A note on Herbert Gintis' "Emergence of a Price System from Decentralized Bilateral Exchange "

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Abstract

In two recent contributions, Herbert Gintis introduces agent-based imitation models built upon evolutionary bargaining games where agents use private prices as strategies. He reports surprising convergence results for simulations performed in exchange economies where goods are strict complement. We investigate analytically these results using the notion of stochastic stability.

Key Words: Exchange economies, Bargaining Games, Equilibrium Selection, Stochastic Stability.

JEL Codes: D51, C62, C63, C78

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1 Introduction

In two recent contributions, (Gintis 2006) and (Gintis 2007), Herbert Gintis introduces agent-based imitation models (see (Dawid 2007)) built upon evolutionary bargaining games where agents use private prices as strategies. He reports surprising convergence results for simulations performed in exchange economies where goods are strict complement. In (Gintis 2007), he focuses on the Scarf economy (see (Scarf 1960)) and his simulations show convergence to equilibrium, whereas this equilibrium is well-known to be unstable for the tâtonnement mechanism. In (Gintis 2006), he studies a variant of the Scarf economy where each good is consumed by each agent. In this setting, the equilibrium price is completely indeterminate. Still, his simulations show convergence to an unique equilibrium.

According to Gintis ((Gintis 2007)), these results suggest, first that *"a highly decentralised Walrasian economy, under a wide range of plausible conditions, has a unique, stable steady state in which the economy is reasonably close to Pareto efficient"* , second that *"the stability of a market system depends on the fact that prices are private information"* and finally[1] that *"a major mechanism leading to convergence of economic behaviour is imitation in which poorly performing agents copy the behaviour of better-performing agents."*

The aim of this note is to illustrate and reinforce these claims by studying analytically a simplified model of (Gintis 2006). As in (Serrano and Volij 2008), the analysis is performed using the notion of stochastic stability (see the exposition of stochastic stability in (Peyton-Young 1993) or (Ellison 2000) and references therein). We prove that the equilibrium towards which Gintis' simulations converge is the only stochastically stable state of a class of Markov chains built upon the iteration of trading and imitation processes.

Moreover, we put forward a characterization of this equilibrium: it is the one which requires the trading of the least share of initial endowments. The intuition for the convergence result follows : the less trading an equilibrium involves, the less welfare is affected when deviation from the terms of trade by some agents introduce rationing. In the language of stochastic stability (using the concepts put forward in (Ellison 2000)), the less trading an equilibrium involves, the larger the number of mutations required to leave its basin of attraction.

If this characterization was to extend to any *highly decentralised Walrasian economy*, one would have at hand a very powerful equilibrium selection device. This note is a first step in that direction.

The paper is organized as follows. In section 2, we present and study the characteristics of the exchange economy considered in (Gintis 2006). In section 3, we introduce a class of dynamics in this exchange economy based on the sequential composition of trading, imitation and mutation processes. In section 4, we prove that the least trade equilibrium is the only stochastically stable state for this class of dynamics.

[1]A third claim stating that *"when even a small fraction of agents are assumed to share the same price system and update in a coordinated manner, as suggested by the tâtonnement mech- anism, the price system becomes highly volatile."* is not discussed here.

2 An exchange economy with strict complementarity

2.0 Notations

In the following, we shall denote by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid \forall i \in \{1 \cdots n\}\ x_i \geq 0\}$ the positive orthant of $\mathbb{R}^n$, and by $\mathbb{R}^n_{++}$ its interior. The ith vector of the canonical basis of $\mathbb{R}^n$ will be denoted by e_i.

2.1 The framework

Given an n-dimensional vector of positive elements $\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{R}^n_{++}$, we consider the exchange economy, denoted by $\mathcal{E}(\omega)$, with the following characteristics.

- There are $m \times n$ agents, where m is an arbitrary positive natural number.
- Each agent has the same utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ defined by:

$$u(x_1, \cdots, x_n) = min(\frac{x_1}{\omega_1}, \cdots, \frac{x_n}{\omega_n}) \tag{1}$$

- For each $i = 1 \cdots n$, there are exactly m agents (hereafter called the agents of type i) which have as initial endowment ω_i units of good i and zero units of every other good. So that the total initial resources in the economy are equal to $m\omega$.

The demand of an agent of type i at positive prices $p \in \mathbb{R}^n_{++}$ is given by:

$$d_i(p) := \text{argmax}_{p \cdot x \leq p \cdot \omega_i e_i} u_i(x) = \frac{p_i \omega_i}{p \cdot w} \omega \tag{2}$$

and its excess demand by :

$$z_i(p) := d_i(p) - \omega_i e_i = \frac{p_i \omega_i}{p \cdot w} \omega - \omega_i e_i \tag{3}$$

So, the aggregate excess demand is:

$$Z(p) = \sum_{i=1}^{n} m z_i(p) = \sum_{i=1}^{n} m(\frac{p_i \omega_i}{p \cdot w} \cdot \omega - \omega_i e_i) = 0 \tag{4}$$

This yields:

Proposition 1 *In the economy $\mathcal{E}(\omega)$, every price $p \in \mathbb{R}^n_{++}$ is an equilibrium price.*

In the following, we denote by Q the space of goods $\mathbb{R}^n_+$. Moreover we consider normalized prices in the simplex $S = \{p \in \mathbb{R}^n_{++} \mid p \cdot \omega = n\}$.

2.2 The minimal trading equilibrium

In order to lift part of the indeterminacy on the outcome of exchange in the economy $\mathcal{E}(\omega)$, Gintis studies evolutionary dynamics in a bargaining game where agents use private prices as strategies (see (Gintis 2006)). More precisely, he considers the following sequence of processes:

1. Trading process: strategies (private prices) of agents are fixed. A bargaining game determines the allocation of the resources $m\omega$ among the agents. During the game, agents agree on a trade only if its value is positive according to their private prices.
2. Imitation process : strategies (private prices) evolve in function of the utility they have yielded during the trading process.
3. Mutation process: strategies randomly mutate.

Gintis shows numerical experiments where the iteration of these processes entails convergence of the economy to the equilibrium associated with the price

$$\bar{p} = (\frac{1}{\omega_1}, \cdots, \frac{1}{\omega_n}) \tag{5}$$

Gintis does not provide any rationale for the selection of this particular price. There is, from our point of view, a very powerful one : the corresponding equilibrium is the one in which the smallest fraction of initial endowments needs to be traded. Namely, one has:

Proposition 2 *The price $\bar{p}$ is the unique minimizer of*[2]*:*

$$\sum_{i=1}^{n} (\|z_i(p)/\omega\|_2)^2 \tag{6}$$

Proof: *We are looking for the minimum over S of*

$$\phi(p) = \sum_{i=1}^{n} (\sum_{j \neq i} (\frac{p \cdot \omega_j}{p \cdot \omega})^2 + (\frac{p \cdot \omega_i}{p \cdot \omega} - 1)^2)$$

$$\phi(p) = (\sum_{i=1}^{n} (\sum_{j=1}^{n} (\frac{p \cdot \omega_j}{p \cdot \omega})^2) + n - 2$$

It is straightforward to see that this quantity is minimized when all the $p \cdot \omega_j$ are equal, that is if and only if the price is $\bar{p}$.

One should note that in the preceding proposition the quantities traded are measured in normalized units (in shares of the initial resources). Therefore, the characterization given in proposition 2 is independent of the units of measurement (as well as of the utility representation). As the price $\bar{p}$ corresponds to the equilibrium which requires the least possible share of goods to be traded, we shall in the following call the associated equilibrium the least trade equilibrium.

[2]The symbol / denotes here the division coordinatewise.

3 A dynamic model based on Gintis' experiments

In the following, we aim at explaining convergence to the least-trade equilibrium in Gintis' experiments, using the notion of stochastic stability (see (Peyton-Young 1993) and (Ellison 2000)). We shall exhibit, in a simplified model of Gintis' experiments, sufficient conditions to ensure that the special price is the unique stochastically stable state.

Our first assumption will be to consider a restricted set of prices $P \subset S$, an arbitrary finite subset of the simplex containing $\bar{p}$. Such a restriction is standard when one aims at applying stochastic stability methods (see e.g (Serrano and Volij 2008)).

3.1 Population

We consider each agent is endowed with a private price in P. The state of our system is represented by a price for each agent, i.e a vector in $P^{n\times m}$, which we shall call a population.

In order to have at hands a simplified representation of a population, we shall use the following notations. Given a population $\pi \in P^{m\times n}$, we denote by $\pi_i \in P^m$ the vector of prices of agents of type i. We shall also refer to such an element as a population of type i. The price of the jth agent of type i is denoted by $\pi_{i,j} \in P$. Moreover, with a slight abuse of notation, we write $p \in \pi_i$ whenever at least an agent of type i in the population π uses the price p. We also denote by $\mu_i(\pi, p) \in \mathbb{N}$ the number of agents of type i using price p in the population π and by $\nu_i(\pi)$ the number of distinct prices used by agents of type i in the population π.

3.2 Trading process

In Gintis' experiments, trading takes place according to a random sequence of bilateral trades. The outcome of such a sequence of bilateral trades is a function of the private prices of agents (considering as given their initial endowments). This gives raise to a combinatorially complex structure of possible outcomes. In order to give a synthetic description amenable to analysis, we shall focus on a class of trading processes fulfilling similar, though more "axiomatic", requirements.

In order to represent the outcome of a sequence of bilateral trades, we define an allocation mechanism as a mapping:

$$\xi : P^{n\times m} \to Q^{n\times m}. \tag{7}$$

In order to take into account the random character of the trading process, we shall consider as given a finite set $X := \{\xi_1, \cdots, \xi_L\}$ of possible allocation mechanisms together with a probability distribution $\rho = (\rho_1, \cdots, \rho_L)$ with full support. The trading process then consists in the choice of an allocation mechanism in X, followed by the corresponding allocation of goods on the basis of prices. Hence, we shall represent the trading process by a probability transition matrix $\mathcal{T}$ with $P^{n\times m}$ columns and $Q^{n\times m}$ rows. The element $\mathcal{T}_{(x,\pi)}$ corresponding to the probability of reaching allocation x when the price distribution is π. That is:

$$\mathcal{T}_{(x,\pi)} = \sum_{\{\ell\in\{1\cdots L\}|\xi_\ell(\pi)=x\}} \rho_\ell \tag{8}$$

Now, we shall restrict attention to the set of allocation mechanisms satisfying the following properties.

- The allocation mechanisms conserves total quantities:

$$\forall \xi \in X\ \forall \pi \in P^{n\times m} : \sum_{i=1}^{n}\sum_{j=1}^{m} \xi_{i,j}(\pi) = m\omega. \tag{9}$$

- The allocation mechanisms are anonymous, that is deliver the same allocation to two agents of the same type who use the same prices:

$$\begin{array}{c}\forall \xi \in X\ \forall \pi \in P^{n\times m}\ \forall i \in \{1\cdots n\} \forall j,k \in \{1\cdots m\} : \\ \pi_{i,j} = \pi_{i,k} \Rightarrow \xi_{i,j}(\pi) = \xi_{i,k}(\pi).\end{array} \tag{10}$$

  With a slight abuse of notation, we shall then denote by $\xi_i(\pi,p) \in Q$ the allocation to agents of type i with price p under the allocation mechanism ξ.

- Now, the main characteristic of the bargaining game studied by Gintis is that a trade is agreed upon only if its value is positive according to the private prices of both contractors. In order to retain, and even reinforce, this property, we shall assume that the allocation mechanism satisfies each agent's private budget constraint:

$$\begin{array}{c}\forall \pi \in P^{n\times m}\ \forall i \in \{1\cdots n\}\ \forall j \in \{1\cdots m\} : \\ \pi_{i,j} \cdot (\xi_i(\pi, \pi_{i,j}) - \omega_i e_i) = 0\end{array} \tag{11}$$

- In (Gintis 2006), trading is iterated many times and hence tends to ensure the achievement of all Pareto improving trades. We shall here ensure the efficiency of the trading process by assuming it yields an optimal allocation (in the sense of Pareto) among those satisfying the preceding conditions. That is, denoting by $\hat{X}$ the set of allocation mechanisms such that (9) to (11) hold and by $>_{PO}$ the relation of Pareto dominance in $\mathcal{E}(\omega)$ [3], we have:

[3] That is one has for $x, y \in Q^{n\times m}$, $x >_{PO} y$ if for all $i \in \{1\cdots n\}$ and $j \in \{1\cdots m\}$ one has $u(x_{i,j}) \geq u(y_{i,j})$ and if for at least one $i' \in \{1\cdots n\}$ and one $j' \in \{1\cdots m\}$ one has $u(x_{i,j}) > u(y_{i,j})$.

$$\forall \pi \in P^{n\times m}\ \forall \xi \in X\ \forall \xi' \in \hat{X} : \neg(\xi'(\pi) >_{PO} \xi(\pi))) \tag{12}$$

Now, one can remark that the allocation mechanism defined by $\xi_{i,j}(\pi) = \omega_{i,j}$ satisfies (9) to (11). Hence $\hat{X}$ is not empty. It is moreover clear that for every $\pi \in P^{n\times m}$, the set $\{x \in Q^{n\times m} \mid \exists \xi \in \hat{X}\ \xi(\pi) = x\}$ is compact. As the utility function u is continuous, one can define an allocation mechanism satisfying (9) to (12) by choosing, for every π, a maximal element for the Pareto order in this set. Therefore:

Proposition 3 *The set of allocation mechanisms $\xi : P^{m\times n} \to Q^{n\times m}$ satisfying* (9) *to* (12) *is not empty.*

Now, conditions (9) to (12) describe a class of mechanisms where trading takes place inside groups of agents using similar prices and until no Pareto improving trade can be performed. Although this does not imply any other form of rationality for the agents than proposing trades which improve their utility and agreeing on trades which have a zero value, this mechanism yields equilibrium allocations whenever all the agents use the same price.

Proposition 4 *Assume the allocation mechanism ξ satisfies conditions* (9) *to* (12)*. If there exists $p \in P$ such that for all i, j, $\pi_{i,j} = p$, one has:*

$$\forall i \in \{1 \cdots n\}\ \forall j \in \{1 \cdots m\} : \xi_{i,j}(\pi) = d_i(p) \tag{13}$$

Proof: *As there exists $p \in P$ such that for all i, j $\pi_{i,j} = p$, it is clear that the allocation ξ such that $\forall i \in \{1 \cdots n\}\ \forall j \in \{1 \cdots m\} : \xi_{i,j}(p) = d_i(p)$ satisfies conditions* (9) *to* (11)*. Moreover, any allocation satisfying* (9) *to* (11) *is such that $\xi_{i,j}(p) \le d_i(p)$ for all i, j. By* (12)*, this ends the proof.*

Finally, we shall avoid implicitly assuming any other form of rationality or determinism than the one postulated in (9) to (12). Ideally, we would like to consider that X consists in the set of such allocation mechanisms. However X must be finite for technical reasons. In order to overcome this technical constraint, we shall assume that the set of allocation mechanisms is sufficiently large to let each agent reach the most desirable outcome with some probability:

$$\begin{array}{c}\forall \pi \in P^{n\times m}\ \forall i \in \{1 \cdots n\}\ \forall j \in \{1 \cdots m\}\ \exists \xi \in X\ \forall \xi' \in \hat{X} : \\ u(\xi_{i,j}(\pi)) \ge u(\xi'_{i,j}(\pi)).\end{array} \tag{14}$$

3.3 Imitation process

In (Gintis 2006), the imitation process clusters the distribution of prices around those yielding high utility through a random sequence of bilateral imitations. Again, we shall not try to provide an analytical description of the fairly complex structure emerging in Gintis' setting. We rather focus on a class of simplified imitation processes which retain characteristic properties.

Let us first define an imitation mechanism as a mapping

$$\phi : P^{n\times m} \times Q^{n\times m} \to P^{n\times m}. \tag{15}$$

It associates to a population $\pi \in P^{m\times n}$, and to an allocation $x \in Q^{n\times m}$, a new population $\phi(\pi, x) \in P^{n\times m}$.

As in the case of the trading, in order to take into account the random character of the imitation process we shall consider as given a finite set Y of imitation mechanisms together with a probability distribution. The imitation process then consists in the choice of an imitation mechanism in Y, followed by the corresponding transformation of prices. Hence, we shall represent the imitation process by a probability transition matrix $\mathcal{I}$ with $P^{n\times m} \times Q^{n\times m}$ columns and $P^{n\times m}$ rows. The element $\mathcal{I}_{((x,\pi),\pi')}$ corresponds to the probability of reaching the population π' from the population π and the allocation of goods x.

Now, we shall restrict attention to the set of imitation mechanisms satisfying the following properties.

- We first endow imitation mechanisms with a "structural" clustering property by assuming that the number of distinct prices within a type decrease at every step (except when each agent use the same price):

$$\begin{array}{c} \forall \phi \in Y \ \forall \xi \in X \ \forall i \in \{1 \cdots n\} \ \forall \pi \in P^{n\times m} \ \nu_i(\pi) \geq 2 : \\ (\{p \in \phi(\pi, \xi(\pi))\} \subset \{p \in \pi\}) \wedge (\{p \in \phi(\pi, \xi(\pi))\} \neq \{p \in \pi\}). \end{array} \tag{16}$$

- In particular, we shall assume that there is a threshold for survival $\overline{m} \leq \frac{m}{2}$ such that when more than half of the population of a certain type uses the same price, any price used by less than $\overline{m}$ agents disappear:

$$\begin{array}{c} \forall \phi \in Y \ \forall \xi \in X \ \forall p' \in P \ \forall i \in \{1 \cdots n\} \ \forall \pi \in P^{n\times m} \ \nu_i(\pi) \geq 2 : \\ (\exists p \ \mu_i(\pi, p) \geq \frac{m}{2}) \Rightarrow (\mu_i(p') < \overline{m} \Rightarrow \mu_i(\phi(\pi, \xi(\pi)), p') = 0) \end{array} \tag{17}$$

It is worth pointing out that conditions (16) would alone ensure that the iteration of the imitation process entails convergence of the population to an uniform distribution of prices. However, in order to avoid washing away any influence of the trading process on the imitation process (see (Dawid 2007)), we must assume some dependency of the imitation process on the utiltiy gained during the trading process. Still, even in a simplified structure as ours, it seems unpracticable to keep track for more than one agent at the same time of the utility obtained via a stochastic trading mechanism. As a proxy, we choose to compare the utility gained by successful agents during the trading process $(u_i(\xi_i(\pi_i, p)))$ to the indirect utility of less successful agents $(u_i(d_i(p')))$. This is justified by the fact that the indirect utility is, under condition (11) the maximal utility an agent can expect assuming the trading process is efficient.

- We shall assume that whenever $\overline{m}$ agents of a given type use a similar price and obtain, during the trading process, an utility higher than the expected utility of the remaining agents, the whole population of the corresponding type might adopt that price.

$$\begin{array}{c} \exists \phi \in Y \ \forall \xi \in X \ \forall p \in P \ \forall i \in \{1 \cdots n\} \ \forall \pi \in P^{n \times m} \ \nu_i(\pi) \geq 2 : \\ ((\mu_i(\pi, p) \geq \overline{m}) \wedge (u_i(\xi_i(\pi, p)) \geq \max_{p' \in \pi_i - \{p\},} u_i(d_i(p'))) \Rightarrow \\ \mu_i(\phi(\pi, \xi), p) = m. \end{array} \tag{18}$$

- Conversely, when more than half of the population uses a price whose indirect utility is higher than the utility yielded by other prices, we shall assume that the whole population adopts that price:

$$\begin{array}{c} \forall \phi \in Y \ \forall \xi \in X \ \forall p \in P \ \forall i \in \{1 \cdots n\} \ \forall \pi \in P^{n \times m} \ \nu_i(\pi) \geq 2 : \\ (\mu_i(\pi, p) \geq \dfrac{m}{2}) \wedge (u_i(d_i(\pi, p)) > \max_{p' \in \pi_i - \{p\},} u_i(\xi_i(\pi, p'))) \Rightarrow \\ \mu_i(\phi(\pi, \xi), p) = m. \end{array} \tag{19}$$

- Finally, a particular case occurs when the distribution of prices is uniform within types but distinct among types. Indeed, agents may then be unable to find trade partners of other types. In order to prevent the permanence of such a situation, we shall assume that if it occurs, the whole population adopts a common price among those used by the various type of agents. Moreover, we shall assume that Y is symmetric with regards to types on this issue. That is:

$$\begin{array}{c} \forall \phi \in Y \ \forall \xi \in X \ \forall \pi \in P^{m \times n} : \\ (\forall i \in \{1 \cdots n\} \ \exists p_i \in P \ \mu_i(\pi, p_i) = m) \Rightarrow \\ (\exists i' \ \forall i \ \mu_i(\phi(\pi, \xi), p_{i'}) = m). \end{array} \tag{20}$$

$$\begin{array}{c} \forall i_0 \in \{1 \cdots m\} \exists \phi \in Y \ \forall \xi \in X \ \forall \pi \in P^{m \times n} : \\ (\forall i \in \{1 \cdots n\} \ \exists p_i \in P \ \mu_i(\pi, p_i) = m) \Rightarrow \\ (\forall i \ \mu_i(\phi(\pi, \xi), p_{i_0}) = m). \end{array} \tag{21}$$

As an example of mechanism satisfying (16) to (19), one can consider any mechanism such that (18) holds and such that the least frequent price is replaced by the most frequent one whenever the right hand side of (18) is not satisfied. It is then straightforward to extend such a mechanism to the whole space of population while satisfying (20) and (21) for a given i_0. Therefore:

Proposition 5 *The set of imitation mechanisms satisfying* (16) *to* (20) *is not empty.*

3.4 Unperturbed dynamics

The unperturbed dynamics of our system consist in the sequential composition of trading and imitation processes. They can then be represented as a Markov chain on $P^{n\times m}$ whose transition matrix is given by $\mathcal{F}$ such that :

$$\mathcal{F}\pi = \mathcal{I}(\mathcal{T}\pi, \pi) \tag{22}$$

Given the assumptions on the imitation process, it is clear that any state where all agents use the same price is absorbing. We shall call such a state uniform and denote by $\upsilon(p) \in P^{n\times m}$ the state where all agents use the price $p \in P$. We let $\mathcal{U}$ stand for the set of uniform states.

Due to (16), from any initial state the system has a positive probability to reach in finite time a state where all agents of the same type use the same price. From there, because of (20), the system reaches necessarily an uniform state. This implies that the uniform states are the only stable states of our system and that the system converges necessarily to an uniform state. Namely:

Proposition 6 *For all $\pi \in P^{n\times m}$, one has:*

$$lim_{t\to+\infty}\ \mathcal{F}^t\pi \in \mathcal{U} \tag{23}$$

3.5 Mutation process

Given the large number of uniform states ($\mathcal{U}$ is isomorphic to P), there remains, if one only focuses on unperturbed dynamics, a huge indeterminacy on the asymptotic behavior of our system. As in Gintis' experiments, the introduction of random perturbations in the dynamics will allow the selection of stochastically stable states.

During the mutation process the price of each agent stays identical with probability $1-\epsilon$ and with probability ϵ is drawn anew uniformly in P. Hence the mutation process can be represented by a probability transition matrix $\mathcal{M}_\epsilon$ with $P^{n\times m}$ rows and $P^{n\times m}$ column. The element $\mathcal{M}(\epsilon)_{(\pi,\pi')}$ corresponding to the probability of reaching the price distribution π' from the price distribution π. By construction this matrix is strictly positive. Its diagonal elements are equal to $(1-\epsilon)^{m\times n}$ while the element (π,π') is a polynomial in ϵ whose non-zero term of least degree is equal to the number of distinct prices between π and π' (we shall in the following denote this degree by $c(\pi,\pi')$).

3.6 ϵ-Perturbed dynamics

The dynamics of our system with ϵ- perturbation are then defined by the composition of the mutation process with the unperturbed dynamics. They are represented as a Markov chain on $P^{n\times m}$ whose transition matrix is given by $\mathcal{F}(\epsilon)$ such that :

$$\mathcal{F}_\epsilon\pi = \mathcal{M}(\epsilon)\mathcal{F}\pi \tag{24}$$

It is clear that

1. $\mathcal{F}(\epsilon)$ is ergodic for each $\epsilon > 0$,
2. $\mathcal{F}(\epsilon)$ is continuous in ϵ and $\mathcal{F}_0 = \mathcal{F}$,
3. there exists a function $c : P^{n\times m} \times P^{n\times m} \to \mathbb{N}$ such that:

$$\forall \pi, \pi' \in P^{n\times m}, \lim_{\epsilon\to 0} \frac{\mathcal{F}(\epsilon)_{(\pi,\pi')}}{\epsilon^{c(\pi,\pi')}} \text{ exists and is strictly positive.}$$

   (This last point follows form the fact that the coefficients of $\mathcal{M}(\epsilon)$ are polynomia in ϵ.)

So that $(P, \mathcal{F}, \mathcal{F}(\epsilon))$ is a model of evolution with noise in the sense of (Ellison 2000).

Ergodicity implies that there exist an unique probability distribution on $P^{m\times n}$ which is invariant for $\mathcal{F}_\epsilon$. That is Π_ϵ such that $\mathcal{F}_\epsilon \Pi_\epsilon = \Pi_\epsilon$.

Moreover for any $\pi \in P^{n\times m}$, one has:

$$\lim_{t\to+\infty} \mathcal{F}_\epsilon \pi = \Pi_\epsilon. \qquad (25)$$

When the perturbation rate is small, the asymptotic behavior of our system is characterized by:

$$\Pi^* = \lim_{\epsilon\to 0} \Pi_\epsilon. \qquad (26)$$

Indeed, in the long-run, the system will spend "infinitely" more time in the neighborhood of a state $\pi \in P^{n\times m}$ such that $\Pi^*(\pi) > 0$ than in the neighborhood of a state π' such that $\Pi^*(\pi') = 0$ (see (Ellison 2000) for a detailed discussion). Following (Peyton-Young 1993), we shall call stochastically stable a state/population $\pi \in P^{n\times m}$ such that $\Pi^*(\pi) > 0$.

4 Stochastic stability of of the least-trade equilibrium

An explanation of the results of Gintis' experiments is provided in this section by showing that $v(\bar{p})$ is the only stochastically stable state of our system.

It is first of all clear (see (Peyton-Young 1993)) that stochastically stable states necessarily are invariant for the unperturbed distribution and hence uniform states. A more precise characterization will be provided by the application of theorem 3 in (Ellison 2000).

Let us first introduce some auxiliary definitions:

- A path from $\pi \in P^{n\times m}$ to $\pi' \in P^{n\times m}$ is a finite sequence of states, $\pi^1, \cdots, \pi^K$ such that $\pi^1 = \pi$ and $\pi^K = \pi'$. The set of paths from π to π' is denoted by $S(\pi, \pi')$. The cost of a path $(\pi^1, \cdots, \pi^K)$ is defined as:

$$c(\pi^1, \cdots, \pi^K) = \sum_{k=1}^{K-1} c(\pi^k, \pi k + 1) \qquad (27)$$

- The basin of attraction of an uniform state $\upsilon(p)$ is:

$$D(p) = \{\pi \in \mathcal{P}^{n\times m} \mid \lim_{t\to+\infty}\mathcal{F}^t\pi = \upsilon(p)\} \tag{28}$$

- The radius of an uniform state $\upsilon(p)$ is defined as the minimal cost of a path leaving $D(p)$ (i.e reaching its complement $D(p)^c$):

$$r(p) = \min c(s) \text{ for } s \in \cup_{\pi\in D(p)^c} S(\upsilon(p), \pi) \tag{29}$$

Now, the application of theorem 3 in (Ellison 2000) to our framework yields that for all $p, p' \in P$, one has:

$$\text{if } r(p) > \min_{s\in S(\upsilon(p'),\upsilon(p))} c(s) \text{ then } \mu^*(\upsilon(p')) = 0 \tag{30}$$

This will suffice to prove our main result:

Theorem 1 *If the number of agents m is sufficiently large, the only stochastically stable state of the system $(P, \mathcal{F}, \mathcal{F}(\epsilon))$ is the uniform state $\nu(\bar{p})$ where $\bar{p}$ is the least-trade equilibrium price:*

$$\bar{p} = (\frac{1}{\omega_1}, \cdots, \frac{1}{\omega_n}).$$

Proof: *Let us mention a priori that the fact that m is sufficiently large is used on the one hand to ensure that m is substantially greater than $\overline{m}$ (see equation* 38*) and on the other hand to ensure that the inequalities we put forward below extend to integral parts.*

1. *We first put forward some properties of the cost function c.*

   - *One has $c(\pi, \pi') = 0$ whenever $\mathcal{F}_{\pi,\pi'} > 0$, that is whenever there is a positive probability to reach π' from π via the unperturbed process.*
   - *Also, $c(\pi, \pi')$ is bounded above by the number of distinct prices between π and π'.*

2. *Let us now consider an arbitrary price $p \in P - \{\bar{p}\}$ and determine an upper bound for $c(s)$ with $s \in S(\upsilon(p), \upsilon(\bar{p}))$.*

   As $p \neq \bar{p}$, there exists $i \in \{1 \cdots n\}$ such that $p_i < \dfrac{1}{\omega_i}$. According to (2) *the indirect utility of the corresponding type of agent is strictly smaller than $\dfrac{1}{n}$:*

$$u(d_i(p)) = \frac{p_i \cdot \omega_i}{n} < \frac{1}{n} \tag{31}$$

Now, let π *be the population such that* $\overline{m}$ *agents of type* i *and* $\overline{m}\frac{p_i\omega_i}{n}$ *agents of type* $i' \neq i$ *use the special price* $\bar{p}$ *while all other agents use price* p. *One has:*

$$c(\nu(p),\pi) = \overline{m}(1 + \frac{n-1}{n}p_i\omega_i) \tag{32}$$

Let us then consider an allocation mechanism ξ *such that :*

- $\xi_i(\pi,\bar{p}) = \omega_i e_i + \frac{p_i\omega_i}{n}\sum_{i'\neq i}(\omega_{i'}e_{i'} - \omega_i e_i)$,
- $\forall i' \neq i, \xi_{i'}(\pi,\bar{p}) = \omega_{i'}e_{i'} + \frac{p_i\omega_i}{n}(\omega_i e_i - \omega_{i'}e_{i'})$,
- $\forall i' \neq i, \xi_{i'}(\pi,p) = \omega_{i'}e_{i'}$

Such an allocation yields the utility $\frac{p_i\omega_i}{n}$ *to the agents of type* i *using* $\bar{p}$. *Moreover, it is clear that such an allocation mechanism satisfies* (9) *to* (11). *Hence according to* (12) *and* (14), *there is a trading mechanism* $\xi' \in X$ *which is at least Pareto equivalent to* ξ. *One then has:*

$$u(\xi_i'(\pi,\bar{p})) \geq u(d_i(p)) \tag{33}$$

In other words, agents using price $\bar{p}$ *achieve an utility higher than the indirect utility associated with price* p. *One then has for any imitation mechanism* ϕ *satisfying* (18) *that* $\mu_i(\phi(\pi,\xi'),\bar{p}) = m$. *Now, by construction one has* $\mathcal{F}_{\pi,\phi(\pi,\xi')} > 0$ *or equivalently:*

$$c(\pi,\phi(\pi,\xi')) = 0. \tag{34}$$

Moreover, according to (16) *and* (20), *it is clear that there exists a path* $s \in S(\phi(\pi,\xi'),\upsilon(\bar{p}))$ *such that* $c(s) = 0$.

One can then conclude that for all $p \in P - \{\bar{p}\}$ *:*

$$min_{s\in S(\upsilon(p'),\upsilon(p))}c(s) < \overline{m}(1 + \frac{n-1}{n}) \tag{35}$$

3. *To end the proof, via* (30), *it remains to show that for* m *large enough:*

$$r(\bar{p}) \geq \overline{m}(1 + \frac{n-1}{n}). \tag{36}$$

We shall prove this by contradiction. Let us assume there exists $\pi \notin D(\bar{p})$ *such that:*

$$c(\upsilon(\bar{p}),\pi) \leq \overline{m}(1 + \frac{n-1}{n}). \tag{37}$$

Now, for m large enough, one has:

$$\frac{m}{2} > \overline{m}(1 + \frac{n-1}{n}) \tag{38}$$

So that for all $i \in \{1 \cdots n\}$

$$\mu_i(\pi, \bar{p}) \geq \frac{m}{2}. \tag{39}$$

According to (17), *if* $\pi \notin D(\bar{p})$, *there must exist* i_0 *and* p *such that*

$$\mu_{i_0}(\pi, p) \geq \overline{m} \tag{40}$$

According to (19), *it must moreover be the case that for some* $\xi \in X$:

$$u(\xi_{i_0}(\pi, p)) \geq u(d_{i_0}(\bar{p})) \tag{41}$$

Now, one has for all $i \in \{1 \cdots n\}$ $u(d_i(\bar{p})) = \frac{1}{n}$. *Hence for* (41) *to hold, it must be that* $u(\xi_{i_0}(\pi, p)) \geq \frac{1}{n}$. *This implies that*

$$\xi_{i_0}(\pi, p) \geq \frac{1}{n} \sum_{i=1}^{n} \omega_i e_i. \tag{42}$$

On the other hand, (10) *and* (11) *yield:*

$$\xi_{i_0}(\pi, p) \leq \frac{1}{\mu_{i_0}(\pi, p)} \sum_{i=1}^{n} \mu_i(\pi, p) \omega_i e_i \tag{43}$$

For (42) *and* (43) *to hold simultaneously, one must have for all* i :

$$\frac{\mu_i(\pi, p)}{\mu_{i_0}(\pi, p)} \geq \frac{1}{n}. \tag{44}$$

As $\mu_{i_0}(\pi, p) \geq \overline{m}$, *this implies for all* i :

$$\mu_i(\pi, p) \geq \frac{1}{n} \overline{m}. \tag{45}$$

And Finally:

$$c(\upsilon(\bar{p}), \pi) \geq \overline{m}(1 + \frac{n-1}{n}). \tag{46}$$

This contradicts (37)

Q.E.D

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