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Convergent calculation of the asymptotic dimension of diffusion limited aggregates: Scaling and renormalization of small clusters

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Diffusion limited aggregation (DLA) is a model of fractal growth that had attained a paradigmatic status due to its simplicity and its underlying role for a variety of pattern forming processes. We present a convergent calculation of the fractal dimension D of DLA based on a renormalization scheme for the first Laurent coefficient of the conformal map from the unit circle to the expanding boundary of the fractal cluster. The theory is applicable from very small (2–3 particles) to asymptotically large ($n\rightarrow\infty$) clusters. The computed dimension is $D=1.713\pm0.003$.

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The model of diffusion limited aggregation was presented in 1981 by Witten and Sander [1] as a computer algorithm. The process begins with fixing one particle at the center of coordinates in d-dimensions, and follows the creation of a cluster by releasing random walkers from infinity, allowing them to walk around until they hit any particle belonging to the cluster. The growing cluster appears to be a random fractal. The model has attracted enormous attention as an example for the spontaneous creation of fractal objects in nature, and also as a paradigm for a family of related ''harmonic'' problems that have to do with the solution of the Laplace equation $\nabla^2 p=0$ with appropriate boundary conditions on moving, ramified boundaries. Among such problems are dielectric breakdown [2], flows in porous media (with D'arcy's law $\boldsymbol{v}=-\boldsymbol{\nabla} p$ and $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$) [3], electrochemical deposition [4], etc.

Numerical estimates of the fractal dimension D of DLA [5] turned out to converge extremely slowly with the number of particles n of the cluster, leading even to speculations [6] that asymptotically the clusters were plane filling (i.e., $D=2$ in two dimensions). To date there are still no controlled calculations of the fractal dimension of DLA. The aim of this Rapid Communication is to close this gap for DLA in two dimensions. We propose a renormalization procedure that culminates with an integral equation whose solution determines the dimension D of DLA.

To derive the wanted integral equation we use the conformal theory that was developed recently [7–11] for fractal growth patterns. In this theory one considers the conformal map from the exterior of the unit circle in the complex plane to the exterior of the growing fractal cluster. At the basis of this approach lies the understanding that once a fractal object is well developed, it is extremely difficult to find a conformal map from a smooth region to its boundary, simply because the conformal map is terribly singular on the tips of a fractal shape. The derivative of the inverse map is the growth probability for a random walker to hit the interface (known as the ''harmonic measure'') which has been shown to be a multifractal measure [12] characterized by infinitely many exponents [13,14]. Accordingly, in the present approach one grows the cluster by iteratively constructing the conformal map starting from a smooth initial interface. Consider $\Phi^{(n)}(w)$ which conformally maps the exterior of the unit circle $e^{i\theta}$ in the mathematical w plane onto the complement of the (simply connected) cluster of n particles in the physical z plane [7–11]. The unit circle is mapped onto the boundary of the cluster. The map $\Phi^{(n)}(w)$ is made from compositions of elementary maps $\phi_{\lambda,\theta}$,

$$\Phi^{(n)}(w)=\Phi^{(n-1)}[\phi_{\lambda_n,\theta_n}(w)], \tag{1}$$

where the elementary map $\phi_{\lambda,\theta}$ transforms the unit circle to a circle with a ''bump'' of linear size $\sqrt{\lambda}$ around the point $w=e^{i\theta}$. An example of a good elementary map $\phi_{\lambda,\theta}$ was proposed in [7], endowed with a parameter a in the range $0<a<1$, determining the shape of the bump. We employ $a=1/2$, which is consistent with semicircular bumps. Accordingly, the map $\Phi^{(n)}(w)$ adds on a new bump to the image of the unit circle under $\Phi^{(n-1)}(w)$. The bumps in the z plane simulate the accreted particles in the physical space formulation of the growth process. Since we want to have *fixed size* bumps in the physical space, say of fixed area λ_0, we choose in the nth step

$$\lambda_n=\frac{\lambda_0}{|\Phi^{(n-1)\prime}(e^{i\theta_n})|^2}. \tag{2}$$

The recursive dynamics can be represented as iterations of the map $\phi_{\lambda_n,\theta_n}(w)$,

$$\Phi^{(n)}(w)=\phi_{\lambda_1,\theta_1}\circ\phi_{\lambda_2,\theta_2}\circ\dots\circ\phi_{\lambda_n,\theta_n}(\omega). \tag{3}$$

The difference between various growth models will manifest itself in the different itineraries $\{\theta_1\cdots\theta_n\}$ [10]. To grow a DLA we have to choose random positions θ_n. This way we accrete fixed size bumps in the physical plane according to the harmonic measure (which is transformed into a uniform measure by the analytic inverse of $\Phi^{(n)}$). The DLA cluster is fully determined by the stochastic itinerary $\{\theta_k\}_{k=1}^n$. In Fig. 1 we present a typical DLA cluster grown by this method to size $n=100\,000$.

This method affords us analytic power that is lacking in the original computer algorithm [1]; the conformal map $\Phi^{(n)}(w)$ is represented in terms of its Laurent expansion,

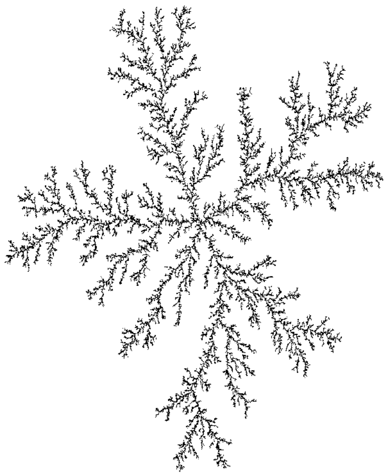

FIG. 1. DLA cluster, $n=100\ 000$.

$$\Phi^{(n)}(w)=F_1^{(n)}w+F_0^{(n)}+F_{-1}^{(n)}w^{-1}+F_{-2}^{(n)}w^{-2}+\ldots \quad . \tag{4}$$

The recursion equations for the Laurent coefficients of $\Phi^{(n)}(w)$ can be obtained analytically, and in particular one shows that [7]

$$F_1^{(n)}=\prod_{k=1}^{n}\sqrt{1+\lambda_k}\quad \text{(with the choice } a=1/2). \tag{5}$$

The first Laurent coefficient $F_1^{(n)}$ has a distinguished role in determining the fractal dimension of the cluster, being identical to the Laplace radius which is the radius of a charged disk having the same field far away as the charged cluster [8]. Moreover, defining R_n as the minimal radius of all circles in z that contain the n-cluster, one can prove that [15]

$$R_n \leqslant 4F_1^{(n)}. \tag{6}$$

Of course, for every realization $\{\theta_i\}_{i=1}^n$ the first Laurent coefficient is a random number depending on $\{\theta_i\}_{i=1}^n$ and on λ_0. It is thus natural to consider the mean of $F_1^{(n)}$ over all the possible realization of growth:

$$\langle F_1^{(n)}\rangle(\lambda_0)\equiv\int_0^{2\pi}d\theta_1\cdots\int_0^{2\pi}d\theta_n F_1^{(n)}(\{\theta_i\}_{i=1}^n,\lambda_0). \tag{7}$$

In light of Eq. (6) one expects that for sufficiently large clusters

$$\langle F_1^{(n)}\rangle(\lambda_0)\sim\sqrt{\lambda_0}\ n^{1/D},\quad n\to\infty. \tag{8}$$

But this is true only for very large values of n. For arbitrary values of n we offer the following proposition, which is central to our developments:

Proposition: For λ_0 of $O(1)$ $\langle F_1^{(n)}\rangle(\lambda_0)$ is a scaling function of the single variable $x\equiv\sqrt{\lambda_0}\ (n+\alpha)^{1/D}$, where n takes on integer values $n=1,2,3,\ldots$, and α is a weak function of λ_0, taking on values of the order of unity.

The origin of the parameter α is evident: it stems from the fact that for varying values of λ_0 the unit circle around which we grow the bumps is contributing to the Laplace radius, changing the effective value of n. We will show, however, that to very good approximation α can be taken as constant.

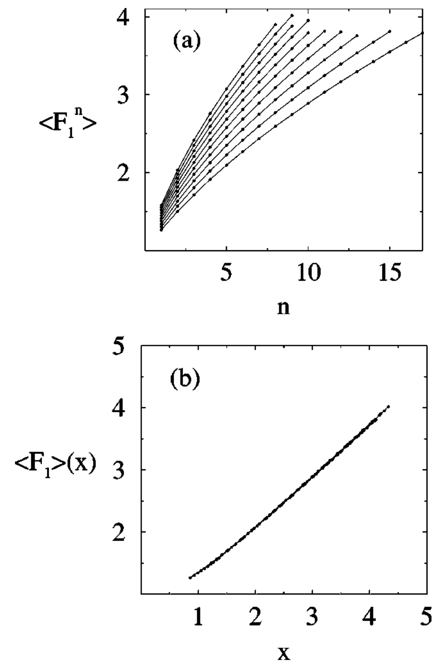

FIG. 2. Panel a: the mean $\langle F_1^{(n)}\rangle$ as a function of n for different values of λ_0. The data shown pertain to the ten values of λ_0 from 0.6 (below) to 1.5 (above). Panel b: the same objects plotted as a function of the scaling variable x with $\alpha=0.2$ and $D=1.71$.

We demonstrate the proposition by a direct calculation of $\langle F_1^{(n)}\rangle(\lambda_0)$. In Fig. 2, panel a we show $\langle F_1^{(n)}\rangle(\lambda_0)$ for ten different values of λ_0 as a function of n. In panel b we show the same data as a function of the scaling variable x using $\alpha=0.2$, $D=1.71$. The data collapse is evident.

We stress three points: (i) The scaling function appears to exist for all values of n starting from $n=1$. It is a *nonlinear* function of the scaling variable, with the attainment of the linear regime (8) not in sight. (ii) α was taken as a λ_0-independent constant; even better data collapse could be obtained with a λ_0-dependent α, and we return to this issue below. (iii) The attainment of data collapse requires a value for D; we have used $D=1.71$ but close-by values would have done equally well by changing α a bit. Thus, we cannot propose the data collapse as an accurate method of computing the fractal dimension D. For this purpose we derive now an integral equation from which both α and D can be computed in a controlled fashion.

Having a scaling function in mind we think about a real-space renormalization group procedure, in which we change the number of particles in the cluster and their size such as to keep the cluster invariant. For n very large, when the radius of gyration is linear in $\sqrt{\lambda_0}\ n^{1/D}$, we can change the size of the particles $\sqrt{\lambda_0}$ by a factor of 2, and their number n by a factor of 2^D. In the asymptotic regime such a renormalization will leave R_n invariant. For small values of n we make use of the discovery that $\langle F_1^{(n)}\rangle(\lambda_0)$ is a scaling function of the single variable $x=\sqrt{\lambda_0}(n+\alpha)^{1/D}$. We demand that upon renormalization the average Laplace radius remains invariant. In other words, the fixed point condition can be written as the following integral equation:

$$\int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n F_1^{(n)}(\lambda_0) = \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_{\bar{n}} F_1^{(\bar{n})}(\bar{\lambda}_0), \tag{9}$$

where for any $\bar{n} > n$ the equation is satisfied by the unique value of $\bar{\lambda}_0$ that solves the equation

$$\sqrt{\bar{\lambda}_0} = \sqrt{\lambda_0}\left(\frac{n+\alpha}{\bar{n}+\alpha}\right)^{1/D}. \tag{10}$$

The way to compute D is then obvious: one computes the integral on the left-hand side (lhs) of Eq. (9) for some value of n, and then finds the unique value of $\bar{\lambda}_0$ for which the rhs with $\bar{n} > n$ equals the lhs. Then

$$D = 2\frac{\ln(\bar{n}+\alpha) - \ln(n+\alpha)}{\ln \lambda_0 - \ln \bar{\lambda}_0}. \tag{11}$$

One should stress that Eq. (9) is all explicit (there is no simulation or randomness left at this point) the integrand on both sides is an explicit function of $\{\theta_1, \ldots, \theta_n\}$ or $\{\theta_1, \ldots, \theta_{\bar{n}}\}$ through Eqs. (5) and (2). We note that this method of calculation is fundamentally different from the standard method of log-log plots of the radius of the cluster versus n [5,6]. These rely on the proportionality of R_n and $n^{1/D}$. Here we may use *small values of n* as we depend neither on the asymptotic linearity of Eq. (8), nor on self-averaging.

Clearly the integral equation will be useful for an actual determination of D only if it converges to the fixed point *quickly* upon increasing n. Otherwise, the calculation of the multidimensional integral will become very cumbersome, maybe even more time consuming than standard numerical simulations. In the rest of this paper we demonstrate that the calculation actually converges very quickly and present the determination of D.

First we determine the value of α. We use $\bar{\lambda}_0 = 1$, and solve Eq. (9) for the simplest case $n=1$, $\bar{n}=2$, demanding

$$\langle F_1^{(2)}\rangle(\bar{\lambda}_0 = 1) = \sqrt{2}\langle\sqrt{1+\bar{\lambda}_2}\rangle = F_1^{(1)}(\lambda_0) = \sqrt{1+\lambda_0}, \tag{12}$$

$$(2+\alpha)^{1/D} = (1+\alpha)^{1/D}\sqrt{\lambda_0}. \tag{13}$$

Computing explicitly (taking $D=1.71$) we found $\langle\sqrt{1+\bar{\lambda}_2}\rangle \approx 1.241$ from which followed $\alpha \approx 0.146$. To bracket the calculations we have employed this value together with $\alpha=0$ and $\alpha=1$.

In solving Eq. (9) we chose invariably $\bar{n} = n+1$, and $\lambda_0 = 1$. The results are shown in Fig. 3, in the form of the value of D versus $(n+\alpha)^{-1}$. Fitting the best quadratic curve to the data with $\alpha=0.146$ we find that it extrapolates for $(n+\alpha)^{-1} \to 0$ to $D=1.7150\ldots$. Obviously, the calculation with $\alpha=0$ and $\alpha=1$ bracket this from above and from below. Nevertheless the best nonlinear fits to these data extrapolate to very close values for $(n+\alpha)^{-1} \to 0$ (see Fig. 3). Taking all the data together we can present a final value of $D=1.713\pm0.003$. We cannot overstress the fact that these results were obtained from solving Eq.(9) with n values ranging from $n=1$ to $n=20$. We believe that this represents a major advance compared to traditional estimates of the dimension of DLA.

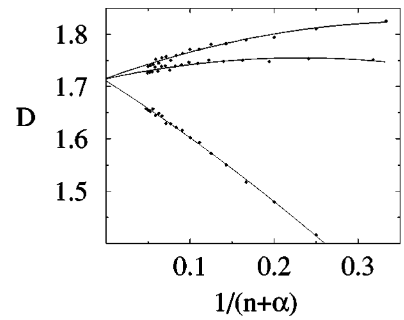

FIG. 3. Estimated value of the fractal dimension D computed from the solution of the integral equation (9). The three sets of points pertain to $\alpha=0$ (upper), $\alpha=0.146$ (middle), and $\alpha=1$ (lower). The lines are the best quadratic fits to the data. The extrapolated values at $1/(n+\alpha)=0$ are 1.7158, 1.7150, and 1.7111, respectively, leading to our final estimate $D=1.713\pm0.003$.

To improve the results even further, and to remove the curvature in the line of D versus $(n+\alpha)^{-1}$ we can endow α with a weak λ_0 dependence. Using the function

$$\alpha(\lambda_0) = 2.048\,11 - 4.080\,71\sqrt{\lambda_0} + 2.234\,46\lambda_0, \tag{14}$$

and using the appropriate value of $\alpha(\lambda_0)$ in Eq. (11) we obtain solutions with essentially constant D for all values of $n>2$. The values of D computed are shown in Fig. 4 as a function of n. With this data we can state without going to the limit $n\to\infty$ that $D=1.713\pm0.005$. This result is invariant with respect to changing n in Eq. (9). It can be computed equally well from $n=3$ and $\bar{n}=4$ as from $n=19$ and $\bar{n}=20$. We should of course stress at this point that the function $\alpha(\lambda_0)$ was *not* computed from first principles; we hope that further theoretical progress will shed light on how to achieve *a priori* determination of the functional form $\alpha(\lambda_0)$.

The most important questions that we need to address now are: (i) why the classical numerical estimates [5,6] of the fractal dimension of DLA converge so slowly, whereas

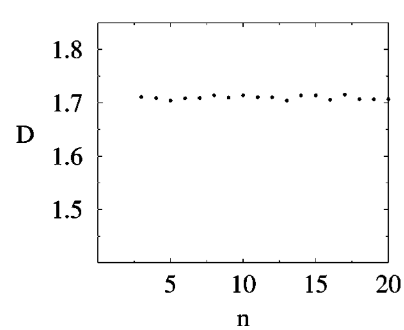

FIG. 4. Estimated value of the fractal dimension D computed from the solution of the integral equation (9), using the function $\alpha(\lambda_0)$ of Eq. (14). Without going to the limit $n\to\infty$ we can state $D=1.713\pm0.005$.

here we can get an excellent estimate of D even with n of the order of unity; (ii) Is the collapse of the scaling data for small n exact or an excellent approximation. The answer to (i) is that in standard numerical experiments the radius of gyration of the grown cluster was plotted in log-log coordinates against the number of particles, with D estimated from the slope. Examining our scaling function $\langle F_1 \rangle(x)$ (see Fig. 2) we note the slow crossover to linear behavior, which may not be fully achieved even for extremely high values of n. In this respect we understand that a reliable estimate of D from radius of gyration calculation requires inhuman effort, as was indeed experienced by workers in the field [6]. In the present formulation the appearance of the *asymptotic* D as a renormalization exponent already at early stages of the growth allows a convergent calculation. The answer to (ii) has to await a first principle theory for $\alpha(\lambda_0)$.

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[1] T.A. Witten and L.M. Sander, Phys. Rev. Lett. **47**, 1400 (1981).

[2] L. Niemeyer, L. Pietronero, and H.J. Wiessmann, Phys. Rev. Lett. **52**, 1033 (1984).

[3] J. Nittman, G. Daccord, and H.E. Stanley, Nature (London) **314**, 141 (1985).

[4] R.M. Brady and R.C. Ball, Nature (London) **309**, 225 (1984).

[5] P. Meakin, Phys. Rev. A **27**, 1495 (1983).

[6] B.B. Mandelbrot, Physica A **191**, 95 (1992).

[7] M.B. Hastings and L.S. Levitov, Physica D **116**, 244 (1998).

[8] B. Davidovitch, H.G.E. Hentschel, Z. Olami, I. Procaccia, L.M. Sander, and E. Somfai, Phys. Rev. E **59**, 1368 (1999).

[9] B. Davidovitch and I. Procaccia, Europhys. Lett. **48**, 547 (1999).

[10] B. Davidovitch, M.J. Feigenbaum, H.G.E. Hentschel, and I. Procaccia, Phys. Rev. E **62**, 1706 (2000).

[11] B. Davidovitch and I. Procaccia, Phys. Rev. Lett. (to be published).

[12] T.C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).

[13] H.G.E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).

[14] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[15] P.L. Duren, *Univalent Functions* (Springer-Verlag, New York, 1983).