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*A semi-analytical solution for the positive degree-day model with stochastic  
temperature variations*

The degree-day model is a parameterisation for the melt rate of snow and ice at the surface of an ice sheet or glacier. It is a simple, empirical relation which states that the melt rate is proportional to the surface-air-temperature excess above 0°C (e.g. Braithwaite and Olesen 1989, Hock 2003). The physical basis of this and related temperature-based melt-index methods was examined by Braithwaite (1995).

It was suggested by Braithwaite (1984) to calculate the number of positive degree-days, *PDD*, from the normal probability distribution around the monthly-mean temperatures in the course of the years. A form, which is based on that and is widely used in current ice-dynamic models, was proposed by Reeh (1991). It reads

$$PDD = \frac{1}{\sigma\sqrt{2\pi}} \int_0^A dt \int_0^\infty dT T \exp\left(-\frac{(T - T_{ac}(t))^2}{2\sigma^2}\right), \quad (1)$$

where  $t$  is the time,  $T$  the actual surface air temperature,  $T_{ac}$  the annual temperature cycle (both in °C) and  $\sigma$  the standard deviation of the temperature from the annual cycle, which accounts for the daily cycle and further, stochastic temperature variations.  $T_{ac}$  is often assumed to vary sinusoidally over time,

$$T_{ac}(t) = T_{ma} + (T_{mj} - T_{ma}) \cos \frac{2\pi t}{A}, \quad (2)$$

where  $T_{mj}$  is the mean July (January) surface air temperature on the extra-tropical northern (southern) hemisphere,  $T_{ma}$  is the mean annual surface air temperature, and  $A = 1$  year. Nonetheless, any other representation of  $T_{ac}$  can be used as a basis for the calculation of the *PDD* integral (1) as well.

In a number of ice-sheet models (e.g. Huybrechts et al. 1991, van de Wal and Oerlemans 1997, Tarasov and Peltier 1999, Marshall et al. 2000, Charbit et al. 2002) the double integral in Eq. (1) is computed numerically. Since this must be done for any grid point separately, it requires a considerable amount of CPU time. Further, the inevitable cut-off of the upper limit  $\infty$  of the temperature integral at some finite value  $T_{\max}$  influences the accuracy of the results. Therefore, Janssens and Huybrechts (2000) proposed an approximative method for solving the temperature integral by fitting an exponential function.

Here, we demonstrate that the temperature integral can be evaluated fully analytically. To our best knowledge, this analytical solution has not been applied to the *PDD* temperature integral before. A similar procedure was carried out by Roe and Lindzen (2001) in the context of determining the precipitation rate over ice sheets. However, they applied an unnecessary absolute value in their equation, they did not show the integration substitutions which lead to the resulting expression, and they missed to evaluate the integral boundaries of the exponential function. Here, we present the complete derivation and the plain resulting equations for the temperature integral of the statistic positive-degree-day model.

We start by adding the expression

$$\int_0^{\infty} dT T_{\text{ac}} \exp\left(-\frac{(T - T_{\text{ac}})^2}{2\sigma^2}\right) - \int_0^{\infty} dT T_{\text{ac}} \exp\left(-\frac{(T - T_{\text{ac}})^2}{2\sigma^2}\right) = 0$$

to the temperature integral in Eq. (1). This yields

$$PDD = \frac{1}{\sigma\sqrt{2\pi}} \int_0^A dt \left\{ \int_0^{\infty} dT (T - T_{\text{ac}}) \exp\left(-\frac{(T - T_{\text{ac}})^2}{2\sigma^2}\right) + \int_0^{\infty} dT T_{\text{ac}} \exp\left(-\frac{(T - T_{\text{ac}})^2}{2\sigma^2}\right) \right\}. \quad (3)$$

The substitutions  $u = (T - T_{\text{ac}})^2/2\sigma^2$ ,  $du = (T - T_{\text{ac}}) dT/\sigma^2$  and  $v = (T - T_{\text{ac}})/\sqrt{2}\sigma$ ,

$dv = dT/\sqrt{2}\sigma$  provide

$$\begin{aligned}
PDD &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^A dt \left\{ \int_{T_{ac}^2/2\sigma^2}^{\infty} du \sigma^2 \exp(-u) + \int_{-T_{ac}/\sqrt{2}\sigma}^{\infty} dv \sqrt{2}\sigma T_{ac} \exp(-v^2) \right\} \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_0^A dt \left\{ [-\sigma^2 \exp(-u)]_{T_{ac}^2/2\sigma^2}^{\infty} + \sqrt{2}\sigma T_{ac} \int_{-T_{ac}/\sqrt{2}\sigma}^{\infty} dv \exp(-v^2) \right\} \\
&= \int_0^A dt \left\{ \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{T_{ac}^2}{2\sigma^2}\right) + \frac{T_{ac}}{\sqrt{\pi}} \int_{-T_{ac}/\sqrt{2}\sigma}^{\infty} dv \exp(-v^2) \right\}. \tag{4}
\end{aligned}$$

With the definition of the complementary error function

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-\tilde{x}^2) d\tilde{x} \tag{5}$$

(e.g. Press et al. 1996, Sect. 6.2;  $\operatorname{erf}(x)$  denotes the conventional error function) this can be rewritten as

$$PDD = \int_0^A dt \left\{ \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{T_{ac}^2}{2\sigma^2}\right) + \frac{T_{ac}}{2} \operatorname{erfc}\left(-\frac{T_{ac}}{\sqrt{2}\sigma}\right) \right\}. \tag{6}$$

This representation has the great advantage that the temperature integral of the original equation (1) has vanished. Therefore, it can be numerically evaluated much faster and yields more precise results, because the upper limit  $\infty$  need not be cut off at some finite value. The error function is not a standard built-in function of Fortran 90 or C. However, it is implemented in a number of modern Fortran or C compilers as well as in the packages MATLAB and Mathematica. Alternatively, it can easily be computed by the routines given by Press et al. (1996) or similar sources. Here, we employ the subroutine *erfcc* by Press et al. (1996, Sect. 6.2) with a fractional error everywhere less than  $1.2 \times 10^{-7}$ .

We investigate briefly the gain in accuracy and CPU time of solving the new semi-analytical representation for *PDD*, Eq. (6), compared to the fully numerical [Eq. (1)] and

other representations. Let us consider a typical situation for the ablation zone in south Greenland with  $T_{\text{ma}} = -10^\circ\text{C}$ ,  $T_{\text{mj}} = 5^\circ\text{C}$  and  $\sigma = 5^\circ\text{C}$ . The time integrals in Eqs. (1) and (6) are solved with a monthly time-step. In addition, the temperature integral in Eq. (1) is computed by the trapezoidal rule with a temperature step of  $\Delta T = 0.5^\circ\text{C}$  and a cut-off temperature  $T_{\text{max}}$  varied between  $0.5\sigma$  and  $4\sigma$ .

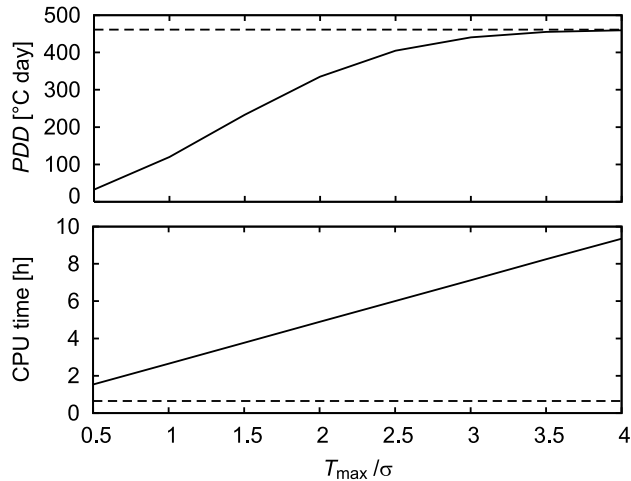


Figure 1:  $PDD$  values and CPU times ( $10^9$  computations) for the numerical solution of (i) Eq. (1) as a function of  $T_{\text{max}}/\sigma$  (solid lines), and (ii) Eq. (6) (dashed lines). For details see main text.

The results are shown in Fig. 1. CPU times refer to computations carried out with a Fortran 90 program, compiled with the Intel Fortran Compiler Version 8 and run on a 3.4-GHz Pentium-4 PC under SuSE LINUX 9.0. They are given for  $10^9$  computations, which is the approximate number of required computations for a simulation of the Greenland ice sheet with 20 km resolution over 250 000 years (two glacial-interglacial cycles) with a time-step of 2.5 years. It is evident that our new method of computing the  $PDD$  integral is by far more efficient than the fully numerical method. In order to be sufficiently accurate, the latter requires at least  $T_{\text{max}} = 3\sigma$ , and for this cut-off temperature it consumes more than ten times more CPU time than the solution of the semi-analytical expression (6).

Table 1 gives some more detailed information about the accuracies and CPU times of the different methods. In addition to our representation of the complementary error

$T_{\max}/\sigma$	$\frac{\Delta PDD}{PDD}$	CPU time
1	-74.01%	4.12
2	-27.28%	7.61
3	-4.438%	11.07
4	-0.335%	14.54
<i>erfcc</i>	(Reference)	
<i>erfcr</i>	-0.002%	0.80
<i>approx</i>	+1.982%	1.14

Table 1: Accuracy (with respect to *erfcc*) and relative CPU time (*erfcc* = 1) for different methods of solving the *PDD* integral. Details are given in the main text. See also Fig. 1.

function by the routine *erfcc* (Press et al. 1996), we have considered the rational approximation by Abramowitz and Stegun (1970, Sect. 7.1.25), here denoted as *erfcr*, which has an absolute error less or equal  $2.5 \times 10^{-5}$ . This approximation was also used by Roe and Lindzen (2001), but they missed to give the values for the parameters. Further, we have implemented the approximative solution of the temperature integral in (1) by Janssens and Huybrechts (2000) (here called *approx*). There is only a tiny difference in accuracy between the *PDD* values resulting from the *erfcc* and the *erfcr* representations of the complementary error function. Obviously, the error due to the numerical integration over the year outweighs the differences in accuracy of the used representations of the complementary error function. The accuracy of the approximation by Janssens and Huybrechts (2000) is clearly lower than that of both other methods, but certainly tolerable in view of the relatively crude assumptions of the positive-degree-day method itself, e.g., the approximation of the synoptic processes by the normal probability distribution (Braithwaite 1984). The CPU time of the method with *erfcc* is slightly longer than that with *erfcr*, because of the higher order representation of the complementary error function in the former. Interestingly, the approximation by Janssens and Huybrechts (2000) is, at least in our implementation, about 14% slower despite its lower accuracy. This is presumably because three different functions (*abs*, *exp*, *max*) must be evaluated in this approximation.

Since the complementary error function is available in nearly every compiler, or,

alternatively, can be easily implemented by several tested subroutines with low computational time and high accuracy, we greatly encourage ice-sheet and glacier modellers to make use of this improved method.

*Epilogue:* Braithwaite's (1984) introduction of the normal distribution in order to reduce the degree-day calculations to monthly rather than daily data reduced data requirements by a factor of 30. The methods presented in this work save more than 90% CPU time. Who is going to make the next saving?

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