i. When these pressure groups agree on an allocation, the result is transformed into practice by the management authority. The negotiations are opened by the scientific institution, which makes a recommendation $r \geq 0$ for the total catch. Each group of the fishing industry tries (i) to get an optimum share of the total harvest h and (ii) to increase h above the catch recommendation r if it is profitable.

The optimum share and the optimum increase $h - r$ may differ between the groups, e.g. due to technical and economic parameters. There is a trade-off between higher profits resulting from higher quotas and deviation costs d_i imposed by exceeding the scientific recommendation. These costs are linked to the legitimation of bargaining positions which challenge scientific advice and increasing transaction costs of fierce negotiations. They may differ between the pressure groups. It is further assumed that the fishing groups act to optimise short-term profits. This is realistic if single fishing firms perceive their impact on the resource as negligible (Banks 1999; Kropp, Eisenack, and Scheffran 2004). This entails that they only account for short-term deviation costs.

Each group i is supplied with a profit function $\pi_i : \mathbb{R}_+ \times \mathbb{R}_+^N \to \mathbb{R}_+$, depending on the fish stock and the overall allocation plan,

$$\pi_i(x, q) = pq_i - c_i(q_i, x) - d_i\Bigg(\sum_{j=1,\dots,N} q_j - r\Bigg), \tag{4.3}$$

where the first term represents revenues on markets, p corresponding to the market price (which is assumed to be exogenous), while $c_i \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $i = 1, \dots, N$ are cost functions, assigning variable costs of a single group to its catch q_i and the amount of fish x. It is economically reasonable to assume that all cost functions c_i are convex and increasing in q_i, while costs are decreasing in x due to higher densities of fish. The deviation costs, described by the functions $d_i \in C^1(\mathbb{R}_+, \mathbb{R}_+), i = 1, \dots, N$ depend on the individual quota q_i, on the quotas allocated to the other pressure groups, and on the scientific catch recommendation r. If $\sum_{j=1,\dots,N} q_j - r$ becomes negative, we assume that d_i vanishes, since deviation costs do not apply if the sum of all quotas is below the recommendation. It is reasonable to assume that each d_i is a monotonically increasing, convex function.

Each group i negotiates for a quota which maximises their profit π_i for given p, x and r. This problem can be described as a non-cooperative game with the so called Nash equilibrium as solution, where each participant takes the decision of the other participants as given (Nash 1951). Since all profit functions $\pi_i, i = 1, \dots, N$ are concave and continuously differentiable with respect to q_i, the Nash equilibrium is given by the equation system

$$\forall i = 1, \dots, N : D_{q_i}\pi_i = 0. \tag{4.4}$$

In the following analysis we restrict this general approach to the case of two specific fishery groups (e.g. artisanal and industrial fishers) and provide a possible functional specification for variable and deviation costs by

$$c_i(q_i, x) := \frac{\alpha_i q_i + \beta_i q_i^2}{x}\,, \tag{4.5}$$

$$d_i(q_1 + q_2 - r) := \begin{cases} 0 \quad \text{if} \quad q_1 + q_2 < r, \\ \kappa_i (q_1 + q_2 - r)^2 \quad \text{otherwise}\,. \end{cases} \tag{4.6}$$

The parameters $\alpha_i, \beta_i, \kappa_i$ $(i = 1, 2)$ are not completely known, but positive.

PROPOSITION 44: *In the Nash bargaining solution for the profit functions given by Eq. (4.3) with Eq. (4.5), Eq. (4.6) and $N = 2$ results in the total catch*

$$h(x, r) = \begin{cases} h_b(x, r) & \textit{if} \quad r \le \hat{r}(x) \quad \textit{and} \quad h_b(x, r) \ge 0, \\ \hat{r}(x) & \textit{if} \quad r \ge \hat{r}(x) \ge 0, \\ 0 & \textit{if} \quad \hat{r}(x) \le 0, \\ 0 & \textit{if} \quad 0 \le r \le \hat{r}(x) \quad \textit{and} \quad h_b(x, r) \le 0, \end{cases} \tag{4.7}$$

$$= \max\big(0, \min(h_b(x, r), \hat{r}(x))\big), \tag{4.8}$$

where

$$\hat{r}(x) = \frac{upx - v}{\beta_1 \beta_2},$$
$$h_b(x, r) = \frac{upx + wxr - v}{\beta_1 \beta_2 + wx},$$

and

$$u := \frac{1}{2}(\beta_2 + \beta_1) > 0,$$
$$w := \beta_1 \kappa_2 + \beta_2 \kappa_1 > 0,$$
$$v := \frac{1}{2}(\alpha_1 \beta_2 + \alpha_2 \beta_1) > 0.$$

The continuous total harvest function h increases monotonically in x and r. The functions $\hat{r}, h_b$ are strictly increasing. Additionally, it holds that

$$h_b(x, \hat{r}(x)) = \hat{r}(x), \tag{4.9}$$

$$\textit{if } \hat{r}(x) \geq 0 : h(x, r) \leq \hat{r}(x). \tag{4.10}$$

The function h (Eq. 4.7), depending on the fish stock x and the harvest recommendation r, is called the total harvest function. The different cases result from that fact that for recommendations $r \geq \hat{r}(x)$ it is not profitable for fishing firms to exceed them. In this case the total harvest equals $\hat{r}(x)$, which is the optimum catch if $\kappa_1 = \kappa_2 = 0$. We refer to this case as non-binding harvest recommendations. However, if $r < \hat{r}(x)$, recommendations are binding, resulting in the catch $h_b(x, r)$.

PROOF: At first we derive Eq. (4.7). Suppose the bargaining is constrained to the case $q_1 + q_2 < r$ where deviation costs vanish. In this case the conditions Eq. (4.4) can be solved independently from each other for q_1 and q_2 by elementary calculations, yielding $h(x, r) = q_1 + q_2 = \hat{r}(x)$. Thus, if $0 \leq \hat{r}(x) \leq r$, the harvest is $\hat{r}(x)$. Now suppose that $0 \leq r \leq \hat{r}(x)$, making the conditions Eq. (4.4) a linear equation system. The solution for q_1 and q_2 yields the harvest $h(x, r) = h_b(x, r)$. Of course, if $\hat{r}(x) \leq 0$ or $h_b(x, r) \leq 0$, then $h(x, r) = 0$ because harvest cannot be negative.

The monotonicity properties of $h, x, \hat{r}, h_b$ and Eq. (4.9) can easily be shown with elementary calculations.

For $r \geq \hat{r}(x)$, Eq. (4.10) is true because $h(x, r) = \hat{r}(x)$. For $r \leq \hat{r}(x)$ and $h_b(x, r) \geq 0$, $h(x, r) = h_b(x, r) \leq h_b(x, \hat{r}(x)) = \hat{r}(x)$ due to monotonicity of h_b and Eq. (4.9). If $0 \leq r \leq \hat{r}(x)$ and $h_b(x, r) \leq 0$, then $h(x, r) = 0 \leq \hat{r}(x)$. This covers all possible cases for Eq. (4.10).

Eq. (4.8) is valid because by Eq. (4.7), harvest does not vanish iff $0 < \hat{r}(x) < r$ or $0 < h_b(x, r) \wedge r \leq \hat{r}(x)$. Using Eq. (4.9) and the monotonicity properties, in the first case

$$h = \hat{r}(x) = h_b(x, \hat{r}(x)) \leq h_b(x, r),$$

i.e. $h = \min(h_b(x, r), \hat{r}(x))$. Similarly, in the second case

$$h = h_b(x, r) \leq h_b(x, \hat{r}(x)) = \hat{r}(x).$$

$\square$

We end up with the ODE

$$\dot{x} = R(x) - h(x, r),$$

where $r \in \mathbb{R}_+$ is a control variable. In the following we will express different management regimes as different strategies for choosing r. To assess them, we need quality criteria – these are provided using the framework of viability theory.

Viability Constraints

Two reasonable viability constraints are defined and investigated for our examination of marine fisheries, and conditions are deduced under which a control rule for r exists, which respects both constraints:

1. Ensure that the biomass of a stock always remains above a minimum level $\underline{x} > 0$:
$$\forall t : x(t) \geq \underline{x}.$$
2. Require that a minimum total harvest $\underline{h} > 0$ can always be realized or exceeded:
$$\forall t : h(t) \geq \underline{h}.$$

We refer to the first criterion as environmental and to the second one as economic viability. We define the set-valued map $F : \mathbb{R}_+ \rightsquigarrow \mathbb{R}$ by

$$F(x) := \{R(x) - h(x,r) | r \in \mathbb{R}_+ \text{ and } h(x,r) \geq \underline{h}\}. \tag{4.11}$$

It assigns to a system state x all possible velocities resulting from a harvest recommendation r which ensures economic viability. We are now looking for a set of states in the interval $[\underline{x}, \infty]$ which is viable with respect to F. In such a set it is possible to choose an open-loop control function $r(\cdot)$ such that both viability constraints are met forever (cf. section 2.4, p. 45). This will serve as the basis to assess whether concrete control rules for r keep this set viable.

To apply the viability theorem (PROP. 10, p. 48), we have to evaluate the regularity of F. This includes determining the fish stocks for which $F(x) = \varnothing$ – which yields the cases where economic viability cannot be met.

PROPOSITION 45: *The set-valued map F defined by Eq. (4.11) equals*

$$F(x) = \begin{cases} [R(x) - \hat{r}(x), R(x) - \max\left(\underline{h}, h_b(x,0)\right)] & \text{if } \hat{r}(x) \geq \underline{h}, \\ \varnothing & \text{otherwise} \,, \end{cases} \tag{4.12}$$

and is Marchaud on every compact set $K \subset \mathbb{R}_+$ where $\exists x \in K : \hat{r}(x) \geq \underline{h}$.

PROOF: At first we show that $F(x) \neq \varnothing$ if and only if $\hat{r}(x) \geq \underline{h}$. The set $F(x)$ does not vanish if and only if there is one $r \in \mathbb{R}_+ : h(x,r) \geq \underline{h}$. Then, from Eq. (4.7), Eq. (4.9) and Eq. (4.10),

$$0 \leq h(x, \hat{r}(x)) = \hat{r}(x) \geq h(x,r) \geq \underline{h}.$$

Now suppose that $\hat{r}(x) \geq \underline{h} > 0$. With choosing $r \geq \hat{r}(x)$ the total harvest function Eq. (4.7) yields $h(x,r) = \hat{r}(x) \geq \underline{h}$, i.e $F(x) \neq \varnothing$.

Next, we determine the concrete form of F. We only have to consider the case $\hat{r}(x) \geq \underline{h}$, and denote the lower and upper bounds of F as $\underline{F}(x)$ and $\bar{F}(x)$, respectively. By definition, $R(x) - h(x,r)$ cannot be below $\underline{F}(x)$, and choosing $r = \hat{r}(x)$ yields exactly $\underline{F}(x)$. Now choose $r = 0 < \hat{r}(x)$. If $h_b(x, 0) > \underline{h} > 0$, then $\forall r \geq 0 : h(x, r) \geq h(x, 0) = h_b(x, 0)$ due to Eq. (4.7) and the monotonicity of the total harvest function. Since r cannot be negative, $R(x) - h(x, r)$ cannot be above $R(x) - h_b(x, 0)$. If, on the other hand, $h_b(x, 0) \leq \underline{h} \leq \hat{r}(x)$, there is one $\underline{r} \in [0, \hat{r}(x)]$ such that $h(x, \underline{r}) = h_b(x, \underline{r}) = \underline{h}$ due to Eq. (4.7), Eq. (4.9) and continuity of h. In summary, continuity of h guarantees that $F(x)$ contains exactly the values between $\underline{F}(x)$ and $\bar{F}(x)$.

That the set-valued map F is Marchaud when it is restricted to K can be verified using the characterisation from p. 43. Obviously, F has convex values and $\mathrm{Dom}(F) = K \cap \{x \in \mathbb{R}_+ \mid \hat{r}(x) \geq \underline{h}\}$ is an intersection of closed sets and nonempty since $\exists x : \hat{r}(x) \geq \underline{h}$. The $\mathrm{Graph}(F)$ is closed because because $\underline{F}$ and $\bar{F}$ depend continuously on x and $\mathrm{Dom}(F)$ is compact. It has linear growth since it is bounded on a compact set. □

Choosing a compact subset of $\mathbb{R}_+$ is only a technicality to account for the case that the recruitment function $R(x)$ may decrease faster than linear for $x > Q$ (when it has negative values). Alternatively, this can be excluded by an additional model assumption. However, since a harvested fish stock will not be above the equilibrium of a non-utilised stock Q, this is irrelevant in our case: we can simply choose a compact set $K \supseteq [0, Q]$, as long as there is one $x \in K$ with $\hat{r}(x) \geq \underline{h}$.

We are now ready to apply the viability theorem, stating that for a Marchaud map a closed set K is viable iff K is a viability domain (cf.PROP. 8, p. 47); in our case

$$\forall x \in K : F(x) \neq \varnothing, \tag{4.13}$$
$$F(\inf(K)) \cap [0, \infty] \neq \varnothing, \tag{4.14}$$
$$F(\sup(K)) \cap [-\infty, 0] \neq \varnothing. \tag{4.15}$$

PROPOSITION 46: *An interval $J = [\underline{x}, b]$ is a viability domain of F iff*

$$\begin{array}{rl} \textit{(i)} & \exists x' \leq \underline{x} : \hat{r}(x') = \underline{h}, \\ \textit{(ii)} & \textit{and } R(\underline{x}) \geq \max\left(\underline{h}, h_b(\underline{x}, 0)\right), \\ \textit{(iii)} & \textit{and } R(b) \leq \hat{r}(b). \end{array}$$

PROOF: Condition (i) is equivalent to Eq. (4.13): It follows from monotonicity of $\hat{r}$ that $\forall x \geq x' : \hat{r}(x) \geq \underline{h}$, such that $F(x) \neq \varnothing$ by PROP. 45. Conversely, if $\forall x \in J : \hat{r}(x) \geq \underline{h}$ there exists an appropriate x' since $\hat{r}$ is continuous increasing and has a positive zero.

Condition (ii) holds iff $\bar{F}(\underline{x}) \geq 0$ which is equivalent to Eq. (4.14) by PROP. 45.

Condition (iii) holds iff $\underline{F}(b) \geq 0$ which is equivalent to Eq. (4.15). □

This proposition can be interpreted as follows. If a fish stock is in a viability domain, it is possible to choose an appropriate harvest recommendation r which keeps the fishery in the domain. Condition (i) guarantees that for sufficient large recommendations r the economic viability criterion can be met. This must be possible for all states in the viability domain. The second condition safeguards that for a fish stock at the lower boundary of the interval, r can be decreased sufficiently to prevent the fish stock from declining further. If the catch has to be decreased below $\underline{h}$ to obtain $\dot{x} \geq 0$, the economic viability constraint would be violated. Condition (iii), being more technical in nature, implies that fishing firms would voluntarily catch more than $R(b)$ if recommendations are non-binding, such that $\dot{x} \leq 0$ in this situation.

Note also that the viability kernel $\mathrm{Viab}_F(K)$ – which is the largest closed viability domain contained in K (cf.PROP. 10, p. 48) – is the largest interval $J \subseteq K$ satisfying the conditions. It has to be an interval, since condition (i) holds for all $x \geq x'$ and never holds for $x < x'$, and the other conditions only apply on the boundary of the viability kernel.

Management

We now use PROP. 46 to assess the viability of two different recommendation strategies for r. Although it is *possible* to keep the system viable if an appropriate strategy is selected, the proposition does not ensure that *every* control strategy is successful. Formally, such a strategy assigns a value for r to a given system state, i.e. a closed-loop control according to the following schemes:

- *Ichthyocentric control*: The harvest recommendation is purely based on an exact estimate of the stock recruitment, i.e.

$$r = R(x).$$

- *Conservative control*: The harvest recommendation is based on economic viability in the sense that recommendations are adjusted to yield $h = \underline{h}$, i.e.

$$r = \underline{r}(x),$$

  where $\underline{r}(x)$ is defined as the smallest $r \geq 0$ such that $h(x, r) \geq \underline{h}$ or as $+\infty$ if no such r exists.

These strategies can be regarded as extreme cases of management where only ecological or where only economic criteria matter. The latter is called "conservative" because this strategy only guarantees a minimum aspiration level for harvest, but avoids increased catches.

Ichthyocentric control: Assuming that the harvest recommendation r equals recruitment presupposes that the scientific institution is able to estimate $R(x)$ correctly. This is a challenging task, since an exact estimation of the stock biomass is bound to fail due to unavoidable measurement deficits (see the discussion of uncertainties above, p. 86). However, let us assume that the estimator is correct. We show that even in this ideal case, the strategy cannot guarantee viability. The main argument stems from the crucial fact that

$$h(x, r) \geq r \Leftrightarrow r \leq \hat{r}(x) \tag{4.16}$$

i.e. that the negotiated harvest is always above the scientific catch recommendation, except in the case of non-binding recommendations. This relation holds by Eq. (4.7) and Eq. (4.9), since $D_r h_b < 1$, and it holds for $0 \leq r \leq \hat{r}(x)$ that $h(x, r) = h_b(x, r) \geq r$, while for $r > \hat{r}(x)$ it holds that $h(x, r) = \hat{r}(x) < r$. Thus, if recommendations are binding, then

$$h(x, r) = h(x, R(x)) > R(x),$$

harvest is above recruitment, resulting in a decreasing fish stock. Comparing with PROP. 46, even if there exists a viability domain, the ichthyocentric strategy necessarily violates the environmental viability criterion if $\underline{h} \leq R(\underline{x}) \leq \hat{r}(\underline{x})$. It is only viable if, contrarily, $\underline{h} \leq \hat{r}(\underline{x}) \leq R(\underline{x})$ holds. The latter means that the realized catch $\hat{r}$ must be significantly lower than the scientific recommendation, a situation which normally does not occur in industrial capture fisheries (Eisenack et al. 2006). This makes it impossible that the fish stock recovers once it is below $\underline{x}$. We can summarise that even in the case of a perfect stock assessment, the ichthyocentric strategy exposes the fishery to a risky development.

Conservative control: If a viability domain $J = [\underline{x}, b]$ exists for the fishery, then $\underline{r}(x) < \infty$ for all $x \in J$, since economic viability is guaranteed. It also holds that $\max\left(\underline{h}, h_b(\underline{x}, 0)\right) \leq R(\underline{x})$ (cf. PROP. 46). By definition of $\underline{r}$,

$$h(\underline{x}, \underline{r}(\underline{x})) = \underline{h} \leq R(\underline{x})$$

if $\underline{r}(x) > 0$; and if $\underline{r}(x) = 0$,

$$h(\underline{x}, \underline{r}(\underline{x})) = h_b(\underline{x}, 0) \leq R(\underline{x}).$$

Together, this has the consequence that conservative control guarantees also environmental viability. If the fish stock is below $[\underline{x}, b]$, i.e. outside the viability domain, environmental or economic viability is no longer sustained. However, it is possible that only the economic criterion is violated, resulting in $h(x, \underline{r}(x)) \leq R(x)$, i.e. x may increase until a viability domain is reached again. In contrast, if only the environmental criterion is violated, there is no chance of a recovery although harvest remains above $\underline{h}$ for some time: the strategy is always viable if the management begins in the viability domain of a fishery, and is thus less risky. Additionally, conservative control allows for a recovery in some fisheries.

Conservative control was claimed to be based purely on economic observations. For binding catch recommendations, the control equals

$$\underline{r}(x) = \frac{\underline{h}(wx + \beta_1\beta_2) + v}{wx} - \frac{up}{w}.$$

This might raise the objection that the parameters u, v, w, which depend on technical and political conditions, may be uncertain to the scientific institutions. In this case, one idea is to approximate conservative control by an adaptive strategy

$$\begin{aligned} \dot{r} &= f(h), \\ f &\in C^1(\mathbb{R}_+, \mathbb{R}_+), \\ f(\underline{h}) &= 0, D_h f < 0. \end{aligned} \tag{4.17}$$

The result is that for harvest above $\underline{h}$, recommendations are decreased and vice versa. This is close to a qualitative control rule which can also work in data-poor settings, and motivates an expanded qualitative model of participatory resource management in the next subsection. It shows how the conservative strategy performs if capital dynamics come into play.

4.3.2 Qualitative Viability Analysis and Control Design

The negotiation model with capital extends the previous model by introducing capital as a further state variable (see Eisenack 2003 for details). As shown in section 4.2 (p. 95), this can fundamentally change the dynamics. If the conservative control strategy is also useful in this setting, this indicates its robustness. The model is transferred to a QDE formulation, again to take account of uncertainty and generality (see p. 86) and to allow for a systematic state space scan in a higher dimensional setting. It is also an example of how viability constraints can be incorporated in the definition of a monotonic landmark ensemble. Finally, it admits a methodological innovation for the design of a closed-loop control in following way: first, a control variable u is included in the model as an open-loop control, and the sign matrix Σ and the constraints C are formulated such that the monotonic landmark ensemble admits all solutions which result from any continuously differentiable control $u(\cdot)$ on $\mathbb{R}_+$. Thanks to the guaranteed coverage theorem (see PROP. 3, p. 34), the resulting state-transition graph contains the abstraction of all trajectories brought about by all possible open-loop controls $u(\cdot)$. This graph can be used to identify the controls which are promising. In a second step the model is refined by introducing qualitative constraints for u – defining a class of closed-loop controls and producing a subgraph of the state-transition graph from the first step. The state-transition graphs of alternative controls can be compared against each other to choose the best option. Using QDEs has two advantages in this context:

- Solving the model for an unconstrained control is possible since the qualitative state space is finite.
- The search space of possible qualitative constraints for u is also finite (as a consequence of the finite qualitative state space) – in contrast to the design of a quantitative closed-loop control.

The Negotiation Model with Capital

As in the previous sections, the basic state variable is the resource stock x, supplied with a logistic recruitment function R and reduced by harvest h. For simplicity, the investigation is focused on stocks $x \leq x_{MSY}$ since non-viable behaviour is more likely to happen in this situation. In addition, the amount of capital k accumulated in the fishery is introduced as a second state variable. Capital is important in this context because (i) it represents the technological efficiency and has an effect on optimum harvest, (ii) inertia is introduced in the model (cf. section 4.2, p. 95), and (iii) it is assumed that capital is an indicator of the political pressure the fishing industry can exercise.

The harvest h is determined in a negotiation process in the fishery council. Again assume full compliance, with catches exceeding the initial catch recommendation r. In contrast to the previous subsection, catch recommendations are always binding. The model is extended in that the outcome of the negotiations, depending on the political power of the fishing industry which is assumed to increase with k such that the so-called negotiation equilibrium is expressed by a total harvest function $h \in C^1(\mathbb{R}^3_+, \mathbb{R}_+), (x, k, r) \mapsto h(x, k, r)$ with $D_x h, D_k h, D_r h > 0$. The stock size x has a positive effect since higher catches are profitable for higher abundance of fish. Catches also increases with r since they are larger when recommendations are less restrictive. Both is in accordance with Eq. (4.7) for binding recommendations.

The dynamics of capital k are described by the investment rate I' and depreciation, given by a (quantitatively unknown) depreciation rate $\delta > 0$ (similar to section 4.2, p. 95):

$$\dot{k} = I(h, k) := I'(h, k) - \delta k.$$

The investment function $I' \in C^1(\mathbb{R}^2_+, \mathbb{R}_+)$ is related to the profit expectations of fishing firms. It is assumed to decrease in k for economic reasons, including the law of diminishing returns (Eisenack 2003). All together, I decreases with k, but increases with h due to better profit expectations.

To design a control strategy for the catch recommendations, we make no assumptions about r at this stage as outlined above. The associated sign matrix with state vector $(x\, k\, h\, r)^t$ (note that $\dot{h}$ is implicitly determined) is:

$$\Sigma = \begin{pmatrix} [+] & 0 & [-] & 0 \\ 0 & [-] & [+] & 0 \\ [?] & [?] & [?] & [?] \\ [?] & [?] & [?] & [?] \end{pmatrix}.$$

The last ambiguous row results from the yet undetermined control rule for r which propagates to the monotonicity properties of $\dot{h}$. The latter are also uncertain due to the unspecified Hessian of h. The same viability criteria as in the previous subsection are introduced. To include them in the QDE we define $\texttt{xmin} = \underline{x}$ as landmark for the state variable x, and $\texttt{hmin} = \underline{h}$ for h. Additional corresponding values are defined by setting the landmarks $\texttt{hv} = R(\underline{x})$ and $\texttt{xv}$ with $R(\texttt{xv}) = \underline{h}$. To define a quantity space for the variables, a choice has to be made about the order of the landmarks. We investigate the more interesting case where $\texttt{xmin} < \texttt{xv}$ and $\texttt{hv} < \texttt{hmin}$, which corresponds to the case where $R(\underline{x}) < \underline{h}$ (see Appendix for the model code). If the results from the previous subsection are robust to extending the model by capital k, we expect from PROP. 46 (p. 106) that stocks below $\texttt{xv}$ are not part of a viability domain.

Results

The state-transition graph of the QDE is very large due to the unconstrained control variable r. To keep it tractable, several abstraction and restriction techniques are applied. At first, we restrict the quantity space to $\texttt{xmin} < x < \texttt{xmsy}$ and $h > \texttt{hv}$, since states where both environmental and economic viability is violated are not our main interest, as well as fish stocks $x > x_{MSY}$ (see above). The QSIM algorithm automatically detects states where this quantity space is left. We further introduce multiple `cornot` constraints to exclude most marginal edges by preprocessing, and apply simple chatter-box abstraction (cf. section 2.2.4, p. 36).

Graph	Vertices	Edges
After chatter-box abstraction and removing marginal edges in runtime	137	407
Removing further marginal edges	137	321
Removing non-analytical states	78	170
Projection on x and h	18	38

Table 4.3: Number of vertices and edges resulting from different restriction techniques applied to the negotiation model with capital.

More simplifications are possible by removing marginal edges (cf. section 3.2, p. 62) and non-analytical states (see Tab. 4.3 for the effect of the methods). The no-return abstraction (cf. section 3.1, p. 52) reveals 2 non-trivial no-return sets. All other no-return sets are final states where the boundaries of the restricted quantity space are hit. To make this structure visible, a simple projection (cf. section 2.2.4, p. 36) is performed with respect to the variables x and h (see Fig. 4.5). Once the system enters the "downstream" no-return set, only problematic final states are possible – environmental or economic viability will necessarily be violated. But even in the "upstream" no-return set it is possible that catches fall below $\underline{h}$. The no-return sets can be distinguished by the qualitative value of x: In the upstream subgraph it is above `xv`, while it is below this landmark in the downstream subgraph. We should bear in mind that the graph covers *all* possible open loop controls $r(\cdot) \in C^1(\mathbb{R}_+ \to \mathbb{R}_+), t \mapsto r(t)$. This means that however $r(\cdot)$ is chosen, and for all ODEs given by $f \in \mathcal{M}(\mu, C)$, viability will be lost once $x <$ `xv`. This very robust result parallels the conclusions of the previous subsection (cf.PROP. 46, p. 106).

Management

To find successful interventions in the sense of structural management (cf. section 4.1, p. 87), the second step of the design method as outlined above is adopted (p. 109). Since r is still unconstrained, we introduce additional constraints to the model which describe the recommendation strategy of the scientific institution – one subsumes conservative control as introduced in the last subsection, and the other one considers economic *and* ecological indicators. The resulting state-transition graphs are compared. It is clear from the above model results that management interventions only make sense for $x \geq$ `xv`. The aim of these interventions is to prevent harvest from decreasing below $\underline{h}$ and to keep the system in the upstream no-return set. For simplification, we restrict the model ensemble by considering only qualitative values for x above `xv` (see Appendix for the modified version of the model including several possible constraints).

Conservative control is implemented in the adaptive version Eq. (4.17) by introducing a quantity space for $\dot{r}$ and the constraint

```
((M- h dr) (hmin 0)).
```

We apply the same restriction and abstraction techniques as before, with the result that the former upstream no-return set splits into two no-return sets (see Fig. 4.6). Observe that eco-

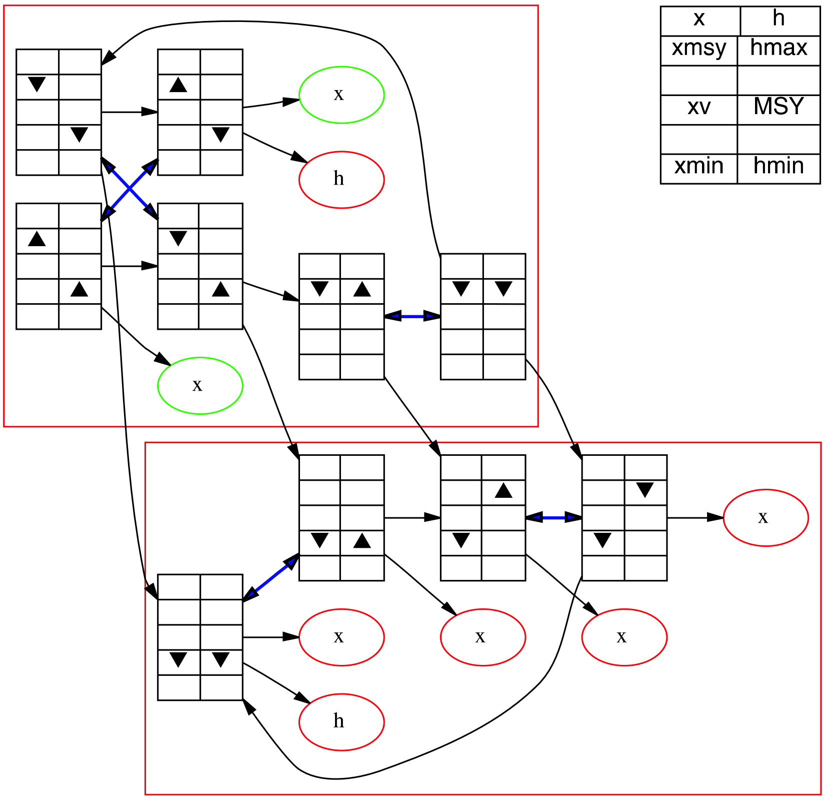

Figure 4.5: Projection of the restricted state-transition graph of the negotiation model with capital with respect to x and h. Final states are indicated by ellipses where a trajectory leaves the pruned state space: the variable transgressing the boundary is printed in an ellipse. Ellipses are red, when the lower boundary is transgressed (x = `xmin` or h = `hmin`), while green ellipses denote a recovering fish stock (x = `xmsy`). Red boxes represent no-return sets. To improve the presentation, final states are printed within the strongly connected no-return sets they are the successors to.

nomic viability cannot be violated in this part of state space. On the other hand, the conservative strategy cannot generally prevent x falling below `xv`. This can be explained if recommendations react too slowly to deviations of h from $\underline{h}$ or if the inertia introduced by k makes the system non-viable. If the knowledge about the fishery only allows for the set-up of a qualitative model as here, it must be admitted that conservative control is also risky. Even if the model is refined by various ordinal assumptions, the ORDAS algorithm presented in section 3.3 (p. 68) does not provide substantial improvements.

Qualitative control subsumes various possibilities for constraints defining control rules. Here, we consider a specific set of constraints which improves the situation:

```
(((M - + -) x dx dh r) (xmsy 0 0 hmin)),
(((M - + -) x k h dh)).
```

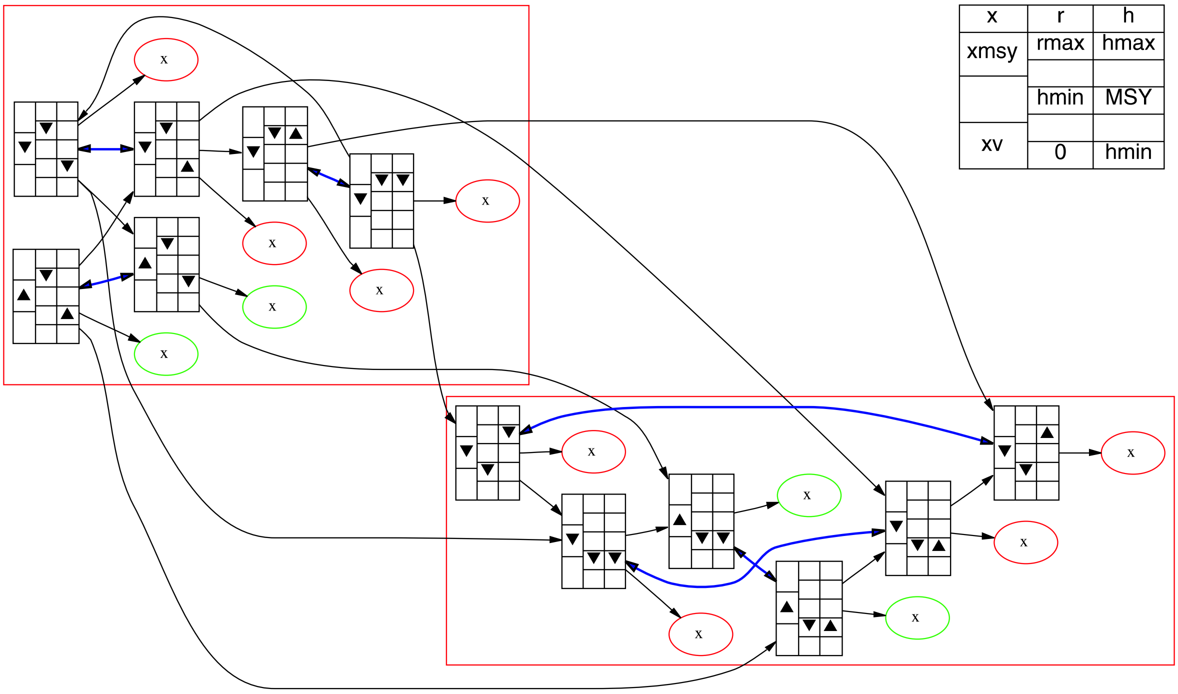

Figure 4.6: Projection of the restricted state-transition graph resulting from conservative control. Red ellipses denote that x drops below xv; other ellipses and clusters are used as in Fig. 4.5.

Consequently,

$$\Sigma = \begin{pmatrix} [+] & 0 & [-] \\ 0 & [-] & [+] \\ [-] & [+] & [-] \end{pmatrix},$$

and we implicitly assume that there exists a control function $u \in C^1(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R}_+)$, $r = u(x, \dot{x}, \dot{h})$ with $D_x u, D_{\dot{h}} u < 0$, $D_{\dot{x}} u > 0$ and $u(x_{MSY}, 0, 0) = \underline{h}$. The function associated with the last row of Σ is denoted by f. Such assumptions need a deeper justification, because they could result in $\mathcal{M}(\mu, C) = \varnothing$. We have to show that there are functions u, f with the given monotonicity properties such that

$$\dot{x} = R(x) - h, \tag{4.18}$$

$$\dot{k} = I(k, h), \tag{4.19}$$

$$h = h(x, k, r), \tag{4.20}$$

$$r = u(x, \dot{x}, \dot{h}), \tag{4.21}$$

$$\dot{h} = f(x, k, h). \tag{4.22}$$

This is only possible if u can be chosen such that by inserting Eq. (4.21) in Eq. (4.20) and differentiating with respect to time yields the same monotonicity properties of $\dot{h}$ as in Eq. (4.22). Showing this directly is problematic since $\dot{h}$ also appears as an argument of u. Thus use an indirect approach. Substituting Eq. (4.18) into Eq. (4.21), and then Eq. (4.21) into Eq. (4.20) yields

$$h = h\big(x, k, u(x, R(x) - h, \dot{h})\big),$$

which can be solved for $\dot{h}$ by the implicit function theorem. We obtain

$$D_x \dot{h} = -\frac{D_x h + D_r h (D_x u + D_{\dot{x}} u D_x R)}{D_r h D_{\dot{h}} u}, \tag{4.23}$$

$$D_k \dot{h} = -\frac{D_k h}{D_r h D_{\dot{h}} u} > 0,$$

$$D_h \dot{h} = \frac{1 + D_r h D_{\dot{x}} u}{D_r h D_{\dot{h}} u} < 0,$$

where Eq. (4.23) has an ambiguous sign – based on the sign assumptions made so far. Thus, the signs do not contradict the last row of Σ, i.e. the monotonicity properties of f. It also becomes clear that the qualitative control rule includes choosing u such that Eq. (4.23) is negative. The restricted and abstracted state-transition graph following from these specifications is given in Fig. 4.7. The graph contains 6 strongly connected no-return sets such that the qualitative control rule introduces much more certainty into the system. The rule also introduces a small invariant set of states where fish stock and harvest never violate the viability constraints. There are no problematic outcomes for one intermediate no-return set as long as it is not left. On the other hand, this cannot be safeguarded by the qualitative control rule since the system may also evolve into an invariant set with non-viable final states. In summary, this rule is an improvement but not a perfect solution. However, an extensive explorative test of various other constraints for r and ordinal assumptions has not revealed a strategy which performs substantially better.

Summarising the results from both models in this section, we have seen that a recommendation strategy based purely on the observation of the fish stock necessarily leads to economic or environmental decline. The situation can be improved substantially by a strategy based on purely economic observations. However, it does not generally work in the more complex setting where capital dynamics come into play and only qualitative observations about the system can be made. This also contributes to the insight that sustainable common property harvesting under uncertainty represents really a difficult problem. At least, the more flexible qualitative control requires only little information about the state of the fishery and is less risky than data-rich ichthyocentric management.

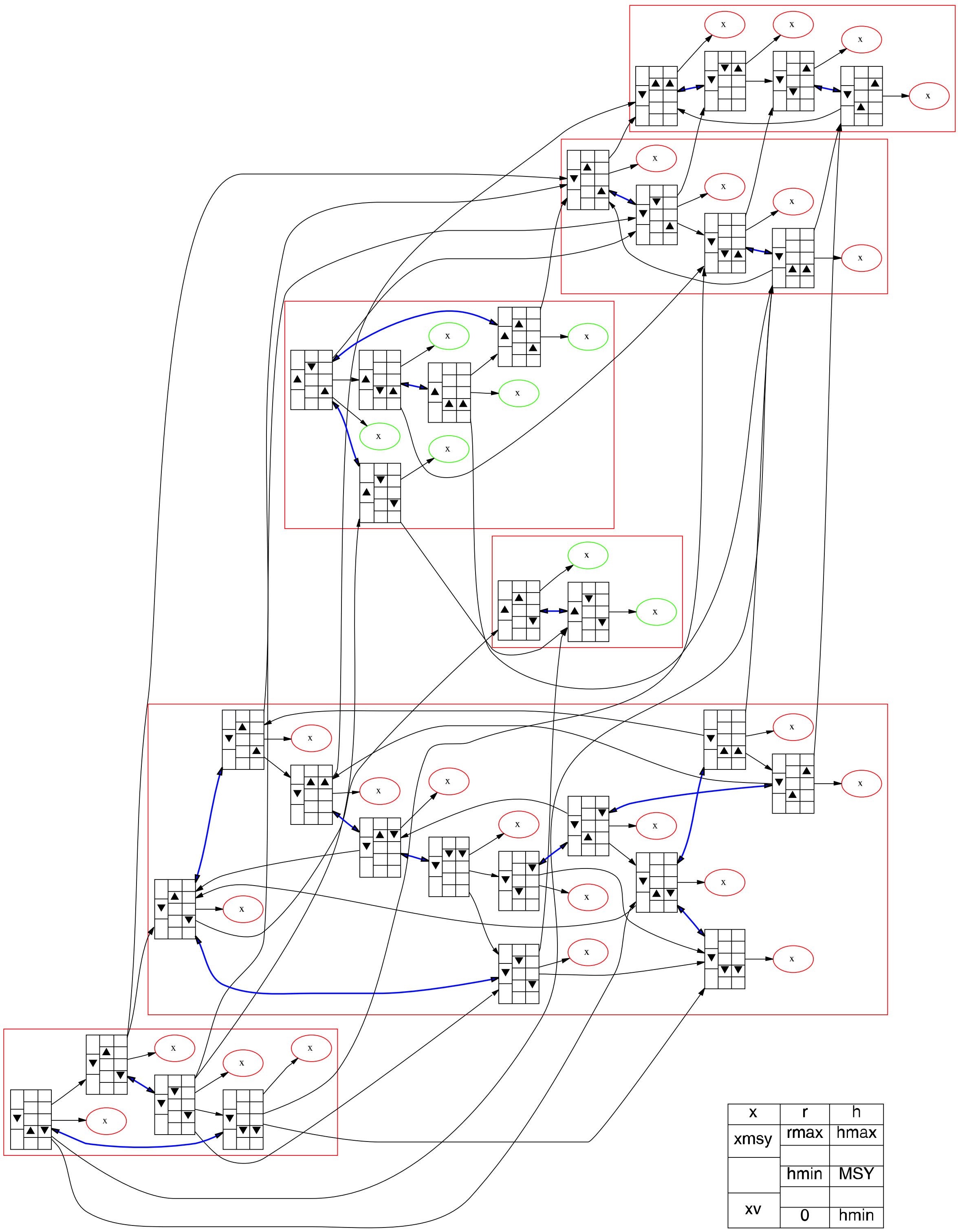

Figure 4.7: Restricted state-transition graph resulting from qualitative control. Ellipses and clusters have the same meaning as in Fig. 4.5.

4.4 Lake Management

In this section I analyse a qualitative model of a lake ecosystem subject to a management system to avoid eutrophication. Based on a model by Carpenter (2003), it is modified to account for various uncertainties. The methods developed in this thesis, in particular ordinal assumptions (cf. section 3.3, p. 68) and quantitative bounds (cf. section 3.4, p. 77) are applied.

Eutrophication of inland waters is a major threat to water quality. It is a process by which a body of water acquires a high concentration of nutrients, especially nitrogen and phosphorus, which promote primary production (Schwörbel 1999). "Many ecosystem services are reduced when inland waters and coastal ecosystems become eutrophic. Water from lakes that experience algal blooms is more expensive to purify for drinking or industrial uses. Eutrophication can reduce or eliminate fish populations." (Millennium Ecosystem Assessement 2005, p. 69). The major cause of eutrophication is excessive inputs of phosphorus from urban runoff and agricultural areas (Lathrop et al. 1998). Despite decades of water research and management, blooms of blue-green algae are still a major water quality problem in lakes and reservoirs. Deterministic models are frequently used to determine consequences of phosphorus input changes, but they involve large prediction uncertainties (Lathrop et al. 1998).

The modification of the original model allows for further insights in the underlying lake management dynamics. It combines an ecosystem model of phosphorus dynamics in lake water and sediment with a management model. The manager observes the phosphorus content in the water column P and in the sediment M qualitatively and takes measures to change phosphorus input using a specified management strategy, e.g. by constructing and operating sewage plants. As in the previous section 4.3 (p. 101), several qualitative constraints describing the management strategy can be assessed for viability (i.e. to avoid eutrophication in our case). We concentrate on one interesting alternative.

The goal of the original contribution is to explore the possibility of anticipating thresholds before they are crossed in a setting of high uncertainties: The levels of phosphorus input are subject to unpredictable variations (e.g. due to weather). The original model is parameterised such that its state is close to a critical level of eutrophication. The lake manager is assumed not to know the values of all parameters of the ecosystem exactly. This leads Carpenter to formulate the relation between input target and actual phosphorus input as well as parameter estimation of the lake manager stochastically. The model contains a non-linear term describing phosphorus release from the sediment to the water column. This non-linearity helps to explain abrupt transitions from a so called clear-water regime to a turbid regime (high phosphorus in water column or eutrophic lake). For simplicity, Carpenter describes this threshold effect by a sigmoid rational function depending on P, although various other processes contribute to phosphorus release and other sigmoid functions cannot be refuted (Wetzel 2001). These problems are good reasons to develop a qualitative version of the model:

- There is uncertainty about functional relationships.
- Several parameters and some state variables (esp. M) are not known exactly and costly to measure.
- The effects of the decisions of the lake manager are not exactly predictable.

The Model

The basic model equations are

$$\dot{P} = -(s+h)P + r\phi M + L, \tag{4.24}$$
$$\dot{M} = sP - bM - r\phi M, \tag{4.25}$$

with L being the exogenous phosphorus influx into the lake which is the control variable of the lake manager. The parameter s describes the rate of sedimentation from the water column, h the outflow rate from the lake. Phosphorus leaves the sediment by two processes: a part is buried with rate b, and another part may be recycled to the water column, depending on the parameter r, the phosphorus content of the sediment M and a dependent recycling parameter ϕ. It is assumed that ϕ is small for low P, increases quickly near a threshold, and converges to a maximum afterwards. The quantitative simulation performed by Carpenter (2003) requires a functional form for ϕ although little is known about an exact quantification. One possibility is

$$\phi(P) = \frac{P^q}{m^q + P^q},$$

with a threshold parameter m. The parameter values used for the original model are given in Tab. 4.4. I translate this to a model ensemble by replacing the function ϕ with the qualitative constraint

```
((S+ P phi) (l1 0) (l2 1)),
```

claiming that the monotonic landmarks ensemble should contain all ODEs with a functions $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+), P \mapsto \phi(P)$ for which values $\lambda_1, \lambda_2 \in \mathbb{R}_+$ exist such that

$$\begin{aligned} \forall P \leq \lambda_1 &: \phi(P) = 0, \\ \forall P \geq \lambda_2 &: \phi(P) = 1, \\ \forall P \in (\lambda_1, \lambda_2) &: \phi(P) \in (0,1) \text{ and } D_P\phi(P) > 0. \end{aligned}$$

Also the linear relationships of the original model are generalised to monotonic functions such that a broad variety of non-linear dependencies are also included in the monotonic landmark ensemble. This is justified by our limited knowledge about the exact relationships, especially for this highly simplified model. The generalisation is based on the signs of the partial derivatives of $\dot{P}$ and $\dot{M}$:

$$\begin{pmatrix} D_P\dot{P} & D_M\dot{P} & D_L\dot{P} \\ D_P\dot{M} & D_M\dot{M} & D_L\dot{M} \end{pmatrix} = \begin{cases} \begin{pmatrix} -(s+h) & 0 & 1 \\ s & -b & 0 \end{pmatrix} & \text{for } P \le \lambda_1, \\ \begin{pmatrix} -(s+h) + rMD_P\phi & r\phi & 1 \\ s - rMD_P\phi & -(b+r\phi) & 0 \end{pmatrix} & \text{for } P \in (\lambda_1, \lambda_2), \\ \begin{pmatrix} -(s+h) & r & 1 \\ s & -(b+r) & 0 \end{pmatrix} & \text{for } P \ge \lambda_2. \end{cases} \tag{4.26}$$

The quantity space of P is supplied with the landmarks $\lambda_1 < \lambda_2 < \lambda_{eu}$, the latter denoting a problematic level of eutrophication. In the first version of the qualitative model, the control variable L is left unconstrained as in the previous section (cf. section 4.3, p. 101) – the resulting state-transition graph will contain the consequences of all continuous differentiable phosphorus input functions $L(\cdot)$. We will test possible constraints for L only in the second step. The sign matrices contributing to the definition of the monotonic landmark ensemble $\mathcal{M}(\mu, C)$ are

$$\mu(v) = \begin{cases} \begin{pmatrix} [-] & 0 & [+] \\ [+] & [-] & 0 \\ [?] & [?] & [?] \end{pmatrix} & \text{for } \mathrm{qmag}_P(v) \le \lambda_1, \\ \begin{pmatrix} [?] & [+] & [+] \\ [?] & [-] & 0 \\ [?] & [?] & [?] \end{pmatrix} & \text{for } \mathrm{qmag}_P(v) \in (\lambda_1, \lambda_2), \\ \begin{pmatrix} [-] & [+] & [+] \\ [+] & [-] & 0 \\ [?] & [?] & [?] \end{pmatrix} & \text{for } \mathrm{qmag}_P(v) \ge \lambda_2. \end{cases}$$

The complete model specification is given in the Appendix.

Results

The state-transition graph of this very general model has 72 vertices and 269 edges. Eliminating marginal edges (cf. section 3.2, p. 62) and states which are non-analytical in P, M or L reduces the graph to 34 vertices and 78 edges (see Fig. 4.8). No-return abstraction (cf. section 3.1, p. 52) reveals that all vertices except the final states form a strongly connected component. This is not surprising due to the generality of a model with unconstrained management. However, it is worth comparing this result with the negotiation model with

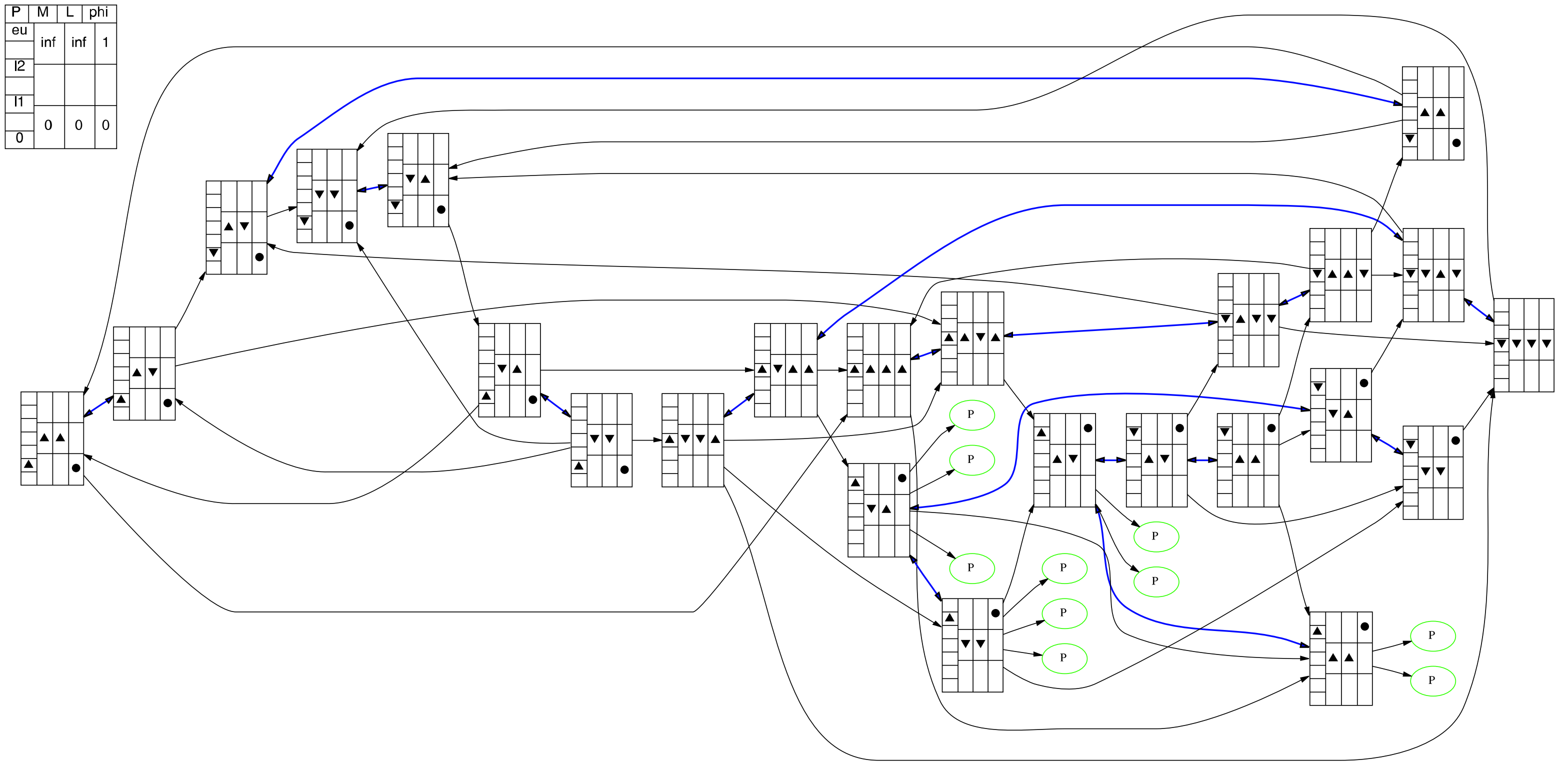

Figure 4.8: Restricted state-transition graph of the lake model with unconstrained management. Final states where phosphorus exceeds the unacceptable threshold λ_{eu} are marked as green ellipses.

capital from section 4.3.2 (p. 109), in particular Fig. 4.5, where even in the unconstrained case no viable control exists for some states.

Management

As in the previous section 4.3 (p. 101) we can in principle test out all possibilities to constrain the control variable L. This corresponds to finding a closed-loop control for the lake manager to influence L, e.g. by the effort put into operating sewage plants, depending on the manager's qualitative observation of the system. I do not describe the whole search process here, but present one of the promising qualitative control rules. It works on a restricted quantity space, obtained by introducing a further landmark $\lambda_* \in (\lambda_1, \lambda_2)$ for P such that

$$\forall P > \lambda_* : D_P\dot{P} < 0 \text{ and } D_P\dot{M} > 0.$$

Only qualitative states v with $\mathrm{qmag}_P(v) \geq \lambda_*$ are considered and the landmark `phil*` $:= \phi(\lambda_*)$ is defined for the quantity space of ϕ. Within the parameterisation of the original model this means that

$$\forall P > \lambda_* : rMD_P\phi(P) < s,$$

which is valid if a sufficiently small upper bound is given for M. Then, since $D_P\phi(\lambda_2) = 0$, such a landmark λ_* always exists if ϕ is differentiable twice. For the original parameter values and a reasonable bound $M < 1500$, the relation is satisfied for $\lambda_* > 4.92 = 2.05m$. Thus, the pruned quantity space restricts the attention to the state space region close to eutrophication, which is in spirit of the original model. In this region the qualitative constraints

```
(((M - + +) P M L dP)),
(((M + - ) P M dM)),
```

are valid. If we take the parameterisation of Eq. (4.24) and Eq. (4.25) as given, and assume that the lake manager can choose a strategy such that L is determined by a monotonic decreasing function $u \in C^2(\mathbb{R}_+, \mathbb{R}_+), P \mapsto u(P)$, differentiating with respect to time yields

$$\dot{L} = D_Pu(P)\big(-(s+h)P + r\phi(P)M + L\big),$$

and

$$\begin{aligned} D_P\dot{L} &= D_{PP}u\dot{P} + D_PuD_L\dot{P}, \\ D_M\dot{L} &= D_PuD_M\dot{P} < 0, \\ D_L\dot{L} &= D_PuD_L\dot{P} < 0. \end{aligned}$$

By assuming $D_{PP}u = 0$, i.e. u to be an affine function, and since $s > rMD_P\phi(P)$ for $P \geq \lambda_*$, the following constraint can be formulated:

```
(((M + - -) P M L dL)),                    (4.27)
```

i.e. $\dot{L}$ is described by a function $u \in C^1(\mathbb{R}^3_+, \mathbb{R}_+), u \mapsto u(P, M, L)$ such that $D_Pu > 0$ and $D_Mu, D_Lu < 0$. This means that the lake manager implicitly considers the changes in M, L

if he only observes P and reacts in an affine way – which is much easier to perform. The essence of this situation is entailed by a monotonic landmark ensemble with

$$\mu \equiv \Sigma := \begin{pmatrix} [-] & [+] & [+] \\ [+] & [-] & 0 \\ [+] & [-] & [-] \end{pmatrix} . \tag{4.28}$$

(the model code is given in the Appendix). The state-transition graph contains 61 vertices and 172 edges, which are restricted to 35 vertices and 54 edges after applying the usual restriction techniques. The no-return abstraction of the restricted graph yields one strongly connected component, 3 no-return sets consisting of one state, and 18 final states where the region with $P \in (\lambda_*, \lambda_{eu})$ is left. However, the large no-return set has successors with eutrophication as well as others where the phosphorus level becomes low, making it difficult to evaluate the proposed control rule as preferable or problematic.

The situation changes when appropriate ordinal assumptions are made. According to section 3.3 (p. 68), these are assumption on the signs of

$$d^{i,j}_{k,l} = D_l f_j \cdot D_k f_i - D_k f_j \cdot D_l f_i,$$

(cf. Eq. 3.2, p. 71). In the case of Eq. (4.28), some of these signs are already prescribed, e.g.

$$[d^{1,2}_{2,3}] = -[d^{1,2}_{3,2}] = -[d^{2,1}_{2,3}] = [d^{2,1}_{3,2}] > 0,$$

but others are free to choose. We assume that

$$\begin{aligned} &d^{1,2}_{1,2}, d^{2,3}_{1,2} > 0, \\ &d^{1,3}_{1,2} = d^{1,3}_{2,3} = d^{1,3}_{1,3} = 0, \end{aligned}$$

and the further ordinal assumptions resulting from symmetry (see Eq. 3.11, p. 73). These assumptions do not make the model ensemble empty since

$$\begin{aligned} d^{1,3}_{1,2} &= D_P\dot{P} D_P u D_M \dot{P} - D_P u D_P \dot{P} D_M \dot{P} = 0, \\ d^{1,3}_{2,3} &= D_M\dot{P} D_P u D_L \dot{P} - D_P u D_M \dot{P} D_L \dot{P} = 0, \\ d^{1,3}_{1,3} &= D_P\dot{P} D_P u D_L \dot{P} - D_L \dot{P} D_P u D_P \dot{P} = 0, \\ d^{1,2}_{1,2} &= h(b + r\phi) + b(s - rMD_P\phi) > 0, \\ d^{2,3}_{1,2} &= -D_P u d^{1,2}_{1,2} > 0, \end{aligned}$$

such that all ordinal assumptions are valid. We are now ready to run the ORDAS algorithm developed in section 3.3 (p. 68). The procedure detects 10 paths of length 2 which contradict at least one of the ordinal assumptions. As some of these paths share vertices, only 6 new vertices and 22 new edges have to be introduced. Although this is an increase in number (see Tab. 4.5), the graph now has 3 strongly connected components and 5 non-final no-return sets consisting of a single state, i.e. the structure fosters stronger predictions of system behaviour (see Fig. 4.9). Even more preferable, there a two invariant sets of vertices which can be clearly evaluated: One admits only final states with eutrophication, the other admits only

Parameter	Value	Units
b	$0.1 \cdot 10^{-2}$	a^{-1}
h	0.15	a^{-1}
r	$1.9 \cdot 10^{-2}$	a^{-1}
s	0.7	a^{-1}
m	2.4	$kg\ m^{-2}$
q	8	dimensionless

Parameter	Interval
b	$[0.05, 0.15] \cdot 10^{-2}$
h	$[0.10, 0.20]$
r	$[1.50, 2.50] \cdot 10^{-2}$
s	$[0.60, 0.80]$
β	$[0.80, 1.20]$
ϕ	$[0.00, 1.00]$

Table 4.4: Left: parameter values of the lake model of Carpenter (2003). Right: interval estimates for model parameters, containing the values of the original model.

Graph	Vertices	Edges
State-transition graph		
after removing marginal edges in runtime	61	172
Removing marginal edges	55	105
Removing non-analytical states	33	54
After applying ORDAS algorithm	41	72

Table 4.5: Number of vertices and edges resulting from different restriction techniques applied subsequently to the lake management model.

final states where P decreases below λ_*. Thus, we cannot conclude that the proposed rule in its general form is always successful, but it is successful once the positive invariant set is reached. In the large no-return set this cannot be predicted from our general assumptions. For crisper results we pick out some interesting states in the graph and investigate the tendency that the system shifts into one or the other successor state by using quantitative bounds (cf. section 3.4, p. 77).

A Linear-Interval Version of the Management Model

We consider the qualitative state v_1 where $\mathrm{qmag}_P(v_1) = \{\lambda_*, \lambda_2\}$, $\mathrm{qdir}_P(v_1) = \mathrm{qdir}_\phi(v_1) = [+]$, and $\mathrm{qdir}_M(v_1) = \mathrm{qdir}_L(v_1) = [-]$; denoted as state (A) in Fig. 4.9. In this state the next successor can determine the fate of the system: If the input level L begins to increase, the large neutral no-return set is entered. In this case there are several succeeding paths of length 2 leading to the negative invariant set or to an eutrophic state. If the phosphorus in the sediment begins to increase, entering the neutral no-return set is just postponed, with the same of risk of reaching the negative invariant set. In contrast, if P decreases, the phosphorus in the water column will necessarily fall below λ_* – no matter which affine control function the lake manager chooses. The question is whether there is a reasonable tendency for the latter outcome.

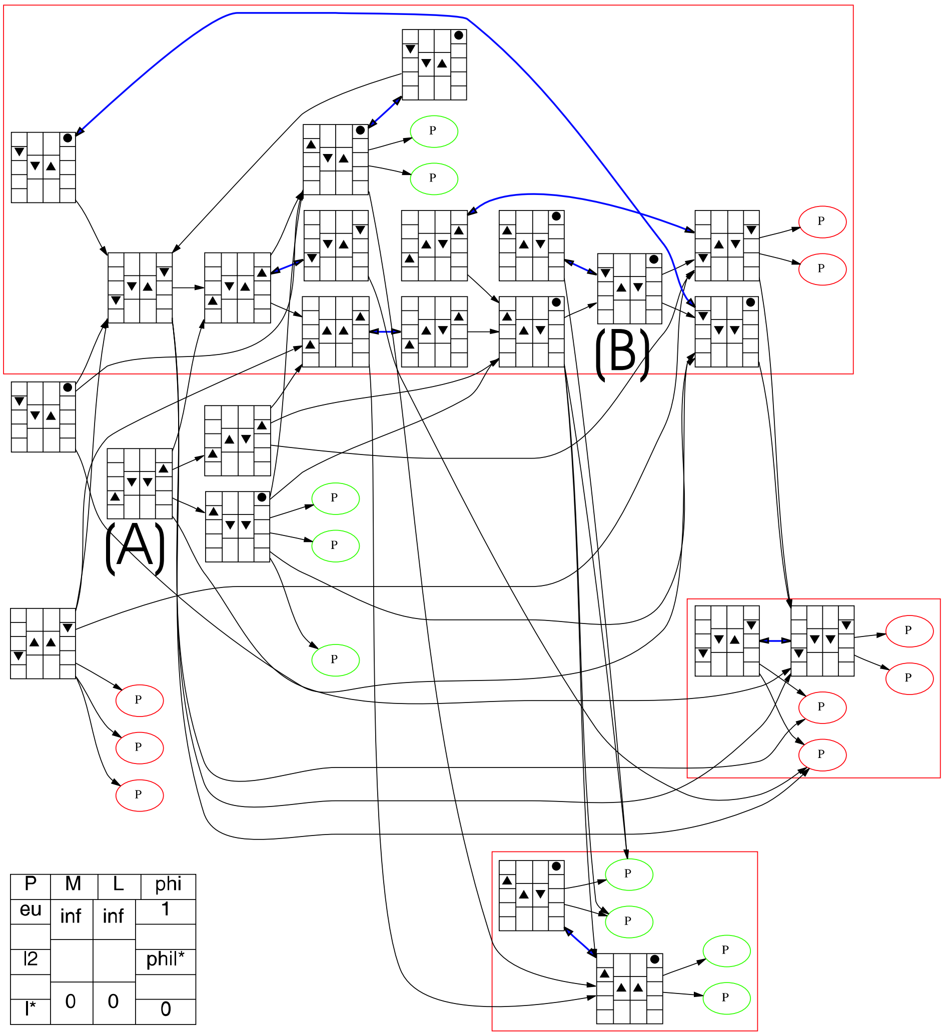

Figure 4.9: Restricted state-transition graph of the lake model with qualitative control rule and ordinal assumptions as discussed in the text. Final states where phosphorus exceeds the unacceptable threshold λ_{eu} are marked as green ellipses, while final states where P falls below λ_* are red.

We now formulate a linear-interval differential inclusion as introduced in section 3.4 (p. 77) which is valid in the qualitative state described above. We compute the absorption basin of each successor state, i.e. all initial velocities which necessarily lead to a given successor state. For this task quantitative intervals for the components of the Jacobian of the system are needed. The values of the original model are replaced by intervals to account for uncertainties. To define the affine closed-loop control function $u(P) = \alpha - \beta P$, also choose an interval for β since the actual phosphorus input L can substantially differ from an input target $u(P)$ (Carpenter 2003). The intervals are given in Tab. 4.4. Little is known about actual values of the non-linear part ϕ. Although by assumption $\mathrm{qmag}_\phi(v) = \{\texttt{phil*}, 1\}$, a tight estimate is difficult since also `phil*` is not known quantitatively. For sake of generality we only assume that $\phi \in [0, 1]$. This has two important consequences: First, in this formulation ϕ is no longer a function of P, but an independent parameter. Secondly, this parameter can

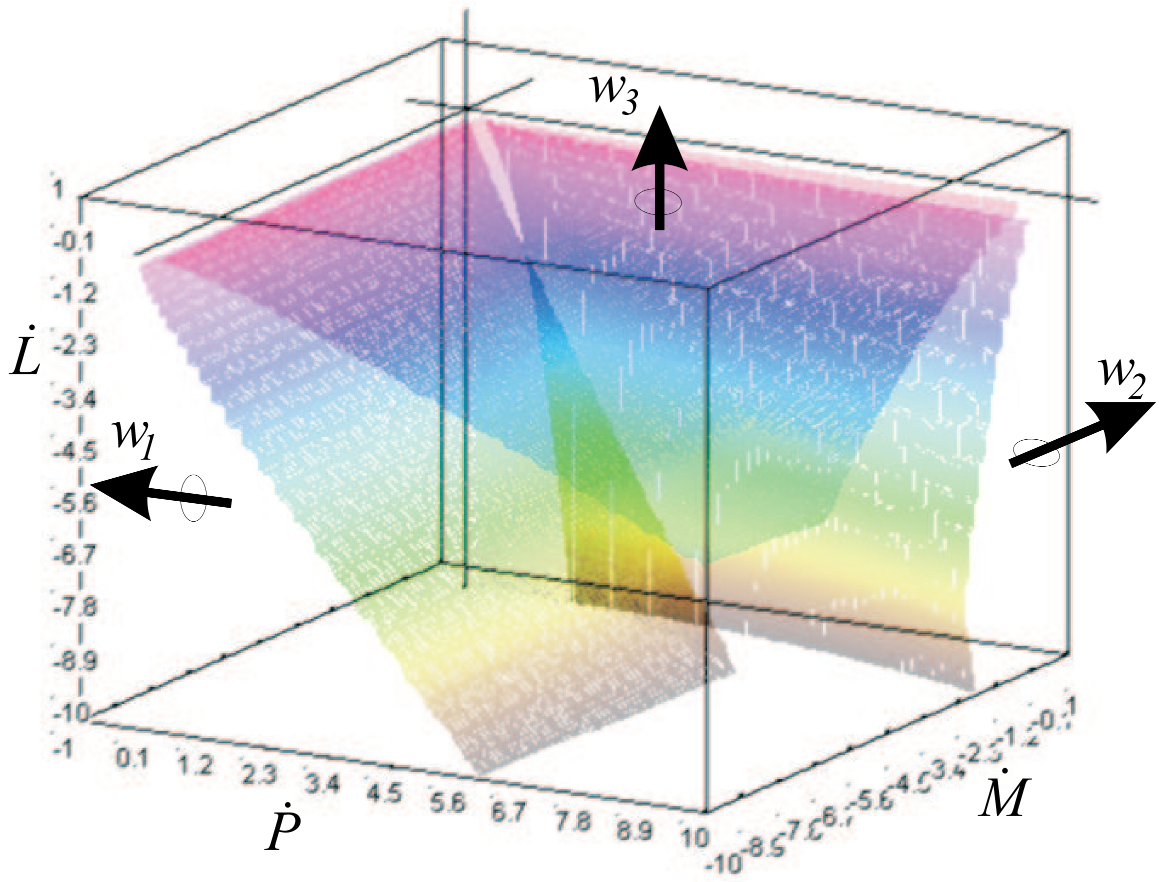

Figure 4.10: The absorption basins $\mathrm{Abs}_F(\bar{K}(v_1), \bar{K}(v_1) \cap \bar{K}(w_1)) \cap Q$, $\mathrm{Abs}_F(\bar{K}(v_1), \bar{K}(v_1) \cap \bar{K}(w_2)) \cap Q$ and $\mathrm{Abs}_F(\bar{K}(v_1), \bar{K}(v_1) \cap \bar{K}(w_3)) \cap Q$.

change arbitrarily in time (as long as $\phi(\cdot)$ remains measurable). The interpretation of the dynamics brought about by a linear-interval differential inclusion is thus different from the QDE dynamics as already discussed in section 3.4 (p. 77). Having said this, the appropriate set-valued map (cf. Eq. 4.26, p. 118) is defined by the interval matrix

$$U = \begin{pmatrix} [-1.00, -0.70] & [0, 0.0250] & 1 \\ [0.60, 0.80] & [-0.0265, -0.0005] & 0 \\ [0.56, 1.20] & [-0.0300, 0] & [-1.2, -0.8] \end{pmatrix}.$$

We use the viability kernel algorithm to compute the absorption basins

$$\mathrm{Abs}_F(\bar{K}(v_1), \bar{K}(v_1) \cap \bar{K}(w_1)),$$
$$\mathrm{Abs}_F(\bar{K}(v_1), \bar{K}(v_1) \cap \bar{K}(w_2)),$$
$$\mathrm{Abs}_F(\bar{K}(v_1), \bar{K}(v_1) \cap \bar{K}(w_3)),$$

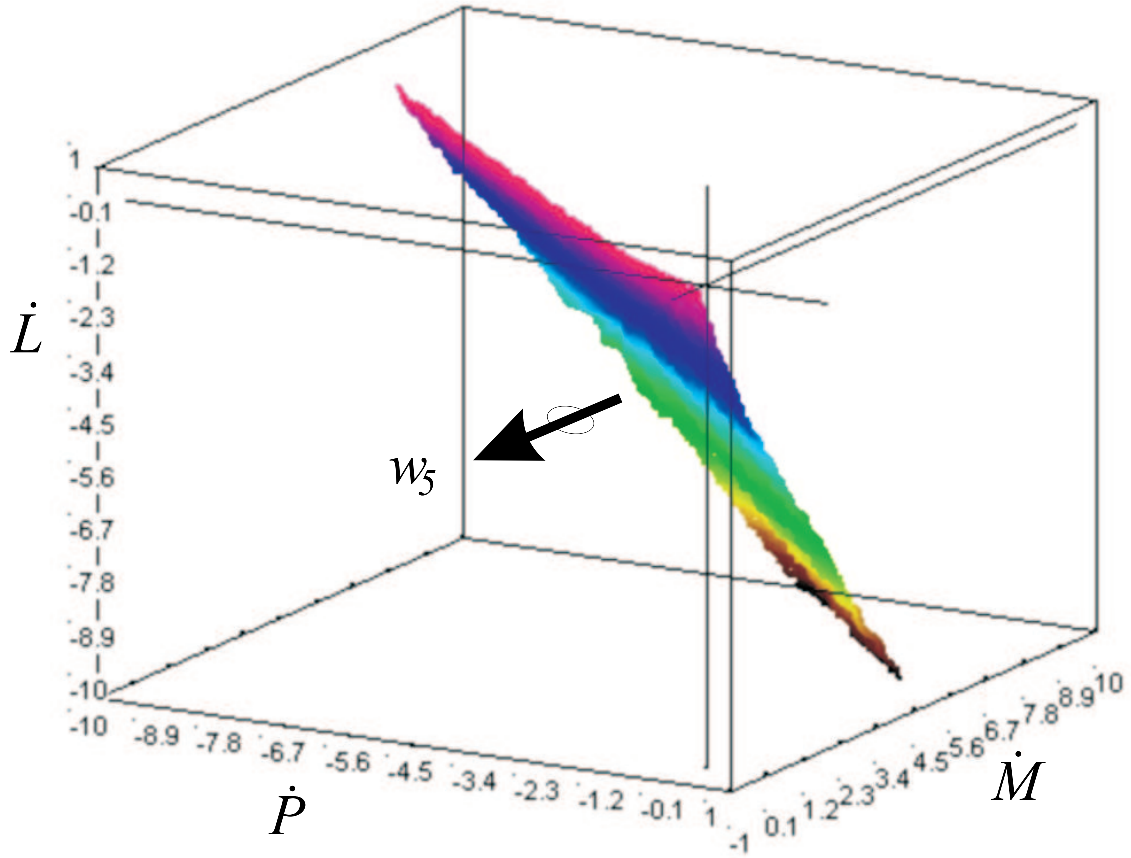

Figure 4.11: The absorption basins $\mathrm{Abs}_F(\bar{K}(v_2), \bar{K}(v_2) \cap \bar{K}(w_4)), \mathrm{Abs}_F(\bar{K}(v_2), \bar{K}(v) \cap \bar{K}(w_5))$, restricted to a cube. The former is too small to be seen in this presentation.

with $F : \mathbb{R}^3 \rightsquigarrow \mathbb{R}^3, (\dot{P}, \dot{M}, \dot{L}) \rightsquigarrow U(\dot{P}\,\dot{M}\,\dot{L})^t$ and

$$\begin{aligned} v_1 &= ([+]\,[-]\,[-])^t, \\ w_1 &= ([-]\,[-]\,[-])^t, \\ w_2 &= ([+]\,[+]\,[-])^t, \\ w_3 &= ([+]\,[-]\,[+])^t. \end{aligned}$$

Recall that for $v \in \mathcal{A}^n, K(v) = \{\dot{x} \in \mathbb{R}^n \mid \mathrm{sgn}(\dot{x}) = v\}$, such that the absorption basins contain all initial values for which the system necessarily shifts from state v_1 to state w_1, w_2 or w_3 (cf. section 3.4, p. 77). The boundary of all three absorption basins, restricted to the cube $Q = [0, 10] \times [-10, 0] \times [-10, 0]$ is depicted in Fig. 4.10. The volume of $\mathrm{Abs}_F(\bar{K}(v_1), \bar{K}(v_1) \cap \bar{K}(w_1)) \cap Q$ is of considerably large compared to the other absorption basins (restricted to Q) – this can be interpreted that within the given parameter ranges there is a considerable chance that the system enters the invariant set where the

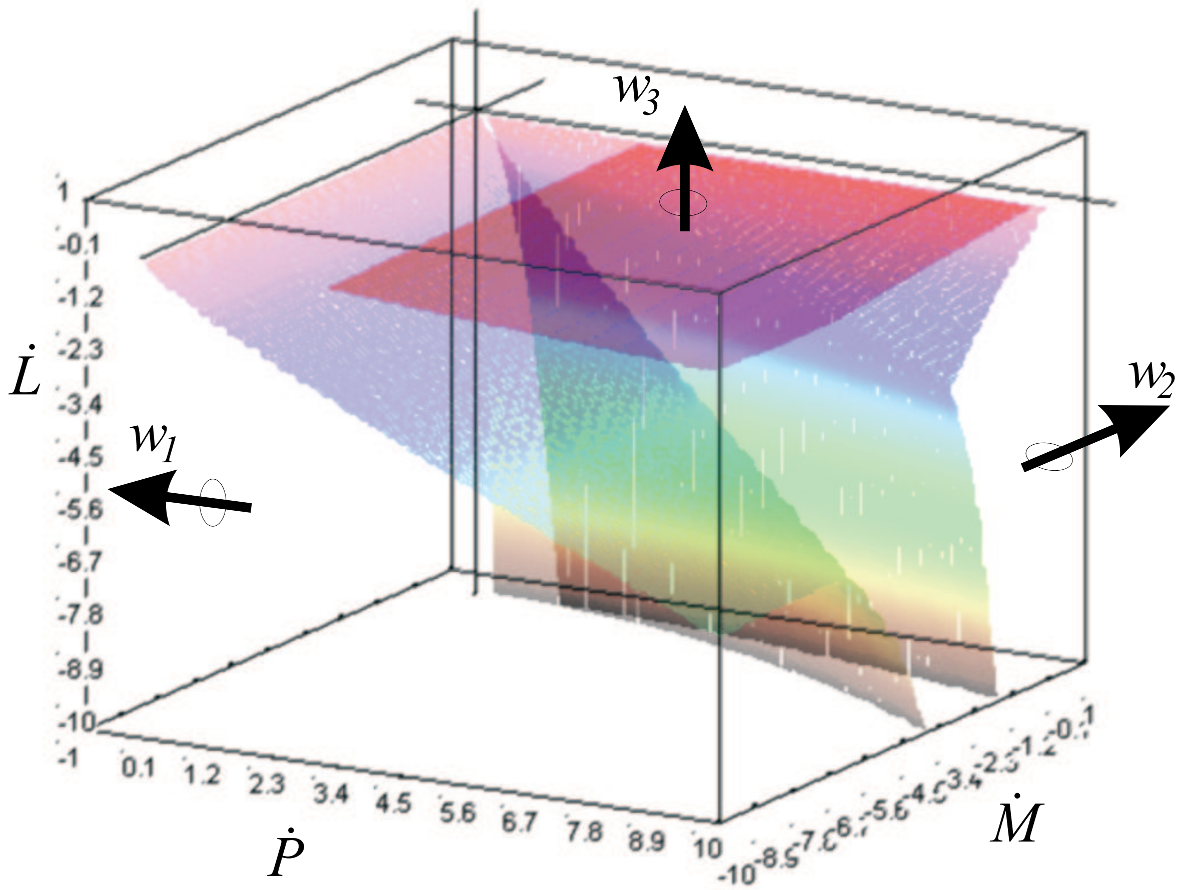

Figure 4.12: The absorption basins for successor states of v_1 for $\beta \in [0.2, 0.4]$.

phosphorus concentrations necessarily decreases. However, none of the three successors can be generally excluded. The volume of $\mathrm{Abs}_F(\bar{K}(v_1), \bar{K}(v_1) \cap \bar{K}(w_2)) \cap Q$, where the phosphorus content in the sediment will begin to increase again is the smallest compared to the other absorption basins. This is different from the state v_2 denoted by (B) in Fig. 4.9, where $\mathrm{qmag}_P(v_2) = \{\lambda_2, \lambda_{eu}\}$, $\mathrm{qdir}_M(v_2) = [+]$, and $\mathrm{qdir}_P(v_2) = \mathrm{qdir}_L(v_2) = [-]$. For one successor w_4, P begins to increase and there is the risk of entering the invariant set with unavoidable eutrophication. There is only one other combination of qualitative directions of the successor states of v_2, which is denoted by w_5: M begins to decrease. Again we use the parameter ranges given in Tab. 4.4, with one exception: since in v_2 unambiguously $\phi(P) = 1$, one obtains

$$U = \begin{pmatrix} [-1.00, -0.70] & [0.0150, 0.0250] & 1 \\ [0.60, 0.80] & [-0.0265, -0.0155] & 0 \\ [0.56, 1.20] & [-0.0300, -0.0120] & [-1.2, -0.8] \end{pmatrix},$$

yielding the absorption basins as indicated in Fig. 4.11. Here, $\mathrm{Abs}_F(\bar{K}(v_2), \bar{K}(v_2) \cap \bar{K}(w_4))$ is significantly smaller than the other absorption basin – we conclude that in state v_2 the linear-interval control strategy performs very well. It is also possible to assess how absorption basins change if the management strategy is modified. Assuming a smaller value for $D_P u$, e.g. $\beta \in [0.2, 0.4]$, the absorption basins associated to v_1 are depicted in Fig. 4.12. In contrast to Fig. 4.10, the volume of the absorption basin leading to an increasing M or L (restricted to Q) becomes much more smaller. There is a large absorption basin of initial velocities from which every trajectory enters the preferable invariant set. Thus, a less "sensitive" lake management is profitable under the conditions of state v_1.

We conclude from this section that making ordinal assumptions can substantially improve the structure of the state-transition graph and that the linear-interval version of the qualitative model gives clear hints for the tendency of alternative system changes in critical states. Both proved to be useful tools for the design of resource management strategies. As in the previous section, the search for a preferable qualitative control is performed on a finite set of possible alternatives. Ordinal assumptions provide further degrees of freedom for management design. Although the search space for management options becomes infinite when using linear-interval differential inclusions, it can be used to refine results already obtained by qualitative reasoning. Thus, the two methods complement each other.

Chapter 5

Conclusions

In this thesis, I have demonstrated with several examples from natural resource management that even very general model ensembles can reveal robust and valuable features. The new restriction and abstraction techniques for model ensembles as well as techniques for viable control design substantially improved the capacity for yielding such results. The framework of model ensembles as the organising principle of the thesis proved to be a common ground for various methods like qualitative differential equations (QDEs), differential inclusions and causal loop diagrams. The achievements of this thesis can be summarised on three levels: theoretical, with respect to the concrete resource use problems investigated in Chapter 4, and for future applications.

- The concept of a model ensemble provides common ground for linking QDEs, differential inclusions and viability theory, and meets methodological demands of sustainability science. Together with the graph theoretical formulation, abstraction and restriction can be investigated in a systematic way.

- The experience with the new abstraction and restriction methods proves an increased applicability of qualitative reasoning methods. They allow the identification of robust properties of infinite ensembles of ODEs as they occur under uncertainty and generality.

- The applications demonstrate that relevant conclusions for natural resource management can be made on the base of model ensembles. Viable control strategies can be designed with these methods.

Theoretical Achievements

The thesis contributes to the theory of qualitative reasoning from different perspectives. In section 2.1, I proposed model ensembles as a framework to embed both qualitative differential equations (QDEs) and differential inclusions. It is designed to meet essential needs of sustainability science and can be expected to be useful for other domains where uncertainty or generality of models have to be faced. It also suggests a base to combine qualitative and quantitative dynamics, and by providing a systematic description of the relation between QDEs and ODEs, we can take a new view on qualitative modelling. In former approaches, a

QDE is portrayed as an abstraction of a single ODE which yields – more or less "involuntarily" – a large set of systems including ODEs which are not intended as valid representations. In contrast, within this thesis the modelling already starts from a set of systems which may all be valid under uncertainty or generality.

This way of formalising the relation between ODEs and QDEs further profited from the use of graph theoretical concepts, which have not been used explicitly in the qualitative reasoning literature previously (section 2.2). These concepts not only allowed for a more concise formalisation, but also for a generic definition of abstraction (section 2.2.4) and the development of new and advanced abstraction techniques.

The no-return abstraction I developed in section 3.1 was motivated by detecting irreversible system development and emerged from a discrete analogon of invariant sets as defined in viability theory. It was shown that sets of qualitative states which cannot be reentered once they are left are closely related to the well-known concept of strongly connected components of directed graphs. It can also be proved how this method can be combined with established abstraction techniques as chatter-box abstraction and projection.

The use of abstraction techniques is limited if the state-transition graph is not very well structured. Further model assumptions are needed to restrict the model ensemble. Some steps in this direction were made in this thesis. I presented automated elimination of marginal edges as one new method in section 3.2. Its power stems from a clear definition of "marginality" which fosters the use of two complementary algorithms, and from the fact that the number of marginal edges tends to be large due for combinatorial reasons.

Another achievement is the restriction of monotonic (landmark) ensembles by making ordinal assumptions as defined in section 3.3. These restrict the model ensemble to a subset where certain expressions involving the coefficients of the Jacobian of the systems are larger or smaller than other expressions. This restriction and the proposed ORDAS algorithm yields stronger structural properties by eliminating paths of length 2.

A hybrid method combining QDEs and differential inclusions was developed in section 3.4. It can be applied if the modeller has information about quantitative intervals for each component of the Jacobian. The viability algorithm computes whether given edges in the state-transition graph can be eliminated under this restriction of the model ensemble and determines velocities which necessarily lead to a given qualitative successor state. This method is limited by the capacities to compute viability kernels. If progress is made in the computation of viability kernels, the results of the hybrid method are still valid such that there is a potential for its future improvement.

In some of the applications in Chapter 4, control problems play a prominent role (section 4.3, section 4.4). QDEs can easily be used to design qualitative control rules. Since the qualitative state space is a finite set, all possible qualitative constraints for the control variable can be explored to identify conditions under which the structure of the state-transition graph significantly improves, e.g. by producing invariant sets. Such constraints are important candidates for management strategies which are robust under uncertainties and transferable to different natural resource use cases.

Management of Natural Resources

The applications yield several robust properties of natural resource use although model ensembles are very general in nature. In section 4.1 about agriculture on marginal land it is shown that the so called poverty-degradation spiral does not necessarily occur for all ODEs which are consistent with the qualitative description of the underlying mechanisms. On the other hand, a slight restriction of the model model ensemble identifies states which lead to a certain outcome for all ODEs: If poverty and agricultural activity are low and decreasing at the same time, the soil cannot degrade at a later stage.

For unregulated fisheries, I demonstrated in section 4.2 that every ODE describing capital dynamics according to some general qualitative assumptions exhibits a phase of overcapitalisation, i.e. every fishery described by the model will invest in fishing gear and technology for some time although harvests are already decreasing. In section 4.3 the effects of participatory fisheries management and of ichthyocentric decision-making were assessed in combination. The results from a general ODE model complemented with viability criteria and from a QDE model both show that participatory management is not sustainable per se and that management which focuses solely on the ecological view necessarily produces trajectories violating environmental or economic viability criteria. A more complex qualitative control strategy was designed which at least outperforms ichthyocentric control.

The example of lake management in section 4.4 demonstrates how ordinal assumptions and numerical bounds can yield new insights into a model which is prone to uncertainties although a theoretical structure and quantitative data are available. The original quantitative model contains one functional relationship for phosphorus recycling, of which little is known empirically. A model ensemble allows generalisation of this relationship. To design a management strategy on this base, ordinal assumptions proved to be essential to bring about a more advantageous structure. The linear-interval version of the model shows, counter intuitively, that in a certain qualitative state a management which is less sensitive to changes in the phosphorus content of the lake is more likely to prevent eutrophication.

All these applications analyse problematic interactions of society and ecology which mainly occur on a regional level. However, although every lake, every fishery and every agricultural system is different, the problems investigated are not singular: They are observed in a more or less similar way at many places on Earth. Using QDEs, their communalities can be precisely formulated as a syndrome of global change without ignoring particularities. If such a description does not yield robust management strategies, more particularities have to be considered and the models have to be refined. In contrast, management strategies which are already robust in the general setting can be transferred between different instances of a problem.

Model Ensembles for Sustainability Science

I conclude with a reflection on how adequate the framework of model ensembles is for sustainability science and about the potential of the new techniques for future use. The set of applications in this thesis demonstrates several advantages and limits. In all case studies, removing marginal edges (section 3.2) was one of the basic techniques to restrict the state-transition graph, thereby revealing more structure and improving the graphical representation. Also the no-return abstraction was used for every model. It reveals important structural features even of large state-transition graphs and can substantially improve the graphical representation. This was particularly important for the control design exercises, where the number of non-trivial no-return sets is a highly valuable indicator for the predictability of a given management strategy. On the other hand, this method also made some limits of monotonic landmark ensembles more visible. Many qualitative models are so weakly constrained that the state-transition graph consists only of one strongly connected component and several final states. This stresses the urgency to restrict the model ensemble – if possible – by making further model assumptions beyond pure QDEs.

Ordinal assumptions are one such possibility which is appealing since it does not require pure quantitative information about the system. This restriction method proved to be valuable for some applications, but the ORDAS algorithm did not produce stronger results for every case. Not all ordinal assumptions a modeller can make yield a graph with more no-return sets. A systematic assessment of all combinatorially possible ordinal assumptions and sign matrices of dimension three reveals that in some cases the ORDAS algorithm does not change the state-transition graph. The stronger restriction of partial derivatives to intervals can be combined with ordinal assumptions. More importantly, this method yields information about the tendency of transitions to a given successor state – a valuable result, since qualitative states have multiple successors in most of the relevant applications.

Considering not a single model but a set of models defined by common structural features takes account of uncertainty and generality at the same time. Using monotonic (landmark) ensembles has a further advantage if some system variables are difficult or impossible to measure quantitatively. Currently, the methods discussed and developed in the thesis reach their limits if the issue of complexity has to be modelled, since they are restricted to a small or intermediate number of variables. It must be noted that simple qualitative management models based only on knowledge expressed by a causal loop diagram are, in many cases, too general to make crisp predictions about systems behaviour. Hence, if there is really not more information available, management of such systems seems hardly possible except for good-natured cases. Using landmarks and restriction by ordinal assumptions or quantitative intervals can substantially improve results. However, for very large models ambiguities due to uncertainty or generality make it difficult to detect robust properties of model ensembles. Therefore, the strength of monotonic (landmark) ensembles lies more in the context of conceptual modelling where dynamical systems are reduced to the most relevant components. The impact of such models should not be underestimated, since they can be used in social learning processes where decision-makers or lay people draw basic conclusions from a model which they can transfer to the much more complex environment they live in. Finally, we have seen that viability theory can be flexibly combined with model ensembles, which allows for complementing conceptual models with normative knowledge – another basic requirement to contribute to decision-making for sustainable development.

For the future, more implications of ordinal assumptions could be investigated. These assumptions are rather strong in comparison to a monotonic (landmark) ensemble. Since in some models only a small number of paths can be excluded by the ORDAS algorithm, this suggests that there are further ways to reduce the state-transition graphs by exploiting ordinal assumptions. One interesting possibility for future consideration could be to expand the representation of qualitative states by including not only signs of derivatives with respect to time, but a (partial) ordering of the velocities. Together with ordinal assumptions on the coefficients of the Jacobian this provides rules about whether the order of velocities can change in time. If this or other new approaches yield a state-transition graph with less ambiguity and more strongly connected components, the usability of the no-return abstraction will increase. It may then be valuable to exploit further concepts from graph theory, which currently does not seem fruitful. A third direction of future research is to explore model ensembles such that QDEs can better cross-fertilise with well established simulation techniques. Within the context of sustainability science, model ensembles offer a promising conceptual framework to discuss uncertainties at the same time with issues of classification of “syndromatic” and “paradigmatic” archetypes. By reasoning about restrictions and subsets of model ensembles, and by investigating how policy options and institutional arrangements change the solution operator, this is central to the design of sustainable futures. I hope this work encourages the appropriate use of qualitative reasoning methods for this task and shows important paths for further development.

Model Code

Code for the Capital Fisher Model

See section 4.2 (p. 95) for details. The variables are transformed to replace $D_k v$ by `mvk` $:= -D_k v$ to meet technical limitations of the script language. Some cases where one variables obtains it upper or lower bound are excluded by the `unreachable-values` directive.

```
(quantity-spaces
 (x   (0 xmsy Q xmax)  "Stock")
 (k   (0 kmax)         "Capital")
 (I   (0 Imax)         "Investment")
 (R   (0 Rmsy Rmax)    "Recruitment")
 (h   (0 MSY hmax)     "Harvest")
 (mvk (0 inf)          "-D_k v")
 (dx  (minf 0 inf)     "dx")
 (dk  (minf 0 inf)     "dk")
 (dI  (minf 0 inf)     "dI"))

(constraints

 ;; derivatives
 ((d//dt x dx))
 ((d//dt k dk))
 ((d//dt I dI))

 ((add dk k I) (0 0 0))
 ((add dx h R) (0 0 0)  (0 MSY Rmsy))
 ((add dI mvk I) (0 0 0))

 ;; auxiliary variables
 ((U- x R) (xmsy Rmsy)  (0 0) (Q 0))
 (((M + - -) h x k mvk))

 ;; algebraic constraints
 (((M + +) k x h))

 ;; sigma (only second, unambiguous row)
  (((M - + ) k I dk)) ; implicit in (add dk k I)
```

```
 ;; exclusion of marginal edges
 ((cornot x h) (xmsy MSY))
 ((cornot x dx) (xmsy 0))
 ((cornot x dk) (xmsy 0))
 ((cornot x dI) (xmsy 0))
 ((cornot h dx) (MSY 0))
 ((cornot h dk) (MSY 0))
 ((cornot h dI) (MSY 0)))

;;; technical directives

(unreachable-values
  (x xmax) (k kmax) (I 0 Imax) (R Rmax) (h hmax) (mvk 0))
```

Code of the Negotiation Model with Capital

The basic model with unconstrained recommendations r and restricted quantitiy space (see section 4.3.2, p. 109).

```
(quantity-spaces
 (x (xmin xv xmsy)  "Stock")
 (k (0 kmax)        "Capital")
 (h (hmin MSY hmax) "Harvest")
 (r (0 hmin rmax)   "Catch Recommendation")
 (dx (minf 0 inf)   "dx")
 (dk (minf 0 inf)   "dk")
 (dh (minf 0 inf)   "dh"))

(constraints
 ((d//dt x dx))
 ((d//dt k dk))
 ((d//dt h dh))

; basic constraints
 (((M + -) x h dx)  (xv hmin 0) (xmsy MSY 0))
 (((M - +) k h dk))
 (((M + + +) x k r h))

; exclude k=0 and h>0
 (((NQ -0 +0) k h) (0 hmin))

; h > r
 (((NQ - +) h r) (hmin hmin))

; exclusion of marginal edges

 ((cornot x h) (xv hmin))
 ((cornot x h) (xv MSY))
 ((cornot x h) (xmin hmin))
 ((cornot x h) (xmin MSY))
 ((cornot x h) (xmsy hmin))
 ((cornot x h) (xmsy MSY))

 ((cornot x r) (xv hmin))
 ((cornot x r) (xmin hmin))
 ((cornot x r) (xmsy hmin))

 ((cornot x dx) (xv 0))
 ((cornot x dx) (xmin 0))
 ((cornot x dx) (xmsy 0))
 ((cornot x dk) (xv 0))
 ((cornot x dk) (xmin 0))
 ((cornot x dk) (xmsy 0))
 ((cornot x dh) (xv 0))
 ((cornot x dh) (xmin 0))
```

```
 ((cornot x dh) (xmsy 0))

 ((cornot h dx) (hmin 0))
 ((cornot h dx) (MSY 0))
 ((cornot h dk) (hmin 0))
 ((cornot h dk) (MSY 0))
 ((cornot h dh) (hmin 0))
 ((cornot h dh) (MSY 0))

 ((cornot r dx) (hmin 0))
 ((cornot r dk) (hmin 0))
 ((cornot r dh) (hmin 0))

 ((cornot dx dk) (0 0))
 ((cornot dx dh) (0 0))
 ((cornot dk dh) (0 0)))

; technical directives

(unreachable-values
  (k kmax) (h hmax) (r 0 rmax))

(weak-cycles)

(with-envisionment)

; initial state of simulation

(make-initial-state
 (x ((xv xmsy) dec))
 (k ((0 kmax) nil))
 (r ((hmin rmax) nil))
 (h ((hmin MSY) nil)))
```

Negotiation Model with Capital and a further restricted qualitative state space. The code contains different constraints for r:

```
(quantity-spaces
 (x (xv xmsy)       "Stock")
 (k (0 kmax)        "Capital")
 (r (0 hmin rmax)   "Catch Recommendation")
 (h (hmin MSY hmax) "Harvest")
 (dx (minf 0 inf)   "dx")
 (dk (minf 0 inf)   "dk")
 (dh (minf 0 inf)   "dh")
 (dr (minf 0 inf)   "dr"))

(constraints
 ((d//dt x dx))
 ((d//dt k dk))
 ((d//dt h dh))
 ((d//dt r dr))

; basic constraints
 (((M + -) x h dx)  (xv hmin 0) (xmsy MSY 0))
 (((M - +) k h dk))
 (((M + + +) x k r h))

; conservative control
; ((M- h dr) (hmin 0))
; qualitative control
  (((M - + -) x dx dh r) (xmsy 0 0 hmin))
  (((M - + -) x k h dh))

; exclude k=0 and h>0
 (((NQ -0 +0) k h) (0 hmin))

; r < h
 (((NQ - +) h r) (hmin hmin))

;; exclusion of marginal edges

 ((cornot x h) (xv hmin))
 ((cornot x h) (xv MSY))
 ((cornot x h) (xmsy hmin))
 ((cornot x h) (xmsy MSY))

 ((cornot x r) (xv hmin))
 ((cornot x r) (xmsy hmin))

 ((cornot x dx) (xv 0))
 ((cornot x dx) (xmsy 0))
 ((cornot x dk) (xv 0))
 ((cornot x dk) (xmsy 0))
 ((cornot x dh) (xv 0))
```

```
 ((cornot x dh) (xmsy 0))
 ((cornot x dr) (xv 0))
 ((cornot x dr) (xmsy 0))

 ((cornot h dx) (hmin 0))
 ((cornot h dx) (MSY 0))
 ((cornot h dk) (hmin 0))
 ((cornot h dk) (MSY 0))
 ((cornot h dh) (hmin 0))
 ((cornot h dh) (MSY 0))
 ((cornot h dr) (hmin 0))
 ((cornot h dr) (MSY 0))

 ((cornot r dx) (hmin 0))
 ((cornot r dk) (hmin 0))
 ((cornot r dh) (hmin 0))

 ((cornot dx dk) (0 0))
 ((cornot dx dh) (0 0))
 ((cornot dk dh) (0 0))
 ((cornot dx dr) (0 0))
 ((cornot dk dr) (0 0))
 ((cornot dh dr) (0 0)))

;;; technical directives

(unreachable-values
  (k kmax) (h hmax) (r 0 rmax))

(weak-cycles)

(with-envisionment)

;;; initial state of simulation

(make-initial-state
 (x ((xv xmsy) dec))
 (k ((0 kmax) nil))
 (r ((hmin rmax) nil))
 (h ((hmin MSY) nil)))
```

Model Code for Lake Management

The basic model with unconstrained phosphorus input L (see section 4.4, p. 116).

```
(quantity-spaces
  (P   (0 l1 l2 eu) "lake phosphorus")
  (M   (0 inf)      "sediment phosphorus")
  (L   (0 inf)      "intake")
  (dP  (minf 0 inf) "dP")
  (dM  (minf 0 inf) "dM")
  (dL  (minf 0 inf) "dL")
  (phi (0 1)        "regime")
  (r   (0 inf)      "recycling"))

(constraints
  ((d//dt P dP))
  ((d//dt M dM))
  ((d//dt L dL))

  (((M + -  ) P M   dM))
  (((M - + +) P r L dP))
  ((mult M phi r) (0 0 0))
  ((S+ P phi) (l1 0) (l2 1))

  ((cornot dP dM) (0 0))
  ((cornot dP dL) (0 0))
  ((cornot dL dM) (0 0)))

(unreachable-values
  (M 0) (L 0) (P 0))

(with-envisionment)

(make-initial-state
  (P ((0 l1) nil))
  (M ((0 inf) nil))
  (L ((0 inf) nil)))
```

Version with qualitative lake management. The quantity space is restricted to $P \geq \lambda_*$.

```
(quantity-spaces
  (P   (l1 l* l2 eu)   "lake phosphorus")
  (M   (0 inf)         "sediment phosphorus")
  (L   (0 inf)         "intake")
  (dP  (minf 0 inf)    "dP")
  (dM  (minf 0 inf)    "dM")
  (dL  (minf 0 inf)    "dL")
  (phi (0 phil* 1)     "regime")
  (r   (0 inf)         "recycling"))

(constraints
  ((d//dt P dP))
  ((d//dt M dM))
  ((d//dt L dL))

  (((M + -  ) P M   dM))
  (((M - + +) P r L dP))
  ((mult M phi r) (0 0 0))
  ((S+ P phi) (l1 0) (l2 1) (l* phil*))
  (((M - + +) P M L dP))

; control
  (((M + - -) P M L dL))

  ((cornot dP dM) (0 0))
  ((cornot dP dL) (0 0))
  ((cornot dL dM) (0 0)))

(unreachable-values
  (M 0) (L 0)  (P l1))

(with-envisionment)

(make-initial-state
  (P ((l* l2) nil))
  (M ((0 inf) nil))
  (L ((0 inf) nil)))
```

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Zusammenfassung

Die Doktorarbeit studiert unendliche Ensembles gewöhnlicher Differentialgleichungen mit gemeinsamen Monotonieeigenschaften, wie sie in der Nachhaltigkeitsforschung auftreten. Es werden neue Verfahren zur Behandlung von solchen Ensembles entwickelt und an verschiedenen Problemen des nachhaltigen Umgangs mit natürlichen Ressourcen erprobt.

Qualitative Differentialgleichungen (QDEs) und Differentialinklusionen werden in den neu formalisierten Rahmen der Modellensembles eingebettet. Unter einem Modellensemble versteht man eine Menge $\mathcal{M}$ von Funktionen f auf einem Zustandsraum $X \subseteq \mathbb{R}^n$, die Anfangswertprobleme $\dot{x} = f(x,t), x(0) = x_0$ definieren. Der mengenwertige Lösungsoperator weist einem Anfangszustand die Lösungen für alle $f \in \mathcal{M}$ zu, und liefert im allgemeinen sehr große Lösungsmengen. Für eine QDE schreibt man eine Matrix von Vorzeichen Σ vor und definiert das monotone Modellensemble $\mathcal{M}$ als die Menge aller Funktionen $f \in C^1(X, \mathbb{R}^n)$, so dass für alle $x \in X$ die Einträge der Jacobimatrix $\mathcal{J}(f)(x)$ dem Vorzeichen nach mit Σ übereinstimmen.

Die erste Anwendung betrifft die Armuts-Degradations-Spirale in Entwicklungsländern. Die zweite und dritte behandelt Fischereimanagement, insbesondere die industrialisierte Hochseefischerei und partizipatorische Managementansätze. Die vierte untersucht Wassermanagement zur Vermeidung von Eutrophierung. Die Anwendungen sind Bestandteil der Syndromforschung, die nach typischen Mustern sozial-ökologischer Veränderungen sucht. Sie stellen besondere Anforderungen an die Modellierung, insbesondere Unsicherheiten im Wissen und der Bedarf nach verallgemeinerbaren Resultaten.

Es wird gezeigt, dass Modellensembles hierfür geeignet sind. Überwiegend kommen QDEs zum Einsatz, die in der neu eingeführten graphentheoretischen Formulierung einen endlichen Zustandsgraphen als Lösung haben. Dies legt den Grundstein für die Entwicklung von vier neuen Verfahren zum Umgang mit großen QDEs. Hierbei kann die Viabilitätstheorie begrifflich wie methodisch für Abstraktions- und Restriktionsverfahren eingesetzt werden. (i) Die graphentheoretische Fassung viabler Mengen führt zur No-return Abstraktion, die einen engen Bezug zu starken Zusammenhangskomponenten aufweist. Damit lassen sich Zustandsgraphen aggregiert darstellen und bezüglich Nachhaltigkeitsfragen evaluieren. (ii) Die Restriktion der zulässigen Lösungen erlaubt es, Kanten von untergeordneter Bedeutung aus dem Zustandsgraphen zu eliminieren. (iii) Die Restriktion auf Systeme, bei denen die Einträge der Jacobimatrix eine vorgegebene partielle Ordnung aufweisen, ermöglicht die Elimination weiterer Pfade. (iv) Zuletzt werden Intervallschranken für die Einträge der Jacobimatrix berücksichtigt.

Die Anwendungen zeigen, dass mit diesen Methoden neue und robuste Eigenschaften auch sehr allgemeiner Modelle zum Management natürlicher Ressourcen gewonnen werden können. Zudem werden ihre Stärken für den Entwurf alternativen Politikoptionen deutlich.