Analysing Influence Diagrams by Linking Qualitative Dynamics and Viability Theory

– Preliminary Version –

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April 22, 2004

Abstract

Influence diagrams are a well-known tool for modelling in interdisciplinary research. If only limited numerical data about the system is available, it is desirable to make predictions for its dynamical behaviour at least from the diagram. One method for this task are qualitative differential equations (QDEs) which are concisely introduced in this paper. However, we typically observe ambiguities in the resulting predictions. To improve this situation we transform QDEs into a certain type of differential inclusion by imposing interval bounds on the influences. We show that some ambiguities can be resolved by computing viability kernels. This can be performed by the viability kernel algorithm due to some regularities of this type of differential inclusion. The procedure is demonstrated by an example from sustainability science.

1 Introduction

Influence diagrams are a well-known tool for modellers, especially for communication with scientists from different domains or with laypersons, and thus in the transdisciplinary field of sustainability science (Petschel-Held et al., 1999; Stave, 2002). In its simplest form, an influence diagram is a directed graph with marked edges. Each vertex represents a variable, and each edge an influence of the source variable on the target variable, which can be marked as positive or as negative (for examples, see Fig. 1). The meaning of "influence" remains vague at this stage, but needs to be made explicit for modelling purposes (see below). An influence diagram, being very general in nature, subsumes a broad range of systems which only share a common sign structure. In the context of sustainability science, such a diagram can be a "candidate" for a generalized pattern of global environmental change, characterising typical – often problematic – interactions between society and nature (Lüdeke et al., 2004). Although different in detail, such interactions may be observed at different places on earth, e.g. the Sahel Syndrome in the Sahel, northeast Brazil or some parts of India.

Once we have set up an influence diagram we want to use it for further reasoning, especially about the dynamical behaviour of systems described by the diagram. For this task we need an

[*]The author acknowledges a grant from the EU Research Training Network HPRN-CT-2002-00281 at CREA, Ecole Poltechnique, and support from Patrick Saint-Pierre and Jean-Pierre Aubin.

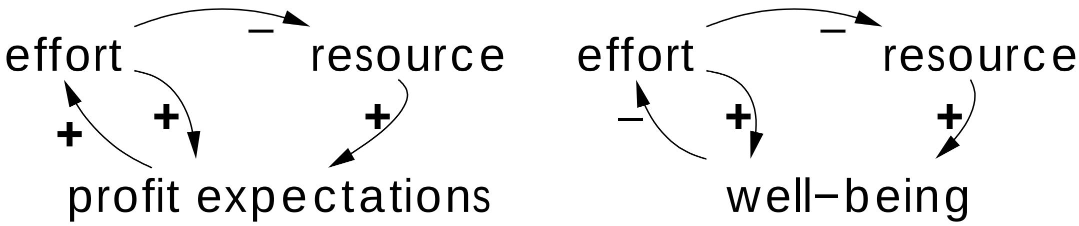

Figure 1: Simplified versions of the influence diagrams for (left) the Overexploitation Syndrome (Cassel-Gintz and Petschel-Held, 2000), and (right) the Sahel Syndrome (Lüdeke et al., 1999).

concise interpretation from the perspective of system dynamics. One is given by an ODE $\dot{x} = f(x), f : \mathbb{R}^n \to \mathbb{R}^n$, where each component of the state space corresponds to one vertex in the diagram (Richardson, 1986). If there is an edge from variable i to variable j marked with sign σ, this denotes that $\mathrm{sgn}\big(D_i f_j(x)\big) = \sigma$, where $D_i f_j(x)$ denotes $\partial f_j(x)/\partial x_i$. If there is no edge, this is interpreted as $D_i f_j(x) = 0$.

One approach to derive possible system behaviour from this interpretation is so-called qualitative physics (e.g. confluences (Kleer and Brown, 1984), qualitative process theory (Forbus, 1984), and qualitative differential equations (Kuipers, 1994)), which is presented in detail in this paper. It allows for the computation of a state-transition graph which covers all evolutions consistent with the influence diagram. In many cases relevant conclusions can be drawn from the graph. However, due to the generality of the diagram, qualitative physics has its limitations in that we usually obtain a large set of possible evolutions. In this case we are often looking for stronger results. In this paper we combine the qualitative physics approch with a set-valued one to integrate additional information. We do this by specifying an analogue to the qualitative physics interpretation of influence diagrams by means of differential inclusions and viability theory (Aubin, 1991), which can be computed using the viability kernel algorithm (Saint-Pierre, 1984; Cardaliaguet et al., 1999).

In the next section qualitative differential equations (QDEs) are introduced, and we give an example for their application to the Overexploitation Syndrome. In the third section the set-valued interpretation is introduced, and we investigate its relation to QDEs. In the last section both methods are combined. For this purpose we need to show some technical properties of the differential inclusions introduced in the third section. The linked method is demonstrated considering again the Overexploitation Syndrome.

2 The Qualitative Physics Approach

The core idea of qualitative differental equations (QDEs) is to scan the phase space of a set of ordinary differential equations which share common monotonicity properties. We assume that the dynamics of the system are governed by an ODE $\dot{x} = f(x), f : \mathbb{R}^n \to \mathbb{R}^n$, where only the signs of the partial derivatives $D_i f_j, i, j = 1, \ldots, n$ are known. Although the full theory of QDEs permits more sign information to be included, e.g. on the partial derivatives of algebraic relations, in this paper we concentrate on the differential part. By $\mathcal{A} := \{+, 0, -\}$ we denote the domain of signs. The sign operator $\mathrm{sgn}(\cdot)$ is extended componentwise to vectors and matrices.

DEFINITION 1: For a given $n \times n$ matrix of signs $\Sigma = (\sigma_{i,j})_{i,j=1,\ldots,n}$, $\sigma_{i,j} \in \mathcal{A}$, and a region $X \subseteq \mathbb{R}^n$ we define the **monotonic system**

$$M(\Sigma) := \{f \in C^1(X, \mathbb{R}^n) \mid \forall x \in X : \operatorname{sgn}\big(\mathcal{J}(f)(x)\big) = \Sigma\},$$

where $\mathcal{J}(f)$ denotes the jacobian of f. The **QDE** of $M(\Sigma)$ is the set of all ordinary differential equations of the form

$$\begin{aligned} \dot{x} &= f(x), \\ f &\in M(\Sigma), \\ x(0) &\in X. \end{aligned} \tag{1}$$

"QDE" is an abbreviation for "qualitative differential equation"; although a set of equations is not an equation we introduce this designation in analogue to (Kuipers, 1994). Since we only require the right hand sides to be continuously differentiable (and monotonic), we cannot guarantee that solutions to an ODE defined by a function $f \in M(\Sigma)$ exist on arbitrary intervals.

Consider an evolution of sign vectors $\operatorname{sgn}\big(\dot{x}(\cdot)\big)$ for an arbitrary function $x(\cdot) \in C^1(\mathbb{R}_+, \mathbb{R}^n)$. Due to continuity, when for one $j \in \{1, \ldots, n\}$ and some $t_1 < t_2$ the relation $\operatorname{sgn}\big(\dot{x}_j(t_1)\big) = -\operatorname{sgn}\big(\dot{x}_j(t_2)\big)$ holds, there is a point $t \in [t_1, t_2]$ with $\dot{x}_j(t) = 0$. If there is a $\tau \in \mathbb{R}_+$ with $\dot{x}(\tau) \neq 0$, there exists a maximal open interval I containing τ, such that $\operatorname{sgn}\big(\dot{x}(\cdot)\big)$ remains constant over I.

DEFINITION 2: Given a function $x(\cdot) \in C^1(\mathbb{R}_+, \mathbb{R}^n)$, we have a sequence of sign jumps (t_i), such that

$$\begin{aligned} &\forall i \exists j \in \{1, \ldots, n\} : \\ &\qquad \dot{x}_j(t_i) = 0 \text{ and } \exists \epsilon > 0 : \\ &\qquad \Big(\forall t \in [t_i - \epsilon, t_i) :\ \dot{x}_j(t) \neq 0 \text{ or } \forall t \in (t_i, t_i + \epsilon] :\ \dot{x}_j(t) \neq 0\Big). \end{aligned}$$

We construct a sequence of sign vectors $\tilde{x} := \Big(\operatorname{sgn}\big(\dot{x}(\tau_i)\big)\Big)$, where we chose $\tau_i \in (t_i, t_{i+1})$. This sequence is called **abstraction** of $x(\cdot)$.

The abstraction may be a finite sequence if there is only a limited number of sign jumps $(t_i)_{i=1,\ldots,m}$. If in this case the solution $x(\cdot)$ of the ODE exists only on some interval $[0, T)$, the abstraction terminates with $\operatorname{sgn}\big(\dot{x}(\tau_m)\big)$, $\tau_m \in (t_m, T)$. Solutions of a QDE are defined in the following way:

DEFINITION 3: The **solution set $\mathcal{S}_\Sigma$ of the QDE of** $M(\Sigma)$ is given by the abstractions of all solutions of ODEs of the form (1).

All elements of $\mathcal{S}_\Sigma$ describe a path in a graph defined by all possible sign-vectors and their successor relation, which can be efficiently computed by the so-called QSIM algorithm (Kuipers, 1994):

DEFINITION 4: The directed **state-transition graph** G of a QDE of $M(\Sigma)$ with solution set $\mathcal{S}_\Sigma$ is given by the vertices $V(G) := \mathcal{A}^n$ and the edges

$$E(G) := \{(v, w) \mid\ \exists \tilde{x} \in \mathcal{S}_\Sigma, i \in N :\ \tilde{x}_i = v \text{ and } \tilde{x}_{i+1} = w\} \subseteq V(G) \times V(G).$$

The vertices are called **qualitative states**.

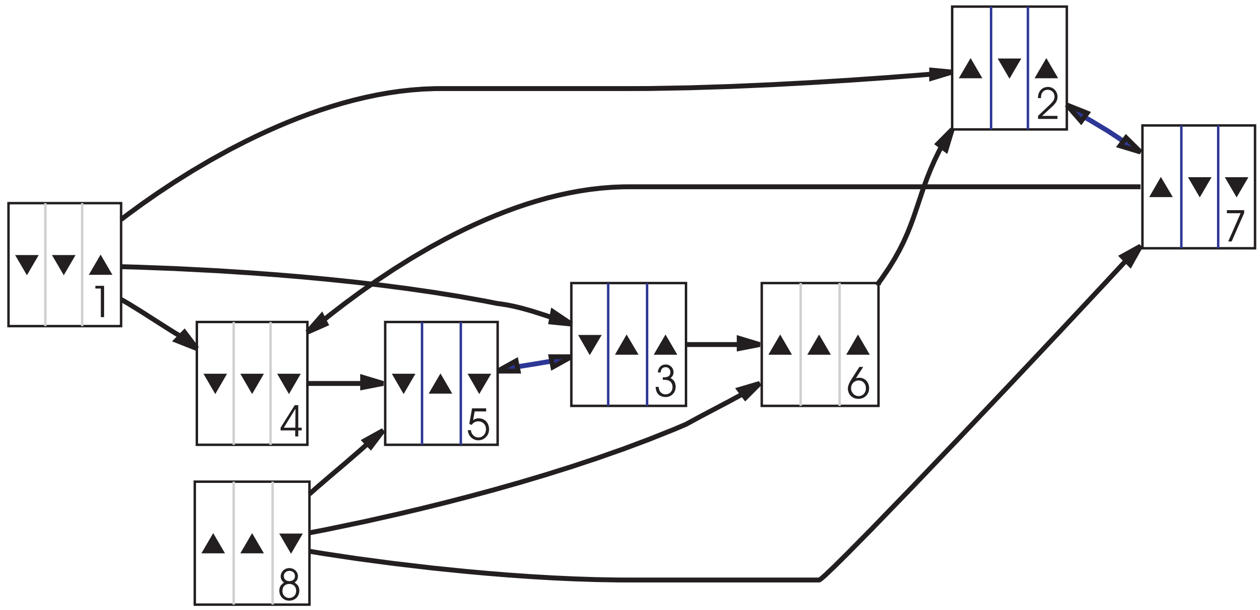

Figure 2: State-transition graph of the Overexploitation Syndrome QDE (computer-generated output). The numbered boxes represent qualitative states. Each column in a box corresponds to a variable (in the order E, R, P). The triangles indicate whether a variable is in- or decreasing. Edges denote possible changes in time. The numbers of the vertices are used for reference in the text. The clarity of the graph has been improved by omitting the vertex corresponding to equilibrium and solutions which remain constant over a time interval, as well as edges classified as so-called marginal cases, where more than one variable changes its sign at the same time.

We now illustrate the qualitative physics approach by the influence diagram of the Overexploitation Syndrome (Cassel-Gintz and Petschel-Held, 2000). This pattern of global change indicates profit-oriented overuse of renewable resources (e.g. forests or fisheries). The three basic variables are the amount of the resource R, the effort E of firms extracting the resource and the profit expectations P of these firms. Obviously, the resource stock decreases for high efforts. High profit expectations result in increasing efforts. The profit expectations are determined by the resource and the effort. It is, of course, debatable, whether there are more influences on this "subjective" variable. We assume that abundant resources improve the situation for resource users. Efforts increase expectations as they are interpreted as technical equipment (physical capital) bought for resource extraction. When higher capital stocks are associated with higher efficiency, this results in – ceteris paribus – accelerated realized profits (Eisenack et al., 2004). As long as we are reasoning from an influence diagram, we do not need to provide a means for measuring profit expectations. We only assume that there is some continuous variable representing this feature of a fishery.

With the state vector $(E, R, P)^t$, the monotonic system for

$$\Sigma = \begin{pmatrix} 0 & 0 & + \\ - & 0 & 0 \\ + & + & 0 \end{pmatrix} \tag{2}$$

defines a QDE containing all possible evolutions of the Overexploitation Syndrome. The resulting state-transition graph is given in Fig. 2.

It has two sources (states 1 and 8), and no sink. The other six states are connected by a cycle. Three of them correspond to a growing resource (5, 3 and 6), while for the rest there is the risk of overuse (2, 7 and 4). This can be interpreted as a "boom-and-bust" cycle: Suppose the system starts with decreasing efforts and profit expectations but an increasing resource stock (state 5), then the state can only change to increasing P (state 3). After that efforts begin to grow (state 6). Consequently, at some time the resource is at its maximum (when the system shifts from state 6 to 2), and again profit expectations change their trend (state 7) before efforts do (state 4).

There are some so-called ambiguities in the graph: Between state 3 and 5 or state 2 and 7 fluctuations in the cycle may occur, where P can change its trend several times before E changes. We do not know from the QDE whether this will really happen or how often it will happen. At the sources (decreasing E and R with increasing P, or the opposite), it cannot be decided to which part of the cycle the system will shift.

To motivate the work of the following sections we now discuss some strengths and weaknesses of the qualitative physics approach. Three main advantages of the method are the ability to handle generality, uncertainty and vagueness: If there is an influence of a given sign, nothing is said about the functional form of the influence – it can be weak, strong, linear, logarithmic etc. By definition, a QDE subsumes a broad range of ODE systems. (1) If a syndromatic pattern is observed in different parts of the world, the specific ODEs may differ, but show a common sign structure. Even the meaning of the variables may alter (R may represent forest for one case, and fish stock for another). (2) Limited knowledge on a single system is respected by the QDE, since we only have to be certain about the signs of the jacobian of the right-hand sides. (3) As every characteristic of the system is expressed in terms of signs, we do not need to measure quantities exactly, allowing for the introduction of variables which are difficult to operationalize (e.g. profit expectations, well-being, political power).

On the other hand, the core of the approach is determinstic. By solving a QDE, we assume a numerically unspecified, but nevertheless quantitative relationship which is invariant in time. Hence, We cannot take the variability of parameters into account. Another theoretical problem is the existence of so-called spurious behaviours (Kuipers, 1994): Not every path in the state-transition graph corresponds to a solution of an underlying ODE.

From the practical perspective, a main challenge are the very large state-transition graphs resulting form larger QDEs. There are usually many states with multiple successors which makes it difficult to draw conclusions for the management of an environmental system. One strategy is to identify relevant structures in the graph automatically, e.g. by applying clustering techniques (Bouwer and Bredeweg, 2002; Eisenack and Petschel-Held, 2002) or using temporal logic algorithms (Clancy, 1997). All these problems are a consequence of the very limited information given in a QDE model. Another strategy is therefore to include stronger model assumptions to eleminate edges or to find conditions for their existence. The latter is the motivation for the set-valued approach in the next section, where we will introduce bounds on the strengths of the influences in the model.

3 The Set-Valued Approach

In this section we introduce a differential inclusion of a certain form which can be used to derive possible evolutions from an influence diagram. The interpretation of the diagram slightly differs from the previous approach. In the follwing definition we regard singletons as intervals.

DEFINITION 5: Let U be a matrix of compact intervals $(u_{i,j})_{i,j=1,\dots,n}$, where each interval either vanishes or doesn't contain 0. A differential inclusion $\dot{x} \in F(x) := Ux$, where the latter denotes interval-valued multiplication, is called of **Q-type**.

The interval-valued multiplication is defined in the usual way by $Ux := \{Mx \mid M \in U\}$, where $M = (m_{i,j})_{i,j=1,\dots,n} \in U$ iff $\forall i, j = 1, \dots, n : m_{i,j} \in u_{i,j}$. The definition guarantees that every coefficient of U has a prescribed sign, which is related to a sign in the influence diagram. It should be noted that the definition does not require a linear relation between x and $\dot{x}$, since any absolutely continuous selection $U^o(\cdot)$ for which $\forall t \geq 0 : U^o(t) \in U$ yields a solution to the differential inclusion $\dot{x} \in F(x)$ via $\dot{x}(t) = U^o(t)x(t)$. It is just required that the possible selections are linearly bounded. Based on a matrix norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$, we define the norm $\|F\| := \max_{M\in U} \|M\|$.

PROPOSITION 1: A Q-type differential inclusion F has compact interval-valued images. It is marchaud, lipschitz, and enjoys the following homogeneity property: $\forall \lambda \in \mathbb{R} : F(\lambda x) = \lambda F(x)$.

PROOF: (i) F has interval-valued images: Each component $F_i, i = 1, \dots, n$ of $F(x)$ has the form $\sum_{j=1,\dots,n} u_{i,j}x_i$, where $u_{i,j}$ are compact intervals. The properties of interval arithmetic imply that this yields an interval. As it results from a contiuous operation on a compact set, it is also compact.
(ii) F is marchaud: $Dom(F) = \mathbb{R}^n$ is obviously closed. It has convex values, because they are interval-valued. It has linear growth, because with $c := \|F\|$, $\|F(x)\| = \|Ux\| \leq \|U\|\|x\| = c\|x\| \leq c(\|x\| + 1)$. Its graph is closed, because F is upper semicontiuous and has closed intervals as images.
(iii) F is lipschitz: Let $x_1, x_2 \in \mathbb{R}^n$. Since U is compact, choose $M \in U$ such that $\|M(x_2 - x_1)\| = \|F\|\|x_2 - x_1\|$, and

$$e := -\frac{M(x_2 - x_1)}{\|M(x_2 - x_1)\|} \in B,$$

with the closed unit ball B. Therefore,

$$M(x_2 - x_1) + c\|x_2 - x_1\|e = M(x_2 - x_1) - \frac{\|F\|\|(x_2 - x_1)\|}{\|M(x_2 - x_1)\|}M(x_2 - x_1) = 0,$$

i.e. $0 \in U(x_2 - x_1) + c\|x_2 - x_1\|B \Rightarrow Ux_1 \subseteq Ux_2 + L\|x_2 - x_1\|B$
$\Rightarrow F(x_1) \subseteq F(x_2) + c\|x_2 - x_1\|B$.
(iv) F is homogeneous: Chose an arbitrary $x \in \mathbb{R}^n, \lambda \in \mathbb{R}$. Due to the properties of interval arithmetic it holds for all $i = 1, \dots, n$ that

$$\begin{aligned} F_i(\lambda x) &= \sum_{j=1,\dots,n} u_{i,j}(\lambda x_i) = \sum_{j=1,\dots,n} \lambda u_{i,j}x_i \\ &= \lambda \sum_{j=1,\dots,n} u_{i,j}x_i = \lambda F_i(x). \end{aligned}$$

■

Consequently, F is a closed process, but it is not a convex process. The viability and invariance theorems can be applied.

We now want to shed some light on the differences and communalites of Q-type differential inclusions to QDEs. Suppose we want to analyse the solution of a QDE, i.e. the set of solutions of ODEs

$$\{y(\cdot) : \mathbb{R}_+ \to X \mid y_0 \in X,\ \exists f \in M(\Sigma) :\ \dot{y} = f(y), \},$$

where an influence diagram gives the sign matrix Σ, by means of a differential inclusion. It may be a first idea to "mimic" it by $\dot{y} \in H(y)$ with

$$H : y \rightsquigarrow \{f(y) \mid f \in M(\Sigma)\}.$$

However, this does not reveal any structure, since $\forall y \in X : H(y) = \mathbb{R}$. (Suppose that $f \in M(\Sigma)$ and $f(y) = z$. Take an arbitrary $z' \in \mathbb{R}$, and define $f' := f + (z' - z)$. Then $f'(y) = z'$ and $\mathcal{J}(f') = \mathcal{J}(f)$.) As it follows from $\dot{y} = f(y)$ that $\ddot{y} = \mathcal{J}(f)(y) \cdot \dot{y}$, we obtain a second order differential inclusion in the direct sum of state and velocity space:

$$\begin{aligned} \ddot{y} &\in H(\dot{y}, y), \\ H : (\dot{y}, y) &\rightsquigarrow \{\mathcal{J}(f)(y) \cdot \dot{y} \mid f \in M(\Sigma)\}. \end{aligned}$$

This can be simplified because it is assumed that $\forall y \in X : \operatorname{sgn}\big(\mathcal{J}(f)(y)\big) = \Sigma$, and even $\forall M \in \mathbb{R}^{n\times n}$ with $\operatorname{sgn}\big(M\big) = \Sigma$ there exists a $f \in M(\Sigma)$ such that $\mathcal{J}(f) = M$:

$$H(\dot{y}, y) = \{M\dot{y} \mid \operatorname{sgn}\big(M\big) = \Sigma\} =: \hat{H}(\dot{y}),$$

i.e. we now turn our attention to the differential inclusion $\ddot{y} \in \hat{H}(\dot{y})$, which is obviously equivalent to a differential inclusion of first order. If the velocities $\dot{y}$ are in a given quadrant, this corresponds to the fact that y is in a monotonic cell

$$K(a) := \{y \in X \mid \operatorname{sgn}\big(f(y)\big) = a\}$$

for $a \in \mathcal{A}^n$ (Aubin, 1996) (or, in the terminology of the previous section, to a qualitative state). As long as a solution evolves in a given monotonic cell, its direction of change remains qualitatively constant. The differential inclusion $\ddot{y} \in \hat{H}(\dot{y})$ "simulates" the QDE of $M(\Sigma)$ in the sense that

$$\forall f \in M(\Sigma)\ \forall y_0 \in X,\ y(\cdot) \text{ solution of } \dot{y} = f(y) \text{ with } y(0) = y_0 :\ \ddot{y} \in \hat{H}(\dot{y}),$$

i.e. it describes not the evolution itself, but its time derivative. Thus, the monotonic cells of the original system map to the quadrants of the velocity space. The new system contains the case of a time-invariant jacobian (as in the definition of monotonic system), but also admits changes in the jacobian, as far as its signs remain constant. However, for a given Σ and $\dot{y}$, the components $i = 1, \ldots, n$ of $\hat{H}(\dot{y})$ evaluate to

$$\hat{H}_i = \begin{cases} 0 & \text{if } \forall j = 1, \ldots, n : \dot{y}_j \cdot \Sigma_{i,j} = 0, \\ \mathbb{R}_+ & \text{else if } \forall j = 1, \ldots, n : \operatorname{sgn}(\dot{y}_j) = \sigma_{i,j} \text{ or } \dot{y}_j \cdot \sigma_{i,j} = 0, \\ \mathbb{R}_- & \text{else if } \forall j = 1, \ldots, n : -\operatorname{sgn}(\dot{y}_j) = \sigma_{i,j} \text{ or } \dot{y}_j \cdot \Sigma_{i,j} = 0, \\ \mathbb{R} & \text{else .} \end{cases}$$

The resulting differenial inclusion is not bounded (except for the trivial case), and some components may even be unconstrained. This is the reason for including interval bounds in Def. (5). We can summarize that Q-type differential inclusions are more general then QDEs in the sense that they also include non-deterministic trajectories, and are more specific in the sense that they are bounded.

4 The Linked Method

In this section we analyse a model of the Overexploitation Syndrome by combining the qualitative physics and the set-valued approach. The starting point is the influence diagram given in Fig. (1) and the resulting state-transition graph (Fig. 2). We include additional information by introducing a Q-type differential inclusion. The problem of multiple successors is generally formulated as a viability problem. More features of model behaviour are extracted for user-selected subgraphs by employing the viabiltiy kernel algorithm (Saint-Pierre, 1984; Cardaliaguet et al., 1999). For a successful application, some general properties of viability kernels of Q-type differential inclusions are derived.

By definition, there is an edge $e = (v, w)$ in the state-transition graph G, if there is a $f \in M(\Sigma)$ and a solution $x(\cdot)$ of the ODE $\dot{x} = f(x)$ which starts with $\operatorname{sgn}\big(f(y(0))\big) = v$ and directly evolves to some t with $\operatorname{sgn}\big(f(y(t))\big) = w$. Conversely, if there is no such solution, there is no corresponding edge in G. If an edge can not be eliminated by this criterion, it is of high value to know *for which velocities in a monotonic cell a neighbouring monotonic cell is reached*. If more than one suceeding monotonic cell is *possible*, it is of special interest to know under which circumstances a given neigbouring monotonic cells is *necessarily* reached. In the context of sustainability science such conditions yield early warning indicators of the form “once the rates of change are in such and such a relation, the following trend will necessarily reverse at a later time”. If the trend reversal is problematic, this results in a clear directive to avoid the critical relation or to change the underlying dynamics fundamentally.

For $a \in \mathcal{A}^n$ define $Q(a) := \overline{\{x \in \mathbb{R}^n \mid \operatorname{sgn}(x) = a\}}$, which are the closed quadrants of the velocity space or their common boundaries, respectively. Given $a, b \in \mathcal{A}^n$, we want to know under which conditions evolutions starting in $Q(a)$ necessarily enter $Q(b)$ directly in finite time. This problem can be posed directly or inverse: (i) Given a Q-type differential inclusion F, find $Abs_F(Q(a), Q(b))$. (ii) Given a matrix of signs, find all consistent Q-type differential inclusions which result in $Abs_F(Q(a), Q(b)) = \varnothing$ or in $Abs_F(Q(a), Q(b)) = Q(a)$. Since $Abs_F(K, C)^c = Viab_F(C^c, K^c)$, the problem can be reduced to the computation of viability kernels.

For a three dimensional system, absorption basins can be easily computed by the viability kernel algorithm, which is designed for a bounded constrained set or target. Since we have to deal with (unbounded) quadrants, or more generally, cones, some remarks are necessary to correctly interpret the results from the algorithm. We start with an observation resulting from the homogeneity of Q-type differential inclusions. By $\mathcal{S}_F(x_0)$ we denote the solution map of F.

PROPOSITION 2: Let F be of Q-type, $x(\cdot) \in \mathcal{S}_F(x_0)$, and $\lambda \in \mathbb{R}$. Then $y(\cdot) := \lambda x(\cdot) \in \mathcal{S}_F(y(0))$.

PROOF: For almost every $t \in \mathbb{R}_+$ the following holds: $\dot{y}(t) = \lambda \dot{x}(t) \in \lambda F(x(t))$. Due to Prop. (1), the last term equals $F(\lambda x(t)) = F(y(t))$. ■

As a consequence, the viability kernel is a cone if the constrained set and the target are unbounded:

PROPOSITION 3: Let $C \subset K \subset \mathbb{R}^n$ be cones, and $\dot{x} \in F(x)$ a differential inclusion of Q-type. Then $D = Viab_F(K, C)$ is a closed cone.

PROOF: The viabiliy kernel

$$D = \{x \in K \mid \exists x(\cdot) \in \mathcal{S}_F(x) :$$
$$\forall t \geq 0 : x(t) \in K \quad (3)$$
$$\text{or } \exists T \geq 0 \; \forall t \in [0,T) : x(t) \in K \text{ and } x(T) \in C\}. \quad (4)$$

We show that $\forall \lambda > 0, x \in D : \lambda x \in D$. Choose $x(\cdot) \in \mathcal{S}_F(x)$ viable in K outside C, and let $y(\cdot) = \lambda x(\cdot)$, which is a solution due to Prop. (2).
In case (3), $\forall t \geq 0 : y(t) \in K$, because K is a cone.
In the case (4), $\forall t \in [0,T) : y(t) \in K$ and $y(T) \in C$, because also C is a cone.
Consequently, $y(0) = \lambda x(0) = \lambda x \in D$. ■

For the absorption basin the analogue result is valid:

PROPOSITION 4: Let $C \subset K \subset \mathbb{R}^n$ be cones, and $\dot{x} \in F(x)$ a differential inclusion of Q-type. Then $D = Abs_F(K, C)$ is a cone.

PROOF: Since the complement of C and K are cones, by Prop. (3) $Viab_F(C^c, K^c)$ is a cone. Hence, also $D = Viab_F(C^c, K^c)^c$ is a cone. ■

These properties allow for the reduction of the problem by one dimension: If the intersection of the absorption basin with an appropriate hyperplane is known, the cone generated by this intersection is the absorption basin. This may help to compute absorption basins for higher dimension systems or to simplify the graphical representation. However, by applying the viability kernel algorithm, we cannot compute $D = Viab(K, C)$ directly, but for example some $D_\lambda := Viab(K \cap \lambda E, C \cap \lambda E)$, where $\lambda > 0$ and E is the unit cube. The viability kernel D can be recovered from the computed D_λ by the following property:

PROPOSITION 5: Let K and C be cones, E the unit cube, $\lambda > 0$, and $D_1 = Viab(K \cap E, C \cap E)$. Then $D_\lambda = \lambda D_1$.

PROOF: (i) For $x \in D_\lambda$, we show that $y := \frac{1}{\lambda}x \in D_1$: Chose $x(\cdot) \in \mathcal{S}_F(x)$ and set $y(\cdot) := \frac{1}{\lambda}x(\cdot)$, which is an element of $\mathcal{S}_F(\frac{1}{\lambda}x)$ due to Prop. (2). If $\forall t > 0 : x(t) \in K \cap \lambda E = \lambda K \cap \lambda E = \lambda(K \cap E)$, then $y(t) \in K \cap E$. If $\exists T > 0 : x(T) \in C \cap \lambda E = \lambda(C \cap E)$, then $y(T) \in C \cap E$, and thus $y \in D_1$.
(ii) We show that each $x \in \lambda D_1$, is viable in $K \cap \lambda E$ outside $C \cap \lambda E$: Chose $x(\cdot) \in \mathcal{S}_F(x)$ and define $y(\cdot) := \frac{1}{\lambda}x(\cdot)$. This function is an element of $\mathcal{S}_F(\frac{1}{\lambda}y(0))$ due to Prop. (2), and $y(0) \in D_1$. If $\forall t > 0 : y(t) \in K \cap E$, then $x(t) \in \lambda(K \cap E) = K \cap \lambda E$. If $\exists T > 0 : y(T) \in C \cap E$, then $x(T) \in \lambda(C \cap E) = C \cap \lambda E$, and consequently $x \in D_\lambda$. ■

By this property we obtain the viability kernel (and analoguely the absorption basin) from some D_μ with $\mu > 0$ by $\bigcup_{\lambda > 0} \lambda D_\mu$.

For the case of the Overexploitation Syndrome a possible Q-type differential inclusion is given by the interval matrix

$$U := \begin{pmatrix} 0 & 0 & [0.7, 0.9] \\ [-0.7, -0.4] & 0 & 0 \\ [0.5, 3.0] & [0.5, 3.0] & 0 \end{pmatrix}$$

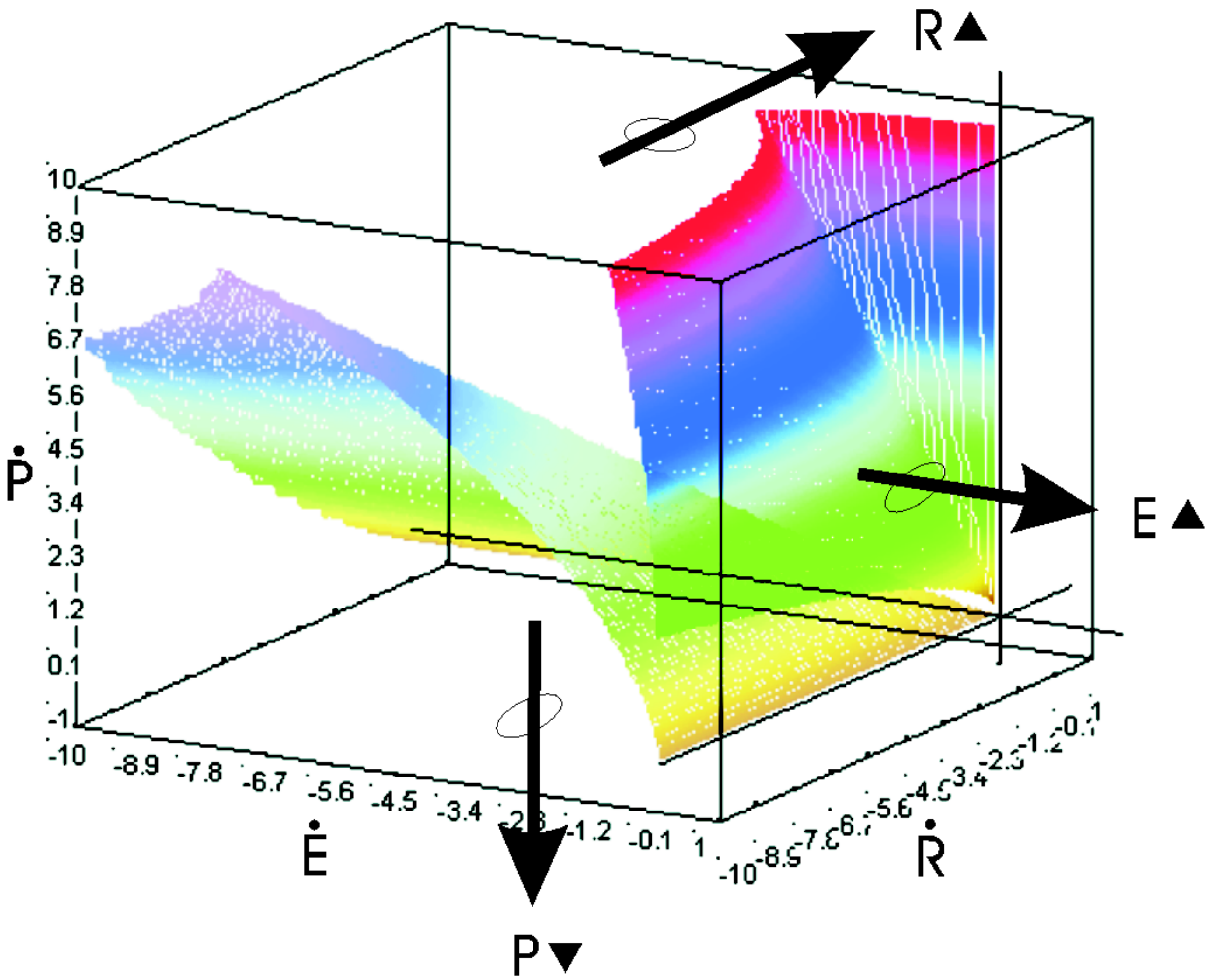

Figure 3: Borders of the absorption basins of $Q(-,-,+)$, corresponding to state 1 in Fig. (2). Large arrows indicate the directions towards the neighbouring monotonic cells. Transitions to state 2 (E▲) necessarily occur from $Abs\big(Q(-,-,+), Q(+,-,+)\big)$, the region between the surface to the right and the plane given by $\dot{E} = 0$. A shift to state 4 (P▼) happens from $Abs\big(Q(-,-,+), Q(-,-,-)\big)$ between the lower surface and the plane given by $\dot{P} = 0$. $Abs\big(Q(-,-,+), Q(-,+,+)\big)$, which would lead to state 3 (R▲), is empty.

via

$$\begin{pmatrix} \ddot{E} \\ \ddot{R} \\ \ddot{P} \end{pmatrix} = U \begin{pmatrix} \dot{E} \\ \dot{R} \\ \dot{P} \end{pmatrix},$$

where $\dot{E}$ denotes the change of efforts, $\dot{R}$ the change of the resource stock, and $\dot{P}$ the change of profit expectations. The signs of the intervals constituting U equal the signs in the influence diagram (Eq. 2). The interval bounds are not empirically justified, but illustrate that their width can be considerabely different – depending on the degree of uncertainty or the generality of the model (for other models they can even differ up to orders of magnitude). Here, the largest uncertainties appear with repect to the (subjective) formation of profit expectations.

We now analyse two exemplary monotonic cells in the state-transition graph (states 1 and 3 in Fig. 2) where multiple successors occur. In the critical situation with decreasing resource stocks

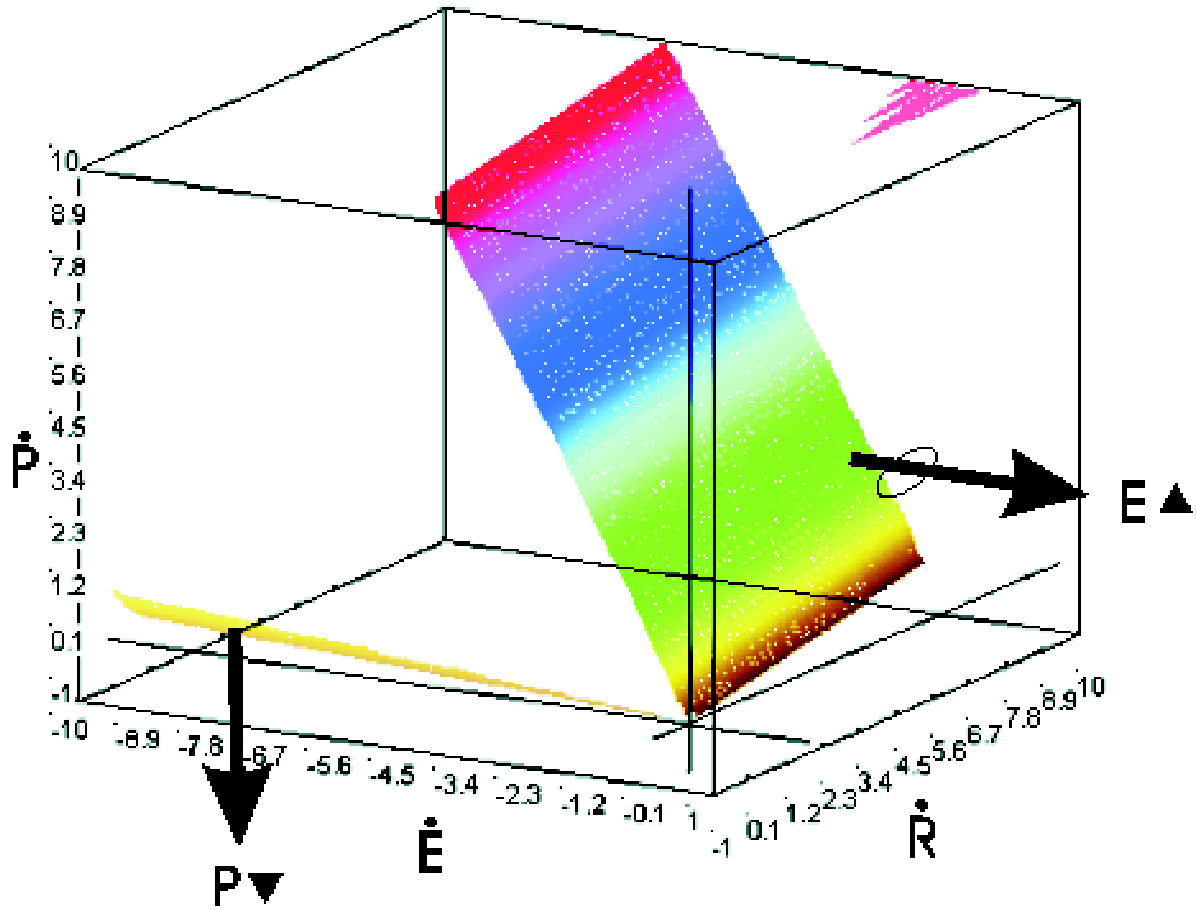

Figure 4: Borders of the absorption basins of $Q(-,+,+)$, corresponding to state 3 in Fig. (2). Large arrows indicate adjacent monotonic cells. $Abs\big(Q(-,+,+),Q(-,+,-)\big)$, corresponding to decreasing profit expectations, is a very small cone, whereas the border of $Abs\big(Q(-,+,+),Q(+,+,+)\big)$ appears to be a plane seperating a large part of the quadrant. The small part of the border to the upper right is an artifact resulting from restricting the quadrant to $10E$, which can be eliminated due to Prop. (5).

and efforts but still increasing profit expectations (state 1), it is of high interest to know whether the system will shift to state 2 (increasing efforts), 3 (increasing resource) or 5 (decreasing profit expectations). The computed absorption basins of state 1 with the respective targets are shown in Fig. 3. One absorption basin is empty. The borders of the other basins are smooth except along one ray from the origin. There are no combinations of velocities that guarantee that the resource stock recovers ($Abs\big(Q(-,-,+),Q(-,+,+)\big)$), but there is a considerable risk that efforts begin to increase after finite time ($Abs\big(Q(-,-,+),Q(+,-,+)\big)$). However, a large part of this quadrant necessarily leads to decreasing profit expectations ($Abs\big(Q(-,-,+),Q(-,-,-)\big)$). It is not a surprise that the region which leads to increasing efforts is smaller if the resource stock is reduced very fast compared to rising profit expectations. But also if $\dot{R}$ comes close to zero, this part of the phase space becomes smaller. Here, it is possible that the resource recovers faster than the efforts. Similarily, the likelihood to change the trend of profit expectations is the largest for an intermediate relation of decrease in effort and resource stock.

Suppose that the resource stock obtains its minimum and starts to decrease again (state 3). Then, two succeeding states are possible: profit expectations decrease (state 5) or efforts increase (state 6). Since there is no edge back to state 1, the corresponding $Abs\big(Q(-,+,+),Q(-,-,+)\big)$ (and even the respective capture basin) has to be empty. The size of the regions necessarily leading to one of the both outcomes is considerably different (Fig. 4). Moreover, as the absorption basins of state 3 intersect the surface between $Q(-,-,+)$ (state 1) and $Q(-,+,+)$ (state 3), in some cases the next change can already be predicted at the time where the resource stock is at its minimum. In the other case that profit expectations start decreasing (which may be more unlikely as the absorption basin is significantly smaller), the state-transition graph implies that the only possible subsequent trend change is that $\dot{P}$ becomes positive again, shifting the system back to state 3.

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