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Resilience. A formal approach to an ambiguous concept.

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FAVAIA Working Paper 5

August 2007



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Resilience

A formal approach to an ambiguous concept

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Diplomarbeit

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August 2007

Abstract

There is no agreed upon definition of the concept of “resilience”, even though the term is increasingly used in the research on effects of environmental changes on natural and social systems. Instead, the current cluster of definitions has rendered the term unclear and confusing. In this context, a formal framework might help clarify the term, improve communication and discussions, as well as it is a prerequisite for any computational approach to its assessment. It is the objective of this thesis to propose such a framework. From the analysis in this thesis follows that the concept of resilience is a relative one, i.e. it depends on the more primitive concepts of the *entity*, which is resilient; the *disturbance* of the entity; the *situation before* and the *situation after* the disturbance occurred; and of the *similarity criteria* to compare the situation before the disturbance occurred to the situation thereafter. These primitive concepts have to be specified precisely in order to make meaningful statements about resilience. Mathematically, the formalization is embedded in the calculus of relations. Resilience is defined extensionally as a relation, and several definitions of such a relation, depending on the interpretation of the primitive concepts, are proposed. In all of these definitions, the mathematical notion of residuation proved useful. We relate the formal framework to two definitions of resilience used in ecological literature.

Acknowledgements

I want to thank everybody, who helped me in writing this thesis. I am especially grateful to Rupert Klein, who made this interdisciplinary kind of thesis possible; Cezar Ionescu, who always took the time to discuss the various stages of the work, and who always had valuable ideas on how to proceed; Stefan Barthel, who was a ready and sympathetic listener; Christian Grümme and Eva Csipak, who proofread the final version of this thesis.

Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 1 |
| 2 | A Calculus of Relations | 5 |
| 2.1 | Definition and Fundamental Operations | 6 |
| 2.2 | Functions as a Special Case of Relations | 10 |
| 2.3 | Power Transpose and Residuation | 12 |
| 3 | Formalization of Resilience | 17 |
| 3.1 | Grammatical and Semantical Investigation | 17 |
| 3.1.1 | Everyday Usage | 17 |
| 3.1.2 | Technical Usage | 18 |
| 3.2 | Formalization of the Primitive Concepts | 19 |
| 3.3 | Resilience as a Relation | 25 |
| 3.3.1 | The Resilience of a State | 26 |
| 3.3.2 | Being Resilient to a Set of Changes in State | 26 |
| 3.3.3 | Comparing States and Systems with Regard to Their Resilience | 28 |
| 3.3.4 | A Note on Change Relations with External Inputs | 30 |
| 3.4 | Formal Definitions of Resilience | 31 |
| 3.4.1 | The Resilience Relation: A First Definition | 32 |
| 3.4.2 | The Resilience Relation: Refined Definitions | 34 |
| 3.4.3 | Common Structure | 45 |
| 3.4.4 | A Note on Non-Deterministic Systems | 48 |
| 4 | Applications | 53 |
| 4.1 | Pimm Resilience | 53 |
| 4.2 | Holling Resilience | 56 |
| 5 | Conclusion | 59 |
| | Bibliography | 63 |

Chapter 1

Introduction

Within the last decades we, as humans, have become aware of the large-scale impacts of our activities on ecosystems such as the loss of biodiversity, land degradation and climate change, just to name a few. Out of the need to understand and characterize the effects of these environmental changes on natural and social systems, a diverse and interdisciplinary research area has since developed, with a wealth of concepts at hand to highlight differences between the behavior of systems facing environmental stresses. However, authors have not necessarily given clear-cut definitions to the concepts they have been using, leading to overlapping definitions of distinct concepts or in contradicting definitions of one and the same concept (e.g. [1, 2, 3]).

One of these concepts with ambiguous meaning, which is increasingly in use in recent years, is “resilience”. Within the ecological literature, it belongs to a cluster of concepts which might be summarized under the (even more confusing) concept of “stability”: when making an inventory of the concepts related to stability and of their definitions used in the ecological literature, Grimm and Wissel [3] collected an impressive number of 70 different concepts including “resilience”, and 163 overlapping and even contradictory definitions of these concepts! In scientific papers and studies, two differing definitions of resilience are prevalent. Often the work of C.S. Holling is cited [4], who defined resilience in his seminal paper “Resilience and Stability of Ecosystems” [5] as “a measure of the ability of these systems to absorb changes . . . and still persist” (p.17). This contrasts the traditional definition of resilience in ecology [6, 7] given by S.L. Pimm as “how fast the variables return towards their equilibrium following a perturbation” ([8] p. 322). Although one of these two definitions is usually cited, different authors emphasize and chart different aspects in their definition

of resilience leading to a multitude of different definitions. In other words, there is no consensus about a definition of resilience.

In addition the concept of resilience has also found its way into the social sciences; there it has been linked to further concepts in the research on global environmental changes such as vulnerability and adaptivity, which themselves do not provide a clear and agreed upon definition. Thus the definition of resilience as well as its relation to other concepts remains unclear and confusing. As a result its operational use has been questioned [1, 9, 10, 11]. There is, however, a shared understanding that there is a need for more precise definitions, because they are a prerequisite for operationalizations. As Schoon [9] puts it: “Much work remains to be done in the creation of a concise and agreed upon definition of each of the concepts [vulnerability, resilience, adaptivity]; how the concepts relate to each other; how to operationalize, evaluate, and compare case studies; how to understand and generate meaningful research cross-cutting concepts” (p. 5).

Against this background, it is the objective of this thesis to contribute to a rigorous and precise basis for analyzing the meaning of resilience as well as for analyzing the interlinkages between resilience and related concepts by proposing a *formal framework for resilience*. In this way this thesis takes up the approach of Ionescu *et al.* [10] who proposed a formal framework as a way of resolving the prevailing confusion about the concept of vulnerability to climate change. The idea of a formal framework is not to provide yet another (formal) definition of resilience; instead mathematical notions and notation are used to define concepts in a general, yet concise way so as to expose their basic structure. The aim is that other definitions might be analysed and compared with the help of the framework. It therefore helps to clarify the meaning of concepts by representing them in a precise manner.

Although a formal framework might represent a challenge to those not accustomed to the language of mathematics, it has several promising consequences. First of all, a formal framework might clarify a concept and limit the potential of analytical inconsistencies such as obscured ambiguities and circularities in the definition by making underlying assumptions explicit and by eliminating inessential features [12]. It thereby provides a systematic and consistent assistance in the process of examining, interpreting and representing the concepts in assessment studies [10]. Second, a formal framework might be tantamount to a bridge of communication between researchers of different scientific disci-

plines and scientific communities as well as between researchers and stakeholders: as mathematical notation is independent of any knowledge domain and is applicable to any system under consideration, a formal framework introduces a standardized terminology and thus improves communication [10]. Last, not least, a formal framework is a prerequisite for any computational approach. In brief, a formal framework helps raise the discussion to a qualitatively new level [12], since there is a shared understanding and language which encourages and improves the debate. Therefore, a formalization of resilience appears profitable.

The thesis is structured as follows. Since the mathematical notion of a *relation* is considered an appropriate and natural setting for the formalization of resilience, a short review of a calculus of relations is presented first (Chapter 2). In Chapter 3 the formalization of resilience is the focus of attention. Chapter 4 relates the framework thus developed back to the definitions of resilience within the ecological literature. A summary of the work, conclusions and recommendations for future work are presented in Chapter 5.

Chapter 2

A Calculus of Relations

In this section we discuss briefly a *calculus of binary¹relations*, which provides the mathematical background for the formalization of resilience in Chapter 3. It does not, however, present a comprehensive summary of the subject. Instead, only the definitions and properties necessary for the formalization are given. Furthermore, since the calculus of relations is not of primary concern in this work, we are not going to give all the proofs in detail. Some proofs, however, are included to demonstrate the calculus, because reasoning with relations differs from the familiar reasoning with functions. For additional details, the reader is referred to [13] and [14], on which our presentation is based.

The calculus of relations was introduced in 1860 by Augustus de Morgan (1806-1871) and was further developed in the second half of the nineteenth-century by Charles Sanders Peirce (1839-1914) and Ernst Schröder (1841-1902) [15, 16]. In 1940 Alfred Tarski (1901-1983) proposed an elegant axiomatization of the calculus [17] which led to the creation of the relation algebras and shaped the subject as we know it today [15, 16]. Since the 1960s relations have been used in a categorical setting and have been applied to various areas of computer science [13].

The benefit of a calculus of relations over a calculus of functions is the possibility to easily encompass such features as non-determinism, partiality, and converse, which are often undefined in a functional context [18, 19, 13].

¹Since we confine ourselves here entirely to the calculus of binary relations, we will omit the adjective “binary” from now on.

2.1 Definition and Fundamental Operations

Definition

In everyday language, a *relation* refers to a certain aspect or a property in which two objects x and y are connected. This is expressed for example by “ x is smaller than y ”, “ x is equal to y ”, or “ x is the mother of y ”. Ever since De Morgan, a relation in mathematics is defined extensionally, meaning that we identify the term with its extension: a relation is a set of ordered pairs [16]. Denoting the Cartesian product of two sets X and Y by $X \times Y$, that is $X \times Y := \{(x, y) \mid x \in X \text{ and } y \in Y\}$, we define a *relation* R over the sets X and Y more precisely as a subset of $X \times Y$; in symbols: $R \subseteq X \times Y$. Given a relation R over X and Y we say that an element $y \in Y$ is *in the relation* R with an element $x \in X$ if $(x, y) \in R$. Instead of $(x, y) \in R$ we often write xRy .

Furthermore, we might understand the relation $R \subseteq X \times Y$ as a mapping from the set Y , which is also called the *domain* or *source*, to the set X , called the *range* or *target*. In this case we write $R : X \leftarrow Y$ and say that the relation is *of type* X *from* Y . If xRy we say that x is a *possible value* of R at y . If $X = Y$ we say that R is an *endo-relation* or an *relation on* X . Notice that, in contrast to the common notation of functions, we write the arrows between the source and the target set backwards. This comes in handy when dealing with relational composition. Often, we write R instead of $R : X \leftarrow Y$, leaving the source and target implicit. In the following, upper case letters R, S, \dots are used to denote relations, whereas upper case letters X, Y, \dots denote sets. Lower case letters x, y, \dots denote the elements of the corresponding sets.

Composition

Relations of appropriate type may be composed. If $R : X \leftarrow Y$ and $S : Y \leftarrow Z$ we define their *composition* $R \cdot S : X \leftarrow Z$ by

$$x(R \cdot S)z \equiv \exists y \in Y : xRy \wedge ySz. \quad (2.1)$$

‘ $R \cdot S$ ’ is read “ R after S ”. It is usually assumed that whenever one writes a composition it is well defined. Composition of relations is associative, i.e.

$$(R \cdot S) \cdot T = R \cdot (S \cdot T) \quad (2.2)$$

for all $R : X \leftarrow Y$, $S : Y \leftarrow Z$, and $T : Z \leftarrow W$. We can therefore drop the brackets in the following.

Moreover, the relation id_X on X defined by

$$id_X := \{(x, x) \mid x \in X\} \quad (2.3)$$

is called the *identity relation* on X . The identity relation is the unit of composition, which means that for $R : X \leftarrow Y$ it holds that

$$id_X \cdot R = R = R \cdot id_Y. \quad (2.4)$$

We will usually omit the subscript, allowing it to be deduced from the context.

Furthermore, if R is an endo-relation, it can also be composed with itself: $R \cdot R$. Since $R \cdot R$ is in turn a relation of the same type, we might compose it with R to get $R \cdot R \cdot R$. More formally, we let R^n denote the n th iterate of R . That is, R^n is the n -fold composition of R with itself, which is defined recursively by $R^n := R^{n-1} \cdot R$, where $R^0 := id$.

Sometimes it is helpful to illustrate reasoning with relations in diagrams. In such a diagram, the relation $R : X \leftarrow Y$ is represented by $X \xleftarrow{R} Y$, and its composition with a relation $S : Y \leftarrow Z$ is represented by $X \xleftarrow{R} Y \xleftarrow{S} Z$. In this way, we might illustrate the equation $id_X \cdot R = R$ as

$$\begin{array}{ccc} X & \xleftarrow{R} & Y \\ & id_X & \\ & & \xleftarrow{R} \\ & & X \end{array}$$

Any two paths between the same pair of sets in this diagram refer to the same relation. If a diagram has this property, we say that the diagram *commutes*, or the diagram is a *commutative diagram*.

Inclusion

Defining a relation $R : X \leftarrow Y$ as a subset of $X \times Y$ allows to compare different relations of the same type under inclusion:

$$R \subseteq S \equiv \forall x, y : xRy \Rightarrow xSy. \quad (2.5)$$

It is easily checked that \subseteq defines a *partial order* on the set of relations. For any X and Y the empty relation $0 := \emptyset$ is the smallest relation among the relations of type X from Y with regard to \subseteq , and the relation $\Pi := X \times Y$ is the largest. Moreover, composition is monotonic with respect to this order; in symbols:

$$R_1 \subseteq R_2 \wedge S_1 \subseteq S_2 \Rightarrow R_1 \cdot S_1 \subseteq R_2 \cdot S_2. \quad (2.6)$$

The proof of this also follows readily from the definition of relational composition.

We illustrate an inequation (i.e. an expression of the form $R \subseteq S$) in a diagram by inserting an inclusion sign at an appropriate point in the diagram. For example,

$$\begin{array}{ccc} X & \xleftarrow{T_1} & Y \\ S_1 \downarrow & \subseteq & \downarrow T_2 \\ Z & \xleftarrow{S_2} & W \end{array}$$

depicts the inequation $S_1 \cdot T_1 \subseteq S_2 \cdot T_2$. In such cases, the diagram is said to *semi-commute*.

Intersection and Union

The *intersection* or *meet* $R \cap S$ of two relations R and S of the same type $X \leftarrow Y$ is defined in set-theoretic terms by

$$x(R \cap S)y \equiv xRy \wedge xSy. \quad (2.7)$$

Equivalently, we might characterize the relation $R \cap S$ by the universal property:

$$U \subseteq (R \cap S) \equiv (U \subseteq R) \wedge (U \subseteq S) \quad (2.8)$$

for all $U : X \leftarrow Y$. In words, $R \cap S$ is the greatest lower bound of R and S with respect to the partial order \subseteq . It can easily be proven that intersection is

- *commutative*, that is $R \cap S = S \cap R$;
- *associative*, that is $(R \cap S) \cap T = R \cap (S \cap T)$;
- *idempotent*, that is $R \cap R = R$.

Since $(xRy \equiv xRy \wedge xSy) \equiv (xRy \Rightarrow xSy)$, intersection can be used to define inclusion:

$$R \subseteq S \equiv (R = R \cap S). \quad (2.9)$$

We might now reexpress monotonicity of composition from Equation (2.6) with respect to \subseteq :

$$R \cdot (S \cap T) \subseteq (R \cdot S) \cap (R \cdot T) \quad (2.10)$$

$$(R \cap S) \cdot T \subseteq (R \cdot T) \cap (S \cdot T). \quad (2.11)$$

This means that composition *distributes weakly* over intersection. To prove that from these two inequations the monotonicity of composition follows, we note first that by Equation (2.9) $S_1 = S_1 \cap S_2$ and $T_1 = T_1 \cap T_2$. Then

$$\begin{aligned}
S_1 \cdot T_1 &= S_1 \cdot (T_1 \cap T_2) \\
&\subseteq \{ \text{Equation (2.10)} \} \\
&\quad (S_1 \cdot T_1) \cap (S_1 \cdot T_2) \\
&\subseteq \{ \text{universal property of intersection} \} \\
&\quad S_1 \cdot T_2 \\
&\subseteq \{ \text{Equation (2.11)} \} \\
&\quad (S_1 \cdot T_2) \cap (S_2 \cdot T_2) \\
&\subseteq \{ \text{universal property of intersection} \} \\
&\quad S_2 \cdot T_2.
\end{aligned}$$

To prove the other direction we argue that by the universal property of intersection $S \cap T \subseteq S$ as well as $S \cap T \subseteq T$. Using monotonicity of composition we get that $R \cdot (S \cap T) \subseteq R \cdot S$ and $R \cdot (S \cap T) \subseteq R \cdot T$. But because of the universal property of intersection this holds if and only if $R \cdot (S \cap T) \subseteq (R \cdot S) \cap (R \cdot T)$, which shows Equation (2.10). Equation (2.11) may be shown in an analogous manner.

For the sake of completeness, we mention that the *union* or *join* $R \cup S$ of two relations $R, S : X \leftarrow Y$ is dually defined by the universal property

$$U \supseteq (R \cup S) \equiv (U \supseteq R) \wedge (U \supseteq S) \quad (2.12)$$

for all $U : X \leftarrow Y$. However, the union of relations will not play a role in the formalization.

Converse and Complement

Every relation $R : X \leftarrow Y$ has a *converse* or *inverse* relation $R^\circ : Y \leftarrow X$ which satisfies

$$bR^\circ a \equiv aRb. \quad (2.13)$$

It follows readily that converse is

- an *involution*, that is, it is its own inverse: $(R^\circ)^\circ = R$;
- *order-preserving*, that is $R \subseteq S \Rightarrow R^\circ \subseteq S^\circ$;
- *contravariant*, that is it reverses composition: $(R \cdot S)^\circ = S^\circ \cdot R^\circ$;

- *distributive over intersection*: $(R \cap S)^\circ = R^\circ \cap S^\circ$.

The converse of a relation should not be confused with its *complement*. Given the relation $R : X \leftarrow Y$, we will denote its complement in $X \times Y$ by $\neg R$. It is defined in set-theoretic notation by

$$\neg R := \{(x, y) \in X \times Y \mid (x, y) \notin R\}. \quad (2.14)$$

In pointwise reasoning the notation $x \not R y$ instead of $x(\neg R)y$ is often used, the most prominent example being \in and \notin . From the definitions follows that complement is

- an involution: $\neg(\neg R) = R$, and
- *order-reversing*, that is $R \subseteq S \Rightarrow \neg R \supseteq \neg S$.

2.2 Functions as a Special Case of Relations

Relations satisfying certain properties are of special interest to us. We call the relation $R : X \leftarrow Y$

- *entire* if $id \subseteq R^\circ \cdot R$, or equivalently, in pointwise notation, if for all $y \in Y$ there is at least one $x \in X$ with xRy ;
- *simple* if $R \cdot R^\circ \subseteq id$, or equivalently, if for all $y \in Y$ there is at most one $x \in X$ with xRy .

If R is both entire and simple we call it a *function*. In the following lower case letters f, g, \dots will always denote functions. In the case of a function $f : X \leftarrow Y$, we use the common notation $x = f y$ instead of $x f y$.

If $f : Z \leftarrow Y$ and $g : Y \leftarrow X$ are functions, then so is their composite $f \cdot g : Z \leftarrow X$. This implies that if f is an endo-relation, the *n*th iterate of f , denoted by f^n , is also a function. We reason:

$$\begin{aligned} id \subseteq f^\circ \cdot f &\Rightarrow \{ \text{monotonicity of composition} \} \\ &g^\circ \cdot id \subseteq g^\circ \cdot f^\circ \cdot f \\ &\Rightarrow \{ \text{monotonicity of composition} \} \\ &g^\circ \cdot g \subseteq g^\circ \cdot f^\circ \cdot f \cdot g \\ &\Rightarrow \{ g \text{ entire, transitivity of inclusion} \} \\ &id \subseteq g^\circ \cdot f^\circ \cdot f \cdot g \\ &\equiv \{ \text{contravariance of converse} \} \\ &id \subseteq (f \cdot g)^\circ \cdot (f \cdot g). \end{aligned}$$

This shows that if f and g are entire, their composition $f \cdot g$ is entire. To show that $f \cdot g$ is simple if f and g are simple we argue:

$$\begin{aligned}
g \cdot g^\circ \subseteq id &\Rightarrow \{ \text{monotonicity of composition} \} \\
&f \cdot g \cdot g^\circ \subseteq f \cdot id \\
&\Rightarrow \{ \text{monotonicity of composition} \} \\
&f \cdot g \cdot g^\circ \cdot f^\circ \subseteq f \cdot f^\circ \\
&\Rightarrow \{ f \text{ entire, transitivity of inclusion} \} \\
&f \cdot g \cdot g^\circ \cdot f^\circ \subseteq id \\
&\equiv \{ \text{contravariance of converse} \} \\
&(f \cdot g) \cdot (f \cdot g)^\circ \subseteq id.
\end{aligned}$$

Functions can be shunted from one side of an inequation to the other:

$$f \cdot R \subseteq S \equiv R \subseteq f^\circ \cdot S \quad (2.15)$$

$$R \cdot f^\circ \subseteq S \equiv R \subseteq S \cdot f. \quad (2.16)$$

To prove Equation (2.15) we argue that

$$\begin{aligned}
f \cdot R \subseteq S &\Rightarrow \{ \text{monotonicity of composition} \} \\
&f^\circ \cdot f \cdot R \subseteq f^\circ \cdot S \\
&\Rightarrow \{ f \text{ entire} \} \\
&R \subseteq f^\circ \cdot S \\
&\Rightarrow \{ \text{monotonicity of composition} \} \\
&f \cdot R \subseteq f \cdot f^\circ \cdot S \\
&\Rightarrow \{ f \text{ simple} \} \\
&f \cdot R \subseteq S.
\end{aligned}$$

Equation (2.16) follows from the first by taking converses and exploiting that converse is order-preserving and contravariant. The two equations are called the *shunting rules*.

We note that from the shunting rules it follows readily that inclusion of functions is equivalent to their equality:

$$(f \subseteq g) \equiv (f = g) \equiv (f \supseteq g), \quad (2.17)$$

since

$$\begin{aligned}
f \subseteq g &\equiv \{ \text{shunting rule Equation (2.15)} \} \\
&\quad id \subseteq f^\circ \cdot g \\
&\equiv \{ \text{shunting rule Equation (2.16)} \} \\
&\quad g^\circ \subseteq f^\circ \\
&\equiv \{ \text{converse is contravariant} \} \\
&\quad g \subseteq f.
\end{aligned}$$

2.3 Power Transpose and Residuation

Power Transpose

There are several possible representations of binary relations. Besides the representation by a subset of the Cartesian product $X \times Y$, or a mapping from Y to X , one can also think of the relation $R : X \leftarrow Y$ as a set-valued function $\mathcal{P}(X) \leftarrow Y$, where $\mathcal{P}(X)$ denotes the *power set* of X , i.e. the set of all subsets of X . We might define an operator Λ that sends a relation to its corresponding set-valued function by

$$(\Lambda R)y := \{x \mid xRy\}. \quad (2.18)$$

This operator Λ is called the *power transpose*. Note that if f is a function then Λf applied to an element returns a singleton set. Since $\Lambda R = \Lambda S$ if and only if $R = S$, the power transpose constitutes an isomorphism between relations and set-valued functions. We furthermore remark that the definition of the power transpose in Equation (2.18) can be restated by the universal property:

$$f = \Lambda R \equiv \in \cdot f = R \quad (2.19)$$

for all $f : \mathcal{P}(X) \leftarrow Y$ and $R : X \leftarrow Y$, where $\in : X \leftarrow \mathcal{P}(X)$ is the *membership relation*. This is summarized in the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{P}(X) & \xleftarrow{\Lambda R} & Y \\
\in & & R \\
& & X \longleftarrow
\end{array}$$

Since $\in \cdot (\Lambda R) = R$ by the universal property of the power transpose, it follows that $\in \cdot (\Lambda R) \cdot f = R \cdot f$ and thus

$$\Lambda(R \cdot f) = \Lambda R \cdot f. \quad (2.20)$$

Equation (2.20) is called the *fusion law*. By taking $f = id$ and $R = \in$ in the universal property of Λ we furthermore have $\Lambda\in = id$. This last equation is called the *reflection law*.

We further note that it follows from the definition of the power transpose and from the definition of intersection that

$$\begin{aligned} (\Lambda(R \cap S))y &= \{x \mid x(R \cap S)y\} \\ &= \{x \mid xRy \wedge xSy\} \\ &= \{x \mid xRy\} \cap \{x \mid xSy\} \\ &= (\Lambda R)y \cap (\Lambda S)y. \end{aligned}$$

Thus the power transpose distributes over intersection, and, by Equation (2.9), we have that it is also monotonic, i.e.

$$R \subseteq S \Rightarrow \Lambda R \subseteq \Lambda S. \quad (2.21)$$

Residuation

The term *residuation* was coined in 1939 by Ward and Dilworth, but the underlying idea has been around since the days of De Morgan and has attracted all contributors to a calculus of relations since [16]. Today, we understand residuation as a form of relational division. The *left residual* S/T of two relations S and T having a common source and the *right residual* $R \setminus S$ of two relations R and S having a common target are defined by the universal properties:

$$(R \subseteq S/T) \equiv (R \cdot T \subseteq S) \equiv (T \subseteq R \setminus S) \quad (2.22)$$

Illustrated in a diagram, this means that S/T is the largest relation (with respect to \subseteq) that makes the triangle

$$\begin{array}{ccc} X & \xleftrightarrow{T} & Z \\ S/T & \subseteq & S \\ & Y \longleftarrow & \end{array}$$

semi-commute. In an analogous manner, $R \setminus S$ is the largest relation such that

$$\begin{array}{ccc} X & \xleftrightarrow{R \setminus S} & Z \\ R & \subseteq & S \\ & Y \longleftarrow & \end{array}$$

semi-commutes. Using pointwise reasoning, the Equation (2.22) translates to:

$$y(S/T)x \equiv (\forall z : xTz \Rightarrow ySz) \quad (2.23)$$

$$x(R \setminus S)z \equiv (\forall y : yRx \Rightarrow ySz). \quad (2.24)$$

The notation $/$ and \backslash respectively was introduced in 1948 by G. Birkhoff [20]. For reasons of clarity, we mention that $/$ and \backslash are in the literature also called *right* and *left division* respectively (e.g. [14]). Note that the adjectives “right” and “left” are interchanged in the names, i.e. S/T is called the “left” residual as well as the “right” division of S with respect to T , and $R\backslash S$ is called the “right” residual and “left” division². In what follows, we favor “residuation” over “division” for reasons that become apparent when directing our attention to the formalization of resilience in Chapter 3.

An immediate consequence of the universal properties of the left and right residuals (Equation (2.22)) are the *cancellation laws*:

$$R \cdot (R\backslash S) \subseteq S \quad (2.25)$$

$$(S/T) \cdot T \subseteq S. \quad (2.26)$$

Since converse is a contravariant operator, the two residuals can also be expressed in terms of each other:

$$S/T = (T^\circ \backslash S^\circ)^\circ \quad \text{and} \quad R\backslash S = (S^\circ / R^\circ)^\circ.$$

In addition, it follows straight from the definition of the residuals that S/T is monotonic in S , i.e.

$$S \subseteq U \Rightarrow S/T \subseteq U/T, \quad (2.27)$$

and anti-monotonic in T , i.e.

$$T \subseteq V \Rightarrow S/T \supseteq S/V. \quad (2.28)$$

In contrast, $R\backslash S$ is anti-monotonic in R and monotonic in S .

We want to note furthermore that in the case of f being a function, the following laws hold:

$$(R\backslash S) \cdot f = R\backslash(S \cdot f) \quad (2.29)$$

$$f^\circ \cdot (R\backslash S) = (R \cdot f)\backslash S. \quad (2.30)$$

²We might try to explain the opposed use of “left” and “right”. Let’s look at the case of the right residual or left division $R\backslash S$, which is defined by the property $R \cdot T \subseteq S \equiv T \subseteq R\backslash S$. In this case, we would say that we “divide” S by the “left” relation of the composition $R \cdot T$, i.e. by R , and thus we refer to it by “left” division. At the same time, we notice that what “remains” or what “rests” of the composition is the right side, i.e. T , and thus we say “right” residual. In the case of the “left” residual or “right” division, this works the other way around.

We are going to make use of them frequently in the next Chapter. To prove Equation (2.29) we reason:

$$\begin{aligned}
T \subseteq (R \setminus S) \cdot f &\equiv \{ \text{shunting of function } f \} \\
&T \cdot f^\circ \subseteq R \setminus S \\
&\equiv \{ \text{universal property of residuation} \} \\
&R \cdot T \cdot f^\circ \subseteq S \\
&\equiv \{ \text{shunting of } f \} \\
&R \cdot T \subseteq S \cdot f \\
&\equiv \{ \text{universal property of residuation} \} \\
&T \subseteq R \setminus (S \cdot f).
\end{aligned}$$

The proof of the Equation (2.30) is done similarly.

Chapter 3

Formalization of Resilience

In this chapter, we turn to the formalization of resilience. We start with an analysis of its meaning in everyday language and within the ecological literature to identify the building blocks of the term (Section 3.1). In the next sections, we are translating these building blocks into mathematical language (Section 3.2), introduce the general idea of formalizing resilience as a relation (Section 3.3), and finally combine the mathematical representations of the building blocks to arrive at a formal definition of resilience (Section 3.4).

3.1 Grammatical and Semantical Investigation

We are going to analyse the structure and meaning of resilience. Thereby, we hope to identify a small set of elements, which need to be understood in order to make meaningful statements about the term. We consider first the everyday and thereafter its technical usage, because technical terminology is usually a refinement of the everyday language. When formulating resilience in mathematical terms in the following sections, we will proceed along the same lines, i.e. we start with a general definition of resilience and refine it later to capture the technical usage of the term. This approach seems promising in the case of resilience since the everyday usage of the term might suggest a common denominator for the different definitions of the term within the ecological literature.

3.1.1 Everyday Usage: Oxford Dictionary of English

We consulted the Oxford Dictionary of English [21] for a definition of the current use of “resilience” in everyday language. It lists “resilience” as a derived term

of “resilient”, for which it provides the following definition:

1. (of a substance or object) able to recoil or spring back into shape after bending, stretching or being compressed,
2. (of a person or animal) able to withstand or recover quickly from difficult conditions.

It provides furthermore the following example sentence: “The fish are *resilient* to most infections” (emphasis in original).

From the definitions and the example sentence it follows first of all that resilience is a relative concept: something (e.g. the fish) *is resilient to* something (e.g. the infections).

Furthermore, in order to decide whether something is back into shape after something, or recovered from something, we need notions of the *situation before* and of the *situation after*, which we compare with one another: if they are similar we would answer in the affirmative, if not, we would answer in the negative. Therefore, the concept of resilience also presupposes a *notion of similarity* between situations.

3.1.2 Technical Usage: Resilience in the Ecological Literature

Within the ecological literature, resilience refers to the ability of ecosystems to recover from potential disturbances. There are, however, two distinct mainstream definitions of resilience, which differ in the manner of evaluating this ability. Both have their roots in theoretical ecology and make thus use of mathematical language [4]. In the following we give a short overview of them. The reader should keep in mind that different authors emphasize and chart different aspects of these definitions, and that for reasons of space this overview cannot be comprehensive.

The traditional definition of resilience in ecology assumes the existence of a (global) equilibrium [6] and is defined by S.L. Pimm in terms of “how fast the variables return towards their equilibrium following a perturbation” ([8], p. 322). In this context it is usually understood that the faster a system returns from the perturbation, the more resilient it is or the higher its resilience to this perturbation. Since this definition by Pimm is often cited, we will call it the

Pimm resilience.

The other mainstream definition of resilience goes back to the work of C.S. Holling [5, 22]. It assumes the existence of multiple equilibria [23] and is defined as “the ability of these [ecological] systems to absorb changes . . . , and still persist” ([5], p. 17). It can be measured by “the magnitude of disturbance that can be absorbed before the system redefines its structure by changing the variables and processes that control behavior” ([24], Chapter “Surprise, Resilience, and Flexibility”). In this context, the term “structure” refers to the domain of attraction of stable equilibria, i.e. the system changes its structure if it does not return to the same equilibrium as before the disturbance [23]. The more the system can be disturbed and still remains its structure, the higher its resilience. There is, however, also a notion of “being resilient to one specific perturbation” in the sense of Holling: a system is resilient to a specific disturbance if it does not change its structure. In this case, not the time of return but only the fact that it returns is of significance. We will call this definition of resilience the *Holling resilience*.

Both definitions rely on the assumption of one or more equilibria of the ecosystem. There are, however, ecologists who question this assumption arguing that ecosystems are continuously changing in response to perturbations and varying external factors [1]. Instead of considering equilibria, the behavior of the system after a disturbance is therefore sometimes compared to a reference dynamic. This reference dynamic is then referred to as the “normal” behavior of the system, which does not need to consist only of an equilibrium. Nevertheless, it includes the equilibrium as a special case. In this manner, we generalize both of the above given definition of resilience to relate not only to equilibria but also to (arbitrary) reference dynamics.

3.2 Formalization of the Primitive Concepts

In general, a definition is supposed to reduce the to be specified term, in our case resilience, to more basic terms. In the last section, we therefore analyzed the dictionary definition and the definitions of resilience within the ecological literature with regard to the more basic terms of resilience and their underlying structure. It followed from this analysis that resilience is a relative concept: it expresses a certain relation between *entities* and *disturbances* of these entities (“an entity *is resilient to* a disturbance”) with respect to what we consider to be

the *situation before* and *after* the disturbance, and with respect to certain *similarity criteria* that are used to compare both situations. We therefore have to understand and formalize the primitive concepts “entity”, “disturbance”, “situation before/after”, and “similarity criteria”, before we formally define resilience.

Entity

As common in mathematics, we interpret an *entity* as a dynamical system in a given state. To develop the basic ideas underlying the concept of resilience, we consider the simplest case of a dynamical system, a discrete, deterministic dynamical system given by a transition function

$$f : X \leftarrow X \quad (3.1)$$

where X is the set of states. Given the current state of the system $x \in X$, the transition function tells us the next state of the system: fx . Having the calculus of relations outlined in Chapter 2 in mind, we want to think of the transition function as a simple and entire relation. This will prove useful when formally defining resilience. The transition function represents for us the “normal behavior in time” of the entity.

To represent the course of a state in time, we introduce the notion of a trajectory: the finite or infinite sequence of states $(x, fx, f^2x, \dots, f^n x)$ or (x, fx, f^2x, \dots) respectively is called the *trajectory* of the *initial state* x . In the former case the trajectory corresponds to the possible behavior of the system in state x within n transitions (i.e. we apply the transition function f n -times to x), in the latter it represents the run of the state unbounded in time. In the case of a finite trajectory $(x, fx, f^2x, \dots, f^n x)$ we call the last state x_n the *endpoint* of the trajectory. To simplify and shorten notation we introduce the relation $\gamma_n : X^{n+1} \leftarrow X$ for $n \in \mathbb{N} \cup \{\infty\}$ (where, as common, $\infty + 1 = \infty$) which maps a state to its trajectory:

$$\gamma_n x = \begin{cases} (x, fx, f^2x, \dots, f^n x) & n < \infty \\ (x, fx, f^2x, \dots) & n = \infty. \end{cases} \quad (3.2)$$

Since f is a function, so is γ_n .

Example 1 We interpret the example sentence for “resilient” provided by the Oxford Dictionary of English (Section 3.1.1; “The fish are *resilient* to infections”) as a simple population model, where the state of the fish is given by their population size. For computational reasons we take $X :=$

\mathbb{R}_+ (instead of the from a biological perspective more reasonable $X = \mathbb{N}$).

We consider the following transition function, which is considered to be the discrete analog of the logistic differential equation [27]:

$$f : \mathbb{R}_+ \leftarrow \mathbb{R}_+$$

that maps a state $x \in \mathbb{R}_+$ to

$$f x := \begin{cases} rx(1 - \frac{x}{K}) & \text{if } x \in (0, K) \\ 0 & \text{otherwise.} \end{cases}$$

where $r, K > 0$. In this model, r refers to the growth rate of the population and K is called the carrying capacity. We might interpret a state x as the population size this year, and its next state $f x$ as the population size in the following year. Thus a trajectory records the size of the population through the years.

The reason we chose a discrete, deterministic dynamical system to explain the basic ideas of the formalization is its structural simplicity. In Section 3.4.4 we are going to show that the definitions proposed for the discrete, deterministic dynamical system are also meaningful in the case of a discrete, non-deterministic dynamical system. Continuous-time and stochastic dynamical systems are not of concern in this thesis, although it might be interesting to relate or extend the definitions developed here to these systems as well.

Disturbance

A *disturbance* is usually thought of either as an influence or a force acting on the entity and thereby it is changing the entity, or else as the outcome of the action of the force on the entity, that is, as the changed entity. In this thesis we focus on the aspect of the changing entity, i.e. we do not want to take the force or cause of the change explicitly into account; only its outcome is of interest.

Thinking of a disturbance in terms of its outcomes means in mathematical terms that we describe a disturbance of an entity by changes in the state of the system. We represent such a change in the system state by an ordered pair (x', x) where x refers to the current state of the system and x' to the outcome of the disturbance. We do not think of x' as a state of the system at a later point in time. Instead, we consider the system “jumping” from the initial state x to the state x' . The next state of the system is not $f x$ anymore, but $f x'$. Note that the order of the pair from the initial state to the disturbed one is again “backwards”, and that if $x' = x$ the ordered pair (x, x) refers to the *no-change*

case. According to this representation of a disturbance of an entity, we might think of every relation of the type $X \leftarrow X$ as a mapping from initial system states to disturbed states, or in set-theoretic terms as a set of changes in state.

Example 2 Referring again to the example sentence in Section 3.1.1, the infections might reduce and thus change the size of the fish population. For example, if the initial state of the population was the size $x = 200$ fish, the disturbed state might be the size $x' = 100$ fish, thus 100 fish died due to the infections. Both of these states refer to the same year, i.e. actually the size of the fish population this year was supposed to be 200, but due to the infections it is just 100. This change in state is represented by the ordered pair $(100, 200)$.

Situation before/after

From the dictionary definition in Section 3.1 follows that an entity is resilient to a disturbance if the situation after the disturbance occurred is similar to the situation before. We therefore need a mathematical interpretation of a “situation” as well as an interpretation of “the situation before the disturbance” and of “the situation after the disturbance”.

Since the notion of a system state is to specify the entity’s situation at a particular instant in time, we might formalize a *situation* by a state of the system. Now, given a change in state (x', x) , a first idea might be to take the initial state x to be the *situation before* the disturbance occurred. Furthermore, as the transition function f represents the “normal behavior” of the system, we can interpret $f x'$ to be the result of the reaction of the system to the change (x', x) . So we might take $f x'$ as a representation of the *situation after* the disturbance occurred.

As we move along with the formalization, we will notice that these interpretations are not all-embracing: just like the interpretation of a “family” depends on the cultural context, the interpretations of “situation” and of the “situation before/after” varies with the focus of the investigation. Thus, because the concept of resilience depends on these interpretations, its formal definition will vary according to the chosen interpretations. To begin with, we stick to the simple interpretations given above.

Example 3 Given the change in state $(100, 200)$ of the fish population

due to the infections from Example 2, the state $x = 200$ represents the situation before the fish were infected. The situation after the change to $x' = 100$ is the population size in the next year, which is for $r = 2$ and $K = 1000$ the size $f x' = 180$ fish.

Similarity Criteria

The *similarity criteria* are used to ascertain whether or not we consider two situations to be similar. Formalizing a situation as a system state, we might represent these similarity criteria as a relation S on the set of states: $S : X \leftarrow X$. Thus, if ySx we understand that the states x and y are similar to each other with respect to the similarity relation S . For S to reflect our intuitive notion of similarity, we expect it to be

1. *reflexive*, that is $id_X \subseteq S$, i.e. every state x is equivalent to itself: xSx ;
2. *symmetric*, that is $S = S^\circ$, i.e. if xSy then ySx .

A relation S that satisfies these conditions is usually called a *tolerance* or *compatibility* relation [28]. Note that although S represents our similarity criteria it will in general not correspond to a *similarity relation* as used inter alia in fuzzy set theory (e.g. [28]).

Example 4 The relation $S := id_X$, i.e.

$$ySx :\equiv y = x,$$

is possibly the simplest tolerance relation we might think of that incorporates the notion of similarity. It does this, however, in a narrow sense, in that every state x is only similar to itself, which means not even a small deviation from it is allowed. We note that the relation S defined in this way is not only reflexive and symmetric, but also transitive: $S \cdot S \subseteq S$, which means in pointwise notation that $xSy \wedge ySz \Rightarrow xSz$. An endo-relation which is reflexive, symmetric and transitive is usually called an *equivalence relation*. Thus, every equivalence relation is by definition a tolerance relation. In general, however, we do not require S to be transitive¹. In the case of the fish population (Example 1), this relation S means that we consider two population sizes similar if and only if they are equal.

Example 5 Suppose we have a metric on X :

$$d : \mathbb{R}_+ \leftarrow X \times X$$

which associates to each pair of states (x, y) a distance $d(x, y)$ between them. Given furthermore a threshold value $T \in \mathbb{R}_+$ for the amount of deviation we allow. We might define S by

$$xSy \equiv d(x, y) \leq T$$

Thus, x is similar to y if their distance $d(x, y)$ is less than the threshold T . S thus defined is a tolerance relation. For the simple population model of Example 1 we could take d to be the Eukclidean metric on \mathbb{R}_+ . Then, according to the just defined tolerance relation, two population sizes are similar if they differ in less than $T \in \mathbb{R}_+$ fish.

Example 6 Suppose we have a partition of X into two subsets S_1 and S_2 , i.e. $S_1 \cup S_2 = X$ and $S_1 \cap S_2 = \emptyset$. We could take S to be defined by

$$xSy \equiv (x \in S_1 \wedge y \in S_1) \vee (x \in S_2 \wedge y \in S_2)$$

If we interpret S_1 as the set of viable states and S_2 as the set of non-viable states, xSy means that x and y are either both viable states or both are non-viable. S is also an equivalence relation. In the case of our population model (Example 1) with parameters $r = 2$ and $K = 1000$, we might interpret the sets $S_1 := (0, K)$ and $S_2 := \{0\} \cup [K, \infty)$ to be the viable and non-viable set respectively, since if $x \in (0, K)$ then $f^n x \in (0, K)$ for all $n \in \mathbb{N}$ (which means that there are always going to be fish), whereas if $x \in \{0\} \cup (K, \infty)$ then $f^n x = 0$ for all $n \in \mathbb{N}$ (which means that there are no fish from the next year on).

Example 7 Another possible relation S might take both the distance between states and their viability into account. Assume as before that we have a metric $d : X \times X \rightarrow \mathbb{R}_+$, a threshold $T \in \mathbb{R}_+$, and a partition of

¹We give an example of why we do not expect the relation which represents the similarity criteria to be an equivalence relation, i.e. why we do not expect it to be transitive as well as reflexive and symmetric. Consider the words “TREE”, “FREE”, “FLEE”, “FLOE” and “FLOW”. We could consider every pair of neighboring words to be similar because they differ in only one letter. However, we would not consider the first and the last similar as they have not even one letter in common. This is not possible if we require transitivity. Transitivity in this example means that if we consider each neighboring pair similar, then we should consider all possible pairs similar, even the pair “TREE” and “FLOW”. In mathematical terms: given the set $O_x := \{y \in X : ySx\}$ and $O_z := \{y \in X : ySz\}$. In the case of an equivalence relation these sets are either equal or disjoint, i.e. either $O_x = O_y$ or $O_x \cap O_y = \emptyset$. In the case of a tolerance relation these might have a nonempty intersection without being equal.

X into the subsets S_1 and S_2 , where we interpret S_1 again as the set of viable states and S_2 as the set of non-viable states. Then the relation S defined by

$$xSy := (x \in S_2 \wedge y \in S_2) \vee (x \in S_1 \wedge y \in S_1 \wedge d(x, y) \leq T)$$

is a tolerance relation. In this case, xSy means that either x and y are both non-viable states (in which case we are not interested any further in their distance since what matters are that both are non-viable) or they are both viable and their distance is less than the threshold T . For the fish population with parameter values $r = 2$ and $K = 1000$ this means that we consider two population sizes x and y to be similar if $x, y \in S_2 = \{0\} \cup [K, \infty)$, or $x, y \in (0, K)$ and they have a difference of no more than T fish.

3.3 Resilience as a Relation

From the analysis of the Oxford Dictionary of English definition followed that the concept of resilience is a relative one: it expresses a certain relation between entities and disturbances of these entities (“an entity *is resilient to* a disturbance”). The mathematical notion of a relation (see Chapter 2) thus plies itself with using it as a basis for our formalization. Moreover, if we use the representations of an entity and a disturbance from the last section, the general concept of resilience might be represented by a relation of type $X \leftarrow X$: we formalized an entity as the discrete, deterministic dynamical system $f : X \leftarrow X$ in a given state $x \in X$, and represented a disturbance of an entity by a change in state. In this way, an ordered pair (x', x) might represent the system f in state x and a change in the state of the system to x' , or shorter, (x', x) might represent a change in the system f from x to x' . Thus, a relation of type $X \leftarrow X$ relates the states of the system f to possible disturbed states; in set-theoretic terms it represents a set of changes in the state of the system.

As a formal definition for resilience we are therefore looking for a relation of type $X \leftarrow X$, which we will denote by Res and which we will call the *resilience relation* of the system f , such that $x' Res x$ represents the fact that the system f in state x is resilient to the change in state to x' (with respect to the other primitive concepts: situation before/after, similarity criteria). If $x' Res x$ we call (x', x) a *resilient change* of the system f . Thus, the resilience relation Res of a system f might be interpreted in set-theoretic terms as the set of all resilient changes of the system f .

In the remaining part of this subsection we will introduce terminology belonging to the realm of resilience. In doing so we are well aware of the fact that we still owe it to the reader to give a formal definition of this resilience relation, which we will return to in the subsequent section. Examples on how to use the definitions of this subsection are provided in the following section as well.

3.3.1 The Resilience of a State

What we are speaking of most of the time is not the resilience Res of a system, but the resilience of a system in a given state as the representation of an entity. We might use the resilience relation of the system, however, to define the resilience of a state extensionally. With the help of the power transpose operator for relations (Section 2) we define:

Definition 1 Let $f : X \leftarrow X$ be a transition function and suppose that $Res : X \leftarrow X$ is the resilience relation of the system f . The set

$$\Lambda Res x$$

where Λ is the power transpose operator, is called *the resilience of the system f in state x* .

Because $\Lambda Res x = \{y \in X \mid y Res x\}$ by the definition of the power transpose, the resilience of the system f in a state x in the sense of Definition 1 is the set of all states $y \in X$ for which the system in state x is resilient to the change in state to y , i.e. for which $y Res x$.

3.3.2 Being Resilient to a Set of Changes in State

Sometimes, we are interested in saying when a system in a given state is resilient to a set of changes in its state, for instance studies usually investigate the behavior of a system facing a specific kind of stressor with varying intensity. We represent such a set of changes in state by a *change relation* C which is, as a set of changes in state, just like the resilience relation, of type $X \leftarrow X$:

$$C : X \leftarrow X. \tag{3.3}$$

We define when a state x of a system f is resilient to a set of changes given by the change relation C :

Definition 2 Let $Res : X \leftarrow X$ be the resilience relation of the system $f : X \leftarrow X$, and let $C : X \leftarrow X$ be the considered change relation. The system f in state x is resilient to C if

$$\Lambda C x \subseteq \Lambda Res x.$$

According to Definition 2 the system f in state x is resilient to the set of changes C if all changes given by C with initial state x are resilient changes.

We go a step further and define when a system is resilient to a set of changes C :

Definition 3 Let $Res : X \leftarrow X$ be the resilience relation of the system $f : X \leftarrow X$. Let $C : X \leftarrow X$ be the considered change relation. The system f is resilient to C if

$$C \subseteq Res.$$

Definition 3 says that a system is resilient to a set of changes if it is in all its states resilient to that set of changes in the sense of Definition 2, because

$$C \subseteq Res \equiv \forall x : \Lambda C x \subseteq \Lambda Res x. \quad (3.4)$$

The set $C \cap Res$ might be interpreted to be the set of considered changes to which the system is resilient to. Since

$$C \subseteq Res \equiv (C = C \cap Res), \quad (3.5)$$

the condition in Definition 3, to require all considered changes to be resilient changes, is therefore equivalent to asking that the set of considered changes to which the system is resilient to, $C \cap Res$, is the set C of considered changes itself.

Since

$$\begin{aligned} \Lambda C x \subseteq \Lambda Res x &\equiv \{ \text{universal property of intersection} \} \\ &\Lambda C x = \Lambda C x \cap \Lambda Res x \\ &\equiv \{ \text{power transpose distributes over intersection} \} \\ &\Lambda C x = \Lambda (C \cap Res) x \end{aligned}$$

we have that if C encompasses only changes with initial state x , i.e. $C = \{(x', x) : x' \in X\}$, then Definition 3 holds if and only if Definition 2 holds, which means that the system is resilient to C if and only if the system in state x is resilient to C . Definition 3 is a meaningful extension of Definition 2.

3.3.3 Comparing States and Systems with Regard to Their Resilience

Often, we want to know which state of a system is more resilient than another in order to decide for instance in which state the system is more likely to change, and in which state we need to be more attentive if a change is not preferred. Using the resilience relation, we might compare two states with regard to their resiliences:

Definition 4 Let $Res : X \leftarrow X$ be the resilience relation of the system $f : X \leftarrow X$. The system f is more resilient in state x than in state y if

$$\Lambda Res x \supseteq \Lambda Res y.$$

According to Definition 4, we call a state x of the system f more resilient than another state y if it holds that if the system in state y is resilient to a change in state to $z \in X$ then the system in state x is also resilient to the change to z . In this case, thus, we know that if the system in state y is resilient to a set of changes $C : X \leftarrow X$ and if the set of disturbed states to which x might be mapped to by C is included in the set of disturbed states to which y might be mapped to by C , in symbols: if $\Lambda C x \subseteq \Lambda C y$, then f is also resilient to C in state x :

$$\begin{aligned} \Lambda C x &\subseteq \{ \text{assumption} \} \\ &\quad \Lambda C y \\ &\subseteq \{ f \text{ in } y \text{ resilient to } C \} \\ &\quad \Lambda Res y \\ &\subseteq \{ f \text{ in } x \text{ more resilient than in } y \} \\ &\quad \Lambda Res x. \end{aligned}$$

If neither $\Lambda Res x \supseteq \Lambda Res y$ nor $\Lambda Res x \subseteq \Lambda Res y$ the two states are not *comparable* with regard to their resilience in the sense of Definition 4.

More generally, we can compare two states of different systems of the same type $X \leftarrow X$ with regard to their resiliences:

Definition 5 Let $Res_f : X \leftarrow X$ and $Res_g : X \leftarrow X$ be the resilience relations of the systems $f : X \leftarrow X$ and $g : X \leftarrow X$ respectively. The system f in state x is more resilient than the system g in state y if

$$\Lambda Res_f x \supseteq \Lambda Res_g y.$$

In words, we call the system f in state x more resilient than the system g in state y if it holds that if the system g in state y is resilient to a change in state to $z \in X$ then the system f in state x is also resilient to the change to z . Thus, in this case, we might show that if the system g in state y is resilient to a set of changes $D : X \leftarrow X$ and $\Lambda D y \supseteq \Lambda C x$ where $C : X \leftarrow X$ is another set of changes, then the system f in state x is resilient to C :

$$\begin{aligned}
\Lambda C x &\subseteq \{ \text{assumption} \} \\
&\quad \Lambda D y \\
&\subseteq \{ f \text{ in } y \text{ resilient to } D \} \\
&\quad \Lambda Res_g y \\
&\subseteq \{ f \text{ in } x \text{ more resilient than in } y \} \\
&\quad \Lambda Res_f x.
\end{aligned}$$

Furthermore, the resiliences of the system f in state x and of the system g in state y are *comparable* with regard to their resiliences in the sense of Definition 5 if either $\Lambda Res_f x \supseteq \Lambda Res_g y$ or $\Lambda Res_f x \subseteq \Lambda Res_g y$.

Sometimes, we might also be interested in comparing the resilience relations of different systems, for instance to decide which dynamical behavior is more resilient in the sense that more disturbances can be absorbed by the system in all its states. We define:

Definition 6 Let $Res_f : X \leftarrow X$ and $Res_g : X \leftarrow X$ be the resilience relations of the systems $f : X \leftarrow X$ and $g : X \leftarrow X$ respectively. The system f is *more resilient than* the system g if

$$Res_f \supseteq Res_g.$$

In words, the system f is more resilient than the system g (with respect to their resilience relations Res_f and Res_g) if all changes (x', x) that are resilient changes of the system g are also resilient changes of the system f . It follows from the definition of the power transpose that the system f is more resilient than the system g in the sense of Definition 6 if and only if it holds for each $x \in X$ that the system f in state x is more resilient than the system g in state x . Furthermore, if f is more resilient than g , and $C \subseteq D$ where $C, D : X \leftarrow X$ are sets of changes, we might easily show that if g is resilient to D then so is f

resilient to C :

$$\begin{aligned}
C &\subseteq \{ \text{premise} \} \\
&D \\
&\subseteq \{ f \text{ resilient to } D \} \\
&Res_g \\
&\subseteq \{ f \text{ more resilient than } g \} \\
&Res_f.
\end{aligned}$$

It is in this sense that f is more resilient than g . The resiliences of the system f and of the system g are *comparable* with regard to their resiliences in the sense of Definition 6 if either $Res_f \supseteq Res_g$ or $Res_f \subseteq Res_g$.

3.3.4 A Note on Change Relations with External Inputs

As mentioned in Section 3.2, we might interpret a disturbance also as a force acting on the entity. We demonstrate briefly how this case might be reduced to the case above where we considered instead only the outcomes of the force acting on the entity as the disturbance.

Consider the system $f : X \leftarrow X$ in a state $x \in X$, and let the set E represent all considered forces acting on the entities; as such E represents the disturbances according to the interpretation of a disturbance examined in this subsection. We call E the set of *external inputs*. As the concept of resilience refers to a relation between entities and disturbances, we are now looking for a relation $Res_E : E \leftarrow X$. We want to define Res_E in terms of the given resilience relation $Res : X \leftarrow X$. The idea is that an external input acts on the system states and thereby changes the states. We represent the external inputs acting on the system states and their resulting changes in state by the following relation:

$$C_E : X \leftarrow X \times E \tag{3.6}$$

which we denote by C_E to express that it is a change relation *with external input* (in contrast to the change relation C – without input – above). Given a state x and an external input e , C_E tells us if the initial state x might be changed to the state y as a result of e , which is the case if $y C_E(x, e)$. We do not expect C_E to be simple (i.e. there might be several possible changes in state due to e), but to be entire. We expect that if the external input e has no influence on a system state x , then the state remains unchanged, i.e. in this case $x C_E(x, e)$.

Example 8 In the case of the fish population, we might as well interpret the infections as the disturbances. Consider for instance the following change relation with input:

$$C_E : \mathbb{R}_+ \leftarrow \mathbb{R}_+ \times E$$

where \mathbb{R}_+ refers again to the size of the fish population and $E = \{e_1, e_2, e_3\}$ is the set of infections, and which is defined by

$$\Lambda C_E(x, e_i) := \left\{ x - \frac{i * 5x}{100}, x - \frac{i * 10x}{100}, x - \frac{i * 15x}{100} \right\}.$$

for $i = 1, 2, 3$. Thus, we consider in our example three infections e_1 , e_2 , and e_3 due to which a certain percentage of the fish population dies.

We now define that a system in a given state is resilient to an external input if it is resilient to all resulting changes in state:

Definition 7 Let $Res : X \leftarrow X$ be the resilience relation (in terms of changes in state) of the system $f : X \leftarrow X$. Let $C_E : X \leftarrow X \times E$ be the change relation with input. The resilience relation $Res_E : E \leftarrow X$ (in terms of external inputs changing the system) is defined by

$$e Res_E x \equiv \forall x' : x' C_E(x, e) \Rightarrow x' Res x$$

Furthermore, we would say that a system f in a state x is resilient to a set of external inputs $E_1 \subseteq E$ if it is in that state resilient to each external input e which is an element of E_1 :

Definition 8 Let $Res_E : X \leftarrow X$ be the resilience relation (in terms of external inputs changing the system) of the system $f : X \leftarrow X$. The system f in state x is resilient to the set of external inputs $E_1 \subseteq E$ if

$$E_1 \subseteq \Lambda Res_E x.$$

The interested reader might develop definitions of further terminology analogously to above.

3.4 Formal Definitions of Resilience

– Resilience as a Residual Relation –

The objective of this section is to propose possible formal definitions of the resilience relation. We begin with combining the mathematical representations

of the primitive concepts (entity, disturbance, situation before/after, similarity criteria) from Section 3.2 to give a first formal definition of resilience (Section 3.4.1). Next, we refine this definition to better agree with the interpretation of resilience within the ecological literature (Section 3.4.2), and we are going to show how these different definitions are related (Section 3.4.3). Lastly, we give interpretations of the definitions of the resilience relations from Section 3.4.1 and 3.4.2 in the case of a discrete, non-deterministic dynamical system (Section 3.4.4).

3.4.1 The Resilience Relation: A First Definition

We interpreted the Oxford Dictionary of English definition of “resilient” in Section 3.1 to mean that an entity is resilient to a disturbance if its situation after the disturbance occurred is similar to the situation before the disturbance. Hence, translating our interpretation of the dictionary definition into the mathematical representations of the primitive concepts “entity”, “disturbance”, “situation before/after”, and “similarity criteria” given in Section 3.2, we record that what we are looking for is a relation $Res : X \leftarrow X$ which satisfies

$$x' Res x \equiv (f x') S x. \quad (3.7)$$

In words, we set $x' Res x$ if the next state $f x'$ of x' under the transition function f is similar to the initial state of the system x : $(f x') S x$. The following diagram illustrates our expectations for the resilience relation

$$\begin{array}{ccc} X & \longrightarrow & \\ Res \downarrow \subseteq & S & \\ X & \xrightarrow{f} & X \end{array}$$

whereat we expect Res to be the largest relation with respect to the partial order \subseteq on relations such that the diagram is still semi-commutative. But this is the case if and only if the relation Res corresponds to the right residual (see Chapter 2) of f with respect to S .

Summarizing our considerations, we formulate our first definition of a resilience relation:

Definition 9 The *static resilience relation* $Res : X \leftarrow X$ of the system $f : X \leftarrow X$ with respect to the tolerance relation $S : X \leftarrow X$ is the right residual of f with respect to S . In symbols

$$Res := f \setminus S.$$

We call $Res = f \setminus S$ the static resilience relation since the focus of the definition is to stay similar to the initial state, i.e. not to change.

Example 9 We come back to the fish population, which we modelled by the transition function f given in Example 1. We are going to illustrate on the one hand the definition of the static resilience relation. On the other hand, we give examples of the definitions of the previous section. Consider the tolerance relation S of Example 7. According to the static resilience relation $Res = f \setminus S$ we consider the fish population resilient to a change in its size (x', x) if $f x' S x$ (i.e. the size of population next year if this year's population size is x' , is similar to the initial size x of the population with respect to S). Assuming that the parameters of f are $r = 2$ and $K = 1000$, and the parameter T of S is 100 (i.e. we consider two viable population sizes to be similar with respect to S if they differ in no more than 100 fish), we have that the fish are resilient to the change $(100, 200)$ since $f 100 = 180$, but they are not resilient to the change $(50, 200)$ since $f 50 = 95$. In fact, the *resilience* of the fish of size $x = 200$ (according to Definition 1) is the set

$$\begin{aligned} \Lambda Res 200 &= \{y \in \mathbb{R} \mid (f y) S 200\} \\ &= \{y \in \mathbb{R} \mid d(f y, 200) \leq 100\} \\ &= \{y \in \mathbb{R} \mid f y \in [100, 300]\} \\ &= [500 - 100\sqrt{20}, 500 - 100\sqrt{10}] \\ &\quad \cup [500 + 100\sqrt{10}, 500 + 100\sqrt{20}] \\ &\approx [53, 184] \cup [816, 947]. \end{aligned}$$

Note that $200 \notin \Lambda Res 200$, which means that the fish population is not resilient to the no-change change $(200, 200)$. Furthermore, the resilience of $x \in S_2 = \{0\} \cup [K, \infty)$, i.e. the resilience of a non-viable population size, is again S_2 since $f x$ is non-viable for non-viable x and viable for viable x . Thus, if $x \in S_1 = (0, K)$ and $y \in S_2$ then changes (x, y) and (y, x) are not resilient.

Consider furthermore the *change relations* $C_1, C_2, C_3 : \mathbb{R}_+ \leftarrow \mathbb{R}_+$ given in set-theoretic terms by $C_1 := \{(100, 200), (150, 200), (0, 800)\}$, $C_2 := C_1 \cup \{(50, 200)\}$ and $C_3 := \{(0, 800), (6, 205), (1000, 9)\}$. Then the fish population of size $x = 200$ is, according to Definition 2, resilient to C_1 (because $\Lambda C_1 200 = \{100, 150\} \subseteq \Lambda Res 200$) and C_3 (because $\Lambda C_3 200 = \emptyset \subseteq \Lambda Res 200$), but not to C_2 (because $50 \in \Lambda C_2 200$ but $50 \notin \Lambda Res 200$). Finally, we give the following example of *comparing states and their resilience*. For all $400 \leq y \leq 600$, we have that $\Lambda Res y \supseteq \Lambda Res 600$ since $\Lambda Res 600 = 500$ and for all $400 \leq y \leq 600$ we have $500 \in \Lambda Res y$ (because $f 500 = 500$). Thus, according to Definition 4, each population size

$400 \leq y \leq 600$ is *more resilient* than the size 600.

Remark 1 Since

$$\begin{aligned} U \subseteq f \setminus S &\equiv \{ \text{universal property of } \setminus \} \\ & f \cdot U \subseteq S \\ &\equiv \{ \text{shunting of functions, Equ. 2.15 } \} \\ & U \subseteq f^\circ \cdot S \end{aligned}$$

we could have defined the static resilience relation of the system f equivalently by the relation $f^\circ \cdot S$. Since, however, the above reasoning does apply to relations in general, we prefer the definition of the static resilience relation in terms of residuation as given in Definition 9.

In short, we defined the static resilience relation $Res := f \setminus S$ of the system f and with respect to the tolerance relation S in such a way that we consider the system f in state x to be resilient to the change in state to x' with respect to the static resilience relation Res , in symbols $x' Res x$, if the next state $f x'$ of x' is similar to the initial state x with respect to the tolerance relation S : $(f x') S x$.

3.4.2 The Resilience Relation: Refined Definitions

The above subsection introduced the “static resilience relation” (Definition 9), our first definition of a resilience relation. This definition fails, however, to account for several aspects of resilience. This is why we are now going to refine Definition 9 to agree better with intuitive interpretations of resilience.

The Dynamic Resilience Relation

In the case of the static resilience relation $Res = f \setminus S$, we attach importance to the fact that the state of the system stays constant: the disturbed state is asked to be similar to the initial one. We could think of it as focussing on the status quo. What could happen, however, is that our entity is changing anyway with time, i.e. it is dynamically evolving. For example, a small child will normally change over time to an adult; in symbols, $f(x) \not\mathcal{S} x$. According to our definition of the static resilience relation in Definition 9 this would mean that the system f in state x is not resilient even if this is its normal behavior.

The problem here lies with the choice of the state which represents the situation before the change (x', x) . Instead of the initial state x we might consider its next state fx appropriate. Thus, the situation after the change, fx' , and the state representing the situation before, fx , refer to the same time step. With this representation of the situation before we expect the resilience relation $Res : X \leftarrow X$ (as common in mathematics, we use the same symbol Res to denote our new resilience relation)² to satisfy

$$x' Res x \equiv (fx') S (fx) \quad (3.8)$$

in order to correspond to our interpretation of “resilient”. In words, we set $x' Res x$ if the next step fx' of the disturbed state x' is similar to the next step fx of x . This comes up to expecting that Res is the largest relation with respect to the partial order \subseteq on relations which makes the following diagram semi-commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ Res \downarrow & \subseteq & \downarrow S \\ X & \xrightarrow{f} & X \end{array}$$

Res satisfies this condition if and only if Res is the right residual of f with respect to $S \cdot f$.

With these considerations we define:

Definition 10 The *dynamic resilience relation* $Res : X \leftarrow X$ of the system $f : X \leftarrow X$ with respect to the tolerance relation $S : X \leftarrow X$ is the right residual of f with respect to $(S \cdot f)$. In symbols:

$$Res := f \setminus (S \cdot f).$$

We call the resilience relation $Res = f \setminus (S \cdot f)$ “dynamic” since it takes into account the evolving character of the state. If the system state does not, however, change with time, i.e. $fx = x$ (in this case we call x a *fixed point* of the system f), then the two definitions coincide in x : the system f in state x is resilient to the change to x' with respect to the static resilience relation if and only if it is

²We decided against using sub- or superscripts to differentiate between the proposed resilience relations in this chapter. One reason is that we think that this would render the notation complex and illegible. The other is that using the same notation Res emphasizes the fact that all of these are “only” alternative versions of the resilience relation. We took special care to make it clear from the context to which definition we refer to.

resilient to that change with respect to the dynamic resilience relation.

Example 10 Coming back to the fish population of Example 1 and the tolerance relation S of Example 7, the dynamic resilience relation $Res = f \setminus (S \cdot f)$ considers a change in the size of the population (x', x) to be a resilient change if the next year's population sizes of x' and x are similar with respect to S . Assuming again that the parameters of f are $r = 2$ and $K = 1000$, and the parameter T of S is 100, we have that this time $(100, 200)$ is not a resilient change, but $(200, 200)$ is (whereas it was just the opposite in the case of the static resilience relation - see Example 9). Since $f 500 = 500$ the resilience of $x = 500$ is the same for the static and dynamic resilience relation. The same holds in the case of non-viable states: the resilience of $x \in S_2$ is the same in the case of the static and the dynamic resilience relation since $f x$ is non-viable for a non-viable population size, but viable for a viable one.

Remark 2 The condition given in Definition 10 under which we call (x', x) a resilient change may remind the reader of the condition under which we consider the function f to be continuous in the state x . However, they do not correspond to each other: whereas we expect that *if* we start near enough to x , for instance in x' , *then* their next states need to be near as well in order to call the function f continuous in x . By contrast, *if* their next states are similar, *then* we call (x', x) a resilient change (in the sense of the dynamic resilience relation). This means that the implications go the other way around.

Remark 3 Since $f \setminus (S \cdot f) = (f \setminus S) \cdot f$ by Equation (2.29) and $f \setminus S = f^\circ \cdot S$ (see Remark 1) we have

$$f \setminus (S \cdot f) = f^\circ \cdot S \cdot f. \quad (3.9)$$

Like in the case of the static resilience relation, we could therefore have defined the dynamic resilience relation by $f^\circ \cdot S \cdot f$. Again, we prefer the representation of Res in terms of residuation over the latter, because it is, in contrast to the above reasoning, also applicable in the case of non-deterministic systems (see Subsection 3.4.4).

Remark 4 The resilience relation $Res = f \setminus (S \cdot f)$ is symmetric because S is symmetric. We reason:

$$\begin{aligned}
 f \setminus (S \cdot f) &\equiv \{ \text{Remark 3} \} \\
 &\quad f^\circ \cdot S \cdot f \\
 &\equiv \{ S \text{ is symmetric, } \circ \text{ is contravariant and an involution} \} \\
 &\quad (f^\circ \cdot S \cdot f)^\circ \\
 &\equiv \{ \text{Remark 3} \} \\
 &\quad (f \setminus (S \cdot f))^\circ.
 \end{aligned}$$

This means that if (y, x) is a resilient change for the system f then so is (x, y) .

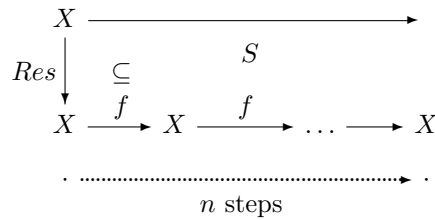
Static and Dynamic Resilience Relation: n-step Extensions

The dynamic resilience relation seems to suit our investigation of Section 3.1 better than the static one. Both of these definitions are, however, a bit impatient: we give the system only one time step to recover from the change in state. We might take this objection into account by giving the system a certain number of transitions to recover from the change in state. More precisely, if we consider the change in state (x', x) we would not necessarily choose the next state of x' but the state $f^n x'$ after n transitions as the “situation after” the change. As the “situation before” the change we either could take again the initial state x if we focus on not changing from the initial state – the static case –, or we could take the state of $f^n x$, as this would be the state the system normally would be in after the same number of transitions – the dynamic case.

Using these representations of the “situation before/after”, we expect that the resilience relation satisfies in the static case (using again the same symbol Res to denote the new resilience relation):

$$x' Res x \equiv (f^n x') S x. \tag{3.10}$$

In words, the system f in state x is resilient to the change in state to x' if the state of x' after n transitions is similar to the initial state x : $(f^n x') S x$. Schematically, this static case is illustrated by the diagram



where we expect Res to be the largest relation with respect to the partial order \subseteq on relations such that the diagram is still semi-commutative. In turn, it holds that Res satisfies this condition if and only if it is a right residual, this time of the n th iterate f^n of f with respect to the tolerance relation S .

We therefore define

Definition 11 The n -step static resilience relation $Res : X \leftarrow X$ of the system $f : X \leftarrow X$ with respect to the tolerance relation $S : X \leftarrow X$ is the right residual of f^n with respect to S . In symbols:

$$Res := f^n \setminus S.$$

The resilience relation $Res := f^n \setminus S$ is called the n -step static resilience relation since it still focuses on staying similar to the initial state but gives the system n transitions after the change in state to get similar to the initial state. If $n = 1$ the n -step static resilience relation of Definition 11 reduces to the static resilience relation of Definition 9.

Example 11 Continuing the running example of the fish population, we get in the case of the n -step static resilience relation $Res = f^n \setminus S$ that the population is resilient to a change (x', x) if after n years the population size of x' , i.e. $f^n x'$, is similar to the initial population size x (with respect to S). With the same parameter values as before ($r = 2$, $K = 1000$, $T = 100$), we have for instance that $(100, 200)$ is a resilient change with respect to the 2-step static resilience relation (as it was for $n = 1$ - see Example 9; because $f^2 100 \approx 295$), but it is not for the $n \geq 3$ - step case (because $f^n 100 > 300$ for $n \geq 3$). The change $(200, 200)$ is not resilient for all $n \in \mathbb{N}$. In fact, in the longer run, i.e. for $n \rightarrow \infty$, no change $(x', 200)$ is a resilient change with respect to the n -step static resilience relation, because either x' is non-viable, or $f^n x' \rightarrow 500$ for $n \rightarrow \infty$ (since $x = 500$ is an attracting fixed point of f). By the same reason, all changes $(x', 500)$ where $x' \in S_1 = (0, K)$ are resilient for some n . Since an extinct population stays extinct over the years and a viable population continues to exist according to our population model, i.e. $f^n x$ is non-viable for non-viable x and viable for viable x , the resilience of non-viable states is again the same as in the static and dynamic case before, i.e. $\Lambda Res x = S_2$ for $x \in S_2$.

Remark 5 By Equation (2.30) we have that since f is a function

$$f^n \setminus S = (f^n)^\circ \cdot S. \quad (3.11)$$

Hence, we could have defined the n -step static resilience relation equivalently by $Res := (f^n)^\circ \cdot S$. We prefer, however, again the definition in terms of residuation since it is more general.

In the static case, we represented the “situation after” the change (x', x) by the n th iterate of x' . To determine if the system is resilient to this change we compared $f^n x'$ to the initial state x . In the dynamic case we now compare $f^n x'$ to the n th iterate $f^n x$ of the initial state x instead of to the initial state itself. $f^n x$ is thereby interpreted as the ‘normal’ state of the system after n transitions. With this representation of the “situation before” we translate our interpretation of the Oxford Dictionary of English definition of “resilient” by expecting that the resilience relation satisfies

$$x' Res x \equiv (f^n x') S (f^n x). \quad (3.12)$$

Accordingly, the system f in state x is resilient to the change in state to x' if the n th iterate of x' and x are similar with respect to S : $(f^n x') S (f^n x)$. This means that we expect Res to be the largest relation with respect to the partial order \subseteq on relations which makes the following diagram semi-commutative:

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & X & \xrightarrow{f} & \dots & \xrightarrow{f} & X \\
 Res \downarrow & & & & \subseteq & & \downarrow S \\
 X & \xrightarrow{f} & X & \xrightarrow{f} & \dots & \xrightarrow{f} & X \\
 \cdot & \xrightarrow{\dots\dots\dots} & \cdot & & & & \cdot \\
 & & & & n \text{ steps} & &
 \end{array}$$

It follows that Res is the right residual of f^n with respect to $(S \cdot f^n)$.

We define:

Definition 12 The n -step dynamic resilience relation $Res : X \leftarrow X$ of the system $f : X \leftarrow X$ with respect to the tolerance relation $S : X \leftarrow X$ is the right residual of f^n with respect to $(S \cdot f^n)$. In symbols:

$$Res := f^n \backslash (S \cdot f^n).$$

We call $Res := f^n \backslash (S \cdot f^n)$ the n -step dynamic resilience relation since it takes into account the evolving character of the system, and gives the system n transitions to recover from the change in state. If $n = 1$ the n -step dynamic resilience relation corresponds to the dynamic resilience relation from Definition 10. If

$f^n x = x$ the n -step dynamic resilience relation and the n -step static resilience relation coincide in x : the system f is resilient to the change (x', x) with respect to the n -step dynamic resilience relation if and only if it is resilient to that change with respect to the n -step static resilience relation.

Example 12 In the case of the fish population model of Example 1, we get in the case of the n -step dynamic resilience relation $Res = f^n \setminus (S \cdot f^n)$ that the population is resilient to a change (x', x) if after n years the population size of x' and of x , i.e. $f^n x'$ and $f^n x$, are similar (with respect to S). With the same parameter values as before ($r = 2$, $K = 1000$, $T = 100$), we have for instance that $(100, 200)$ is a resilient change for $n \geq 3$ (in contrast to the n -step static case - see Example 11). This time $(200, 200)$ is a resilient change for all $n \in \mathbb{N}$. In the longer run, every change (x', x) for which $x', x \in (0, K)$ is a resilient change with respect to the n -step dynamic resilience relation since for all $y \in (0, K)$ it holds that $f^n y \rightarrow 500$ for $n \rightarrow \infty$. For a fixed $n \in \mathbb{N}$ the n -step static and dynamic resilience of the population size $x = 500$ are the equal since $x = 500$ is a fixed point of f . For non-viable sizes we even have that for all $n \in \mathbb{N}$ their n -step static and dynamic resilience coincide.

Remark 6 It follows from Equation (2.30) and Remark 1 that:

$$f^n \setminus (S \cdot f^n) = (f^n)^\circ \cdot S \cdot f^n \quad (3.13)$$

Thus, we could have defined the n -step dynamic resilience relation by $(f^n)^\circ \cdot S \cdot f^n$. As before, we prefer the definition in terms of residuation since it is more general.

Remark 7 $Res := f^n \setminus (S \cdot f^n)$ is symmetric since S is symmetric. We reason:

$$\begin{aligned} f^n \setminus (S \cdot f^n) &\equiv \{ \text{Remark 6} \} \\ &\quad (f^n)^\circ \cdot S \cdot f^n \\ &\equiv \{ S \text{ is symmetric, } \circ \text{ is contravariant and an involution} \} \\ &\quad ((f^n)^\circ \cdot S \cdot f^n)^\circ \\ &\equiv \{ \text{Remark 6} \} \\ &\quad (f^n \setminus (S \cdot f^n))^\circ. \end{aligned}$$

This means that if (y, x) is a resilient change for the system f then so is (x, y) .

Static and Dynamic Resilience Relation: Transitional Extensions

The n -step static and dynamic resilience relations seem to suit our investigations of Section 3.1 better than just the static or dynamic resilience relation. Both

of these definitions neglect, however, what happens up to time n , or what happens thereafter. Put differently, we might be interested not just in comparing states, but in fact in comparing whole trajectories. This means that we move on from interpreting a situation as a state to interpreting it as a trajectory. Consequently, we need to represent the similarity criteria as a tolerance relation on the set of sequences of states: $S : X^{n+1} \leftarrow X^{n+1}$ where $n \in \mathbb{N} \cup \{\infty\}$ (using the same symbol to denote the new tolerance relation).

Example 13 Suppose that S^* is a tolerance relation on states. Using S^* we might define a tolerance relation $S : X^{n+1} \leftarrow X^{n+1}$ by

$$(x_0, x_1, x_2, \dots, x_n) S (y_0, y_1, y_2, \dots, y_n) \equiv \forall 1 \leq k \leq n : x_k S^* y_k.$$

for $n \in \mathbb{N}$, or

$$(x_0, x_1, x_2, \dots) S (y_0, y_1, y_2, \dots) \equiv \forall k \geq 1 : x_k S^* y_k.$$

for $n = \infty$. In words, two trajectories are similar with respect to the tolerance relation S if all the states of the trajectories with the same subindex, possibly except the initial states x_0 and y_0 , are similar with respect to the tolerance relation S^* on the states.

As before, we might consider a static case, where we focus on staying similar to the initial case x , and a dynamic case, which takes into account the evolving character of the system.

We resume our definitions:

Definition 13 The n -step transitional static resilience relation $Res : X \leftarrow X$ of the system $f : X \leftarrow X$ with respect to the tolerance relation $S : X^{n+1} \leftarrow X^{n+1}$ is the right residual of γ_n with respect to $(S \cdot const_n)$, where $\gamma_n : X^{n+1} \leftarrow X$ is the relation that maps a state to its trajectory, and where the relation $const_n : X^{n+1} \leftarrow X$ maps a state x to the constant sequence of $(n + 1)$ x 's. In symbols:

$$Res := \gamma_n \setminus (S \cdot const_n).$$

Thus, according to the resilience relation in Definition 13, the system f is resilient to the change (x', x) if the trajectory of x' is similar to the constant sequence of x 's of the same length as the trajectory with respect to the tolerance relation S . This is illustrated by the following diagram

$$\begin{array}{ccc} X & \xrightarrow{const_n} & X^{n+1} \\ Res \downarrow & \subseteq & \downarrow S \\ X & \xrightarrow{\gamma_n} & X^{n+1} \end{array}$$

in which we expect Res to be the largest relation with respect to the partial order \subseteq such that the diagram is semi-commutative.

Example 14 Suppose that f is the transition function of the fish population from Example 1, and that S^* is the tolerance relation from Example 7. Let S be the binary tolerance relation defined on X^{n+1} according to the definition given in Example 13. By Definition 13 we have that the system f in a state x is n -step transitional static resilient to the change in state to x' if

$$(x', f x', f^2 x', \dots, f^n x') S (x, x, x, \dots, x) \equiv \forall 1 \leq k \leq n : (f^k x') S^* x$$

which is the case if the change (x', x) is for all $1 \leq k \leq n$ k -step static resilient. It follows that if (x', x) is n -step transitional static resilient then it is m -step transitional static resilient for all $m \leq n$. With regard to the fish population and the same parameter values as before ($r = 2$, $K = 1000$, $T = 100$), this means that the population is n -step transitional static resilient to a change in its size (x', x) if the population size of x' in the following n years is similar to the size x , i.e. if x is non-viable, x' is non-viable as well, or $f^k x'$ differs for all $1 \leq k \leq n$ in no more than 100 fish from x . From this and Example 11 it follows that $(100, 200)$ is 2-step transitional static resilient but not $3 \leq n$ -step resilient. In fact all $(x', 200)$ are not 3-step transitional static resilient (and thus not for $n \geq 3$ as well). The resilience of non-viable states is again S_2 for all $n \in \mathbb{N}$.

In the dynamic case we are comparing the trajectory of the disturbed state to the trajectory of the initial state:

Definition 14 The n -step transitional dynamic resilience relation $Res : X \leftarrow X$ of the system $f : X \leftarrow X$ with respect to the tolerance relation $S : X^{n+1} \leftarrow X^{n+1}$ is the right residual of γ_n with respect to $(S \cdot \gamma_n)$, where $\gamma_n : X^{n+1} \leftarrow X$ is the relation which maps a state to its trajectory. In symbols:

$$Res := \gamma_n \setminus (S \cdot \gamma_n).$$

According to this resilience relation, we consider a system f in a state x resilient to a change in state to x' if the trajectory of x' is similar to the trajectory of x with respect to the tolerance relation S : $(\gamma_n x') S (\gamma_n x)$. The following diagram

illustrates Res from Definition 14:

$$\begin{array}{ccc}
 X & \xrightarrow{\gamma_n} & X^{n+1} \\
 Res \downarrow & \subseteq & \downarrow S \\
 X & \xrightarrow{\gamma_n} & X^{n+1}
 \end{array}$$

where Res is the largest relation with respect to the partial order \subseteq such that the above diagram is still semi-commutative.

If, in the case of $n < \infty$, the tolerance relation S takes only the endpoints of trajectories into account then the transitional static and dynamic resilience relations reduce to the n -step static and dynamic resilience relations respectively. More generally, if, for $m \leq n \leq \infty$ where $m < \infty$, S takes only the m th state x_m of sequences (x_0, x_1, \dots, x_n) , or (x_0, x_1, \dots) if $n = \infty$, into account, the transitional static and dynamic resilience relation reduce to the m -step static and dynamic resilience relation respectively. Furthermore, if x is a fixed point of the system, i.e. if $f x = x$, then the n -step transitional dynamic resilience relation coincides in x with the static one.

Example 15 As before, suppose that f is the transition function of the fish population from Example 1, and S^* is the tolerance relation from Example 7. Let S be the tolerance relation defined on X^{n+1} according to Example 13. By Definition 13 we have that the system f in a state x is n -step transitional dynamic resilient to the change in state to x' if

$$(x', f x', \dots, f^n x') S(x, f x, \dots, f^n x) \equiv \forall 1 \leq k \leq n : (f^k x') S^*(f^k x)$$

which is the case if the change (x', x) is for all $1 \leq k \leq n$ k -step dynamic resilient. It follows that if (x', x) is n -step transitional dynamic resilient then it is m -step transitional dynamic resilient for all $m \leq n$. With regard to the fish population and the same parameter values as before ($r = 2$, $K = 1000$, $T = 100$), this means that the population is n -step transitional dynamic resilient to a change in its size (x', x) if the population size of x' and x in the following n years are similar with respect to S^* , i.e. if x is non-viable, x' is non-viable as well, or $f^k x'$ differs for all $1 \leq k \leq n$ in no more than 100 fish from $f^k x$. Thus, by Example 12 it follows that $(100, 200)$ is not a resilient change with respect to the n -step transitional dynamic resilience relation for all $n \in \mathbb{N}$, but $(200, 200)$ is. For each fixed $n \in \mathbb{N}$, the resilience of $x = 500$ is again the same for the static and dynamic case (because it is a fixed point). In the case of the non-viable population sizes we note that again their resilience is S_2 for all $n \in \mathbb{N}$. It follows that the resilience of a non-viable population size is S_2 no matter

which resilience relation we use.

Remark 8 The condition under which we call (x', x) a resilient change with respect to the ∞ -step transitional dynamic resilience relation may remind the reader of the condition under which we call the system f to be (Liapunov)-stable in a fixed point, especially if the tolerance relation S is defined as in Example 13. However, as in the case of the dynamic resilience relation and continuity, they do not correspond to each other: we expect that *if* we start near the fixed point x , say in x' , *then* the trajectories of x and x' are near in order to say that the fixed point x is stable. By contrast, *if* their trajectories are near, *then* we call (x', x) a resilient change in the sense of the ∞ -step transitional dynamic resilience relation. Again, the implication goes the other way around.

Remark 9 As in the case of the previous definitions, the shunting rules can be used to prove the following equivalences:

$$\gamma_n \setminus (S \cdot constn) = \gamma_n^\circ \cdot S \cdot constn \quad (3.14)$$

$$\gamma_n \setminus (S \cdot \gamma_n) = \gamma_n^\circ \cdot S \cdot \gamma_n. \quad (3.15)$$

(since γ_n is a function since f is one, and $constn$ is a function per definition). However, this reasoning only holds in the case of deterministic systems, whereas the definitions in terms of residuation are meaningful even in the case of non-deterministic systems (see Subsection 3.4.4).

Remark 10 By a similar reasoning as in Remark 4 and in Remark 7, we have that $Res := \gamma_n \setminus (S \cdot \gamma_n)$ is symmetric since S is symmetric. Thus, if (y, x) is a resilient change of the system f with respect to the n -step transitional dynamic resilience relation, then so is (x, y) .

The aim of these somewhat pedantic definitions was to show how we might put the mathematical representations of the primitive concepts together to define a meaningful resilience relation. We approached the definitions in such a way that we refined and generalized them in each step such that the later definitions reduce to the former ones in special cases. In all cases, the mathematical notion of residuation³ proved very useful. The definitions differ in what we named the

³The notation *Res* seems therefore fitting as it is indicative of the concept *Resilience* and the mathematical notion *Residuation* used throughout the definitions. Interestingly, the same abbreviation only in lower case letters, i.e. *res*, was used in medieval Europe to denote the *unknown* in algebraic equations [29]. Thus, *Res* seems to be an even more fitting notation for the resilience relation as it is what we are looking for, i.e. the unknown, of this thesis.

“situation before/after”.

As explained in the introduction, it is not the aim of the work to point to “the right” definition. Instead we intend a mathematical description of the conceptual space as is used in real-world discussions and scientific research. With the use of the primitive concepts and their mathematical representations, the interested reader might check if our definitions capture his or her notion of resilience.

3.4.3 Common Structure

So far, we explored different formal definitions that might represent whether or not an entity is resilient to a disturbance with respect to certain interpretations of the situation before and after, and with respect to certain similarity criteria. Step by step we refined and generalized our definitions and reached Definition 14, which is the most general of the definitions in that it reduces to the other definitions under certain conditions. Our objective at this point is to show that all of these definitions are “in a way” particular instances of the static resilience relation (Definition 9). By “in a way” we mean that we might have to consider a different transition function, a different change in state, and a different tolerance relation. In symbols, we show that for each transition function $f : X \leftarrow X$, change (x', x) , tolerance relation $S : X \leftarrow X$ or $S : X^{n+1} \leftarrow X^{n+1}$, and resilience relation Res^* , where $*$ refers to one of the attributes “ n -step static”, “ n -step dynamic” (which includes the case of just “dynamic” for $n = 1$), “ n -step transitional static”, or “ n -step transitional dynamic”, there is a transition function $\tilde{f} : \tilde{X} \leftarrow \tilde{X}$, a change (\tilde{x}', \tilde{x}) , and a tolerance relation $\tilde{S} : X \leftarrow X$ such that

$$\tilde{x}' \tilde{Res} \tilde{x} \equiv x' Res^* x \quad (3.16)$$

where \tilde{Res} refers to the static resilience relation of \tilde{f} with respect to \tilde{S} , i.e. $\tilde{Res} := \tilde{f} \setminus \tilde{S}$.

n-Step Static Resilience Relation

In the case of the n -step static resilience relation, Equation (3.16) means that we expect that

$$\tilde{x}' \tilde{f} \setminus \tilde{S} \tilde{x} \equiv x' f^n \setminus S x. \quad (3.17)$$

Thus, if $\tilde{f} = f^n$ and the change as well as the tolerance relation stay unchanged, i.e. $\tilde{x}' = x'$, $\tilde{x} = x$ and $\tilde{S} = S$, then Equation (3.16) holds in the case of $Res^* =$

n -step static resilience relation: the system f in state x is resilient to the change in state to x' with respect to the tolerance relation S and the n -step static resilience relation if and only if the system f^n in state x is resilient to the change in state to x' with respect to the tolerance relation S and the static resilience relation.

Since by Equation (2.30)

$$f^n \setminus S = f^{n-1 \circ} \cdot (f \setminus S), \quad (3.18)$$

a second possibility is to choose $\tilde{x}' := f^{n-1} x'$ and leave the rest unchanged, i.e. $\tilde{f} = f$, $\tilde{S} = S$ and $\tilde{x} = x$, i.e. the change (x', x) is a resilient change for the system f with respect to the tolerance relation S and the n -step static resilience relation if and only if the change $(f^{n-1} x', x)$ is a resilient change for the system f with respect to tolerance relation S and the static resilience relation.

n-Step Dynamic Resilience Relation

In order for Equation (3.16) to hold in the case of the n -step dynamic resilience relation, the following equation needs to be satisfied:

$$\tilde{x}' \tilde{f} \setminus \tilde{S} \tilde{x} \equiv x' f^n \setminus (S \cdot f^n) x. \quad (3.19)$$

Since by Equation (2.29)

$$f^n \setminus (S \cdot f^n) = (f^n \setminus S) \cdot f^n, \quad (3.20)$$

Equation (3.16) holds in the case of the n -step dynamic resilience relation if $\tilde{f} := f^n$, $\tilde{x} := f^n x$, and the rest pass on, i.e. $\tilde{x}' := x'$, $\tilde{S} := S$. This means that the change (x', x) is a resilient change for the system f with respect to the tolerance relation S and the n -step dynamic resilience relation if and only if the change $(x', f^n x)$ is a resilient change for the system f^n with respect to the tolerance relation S and the static resilience relation.

Another possibility arises again from Equation (2.30); since

$$f^n \setminus (S \cdot f^n) = f^{n-1 \circ} \cdot (f \setminus S) \cdot f^n, \quad (3.21)$$

we could also take $\tilde{x}' := f^{n-1} x'$, and $\tilde{x} := f^n x$, and carry f and S over, i.e. $\tilde{f} := f$ and $\tilde{S} := S$. This means that the change (x', x) is a resilient change for the system f with respect to the tolerance relation S and the n -step dynamic resilience relation if and only if the change $(f^{n-1} x', f^n x)$ is a resilient change

for the same system f and the same tolerance relation S , but with respect to the static resilience relation.

n-Step Transitional Static Resilience Relation

The following equation needs to be satisfied in order that Equation (3.16) holds in the case of the n -step transitional static resilience relation:

$$\tilde{x}' \tilde{f} \setminus \tilde{S} \tilde{x} \equiv x' \gamma_n \setminus (S \cdot \text{const}_n) x. \quad (3.22)$$

We define

$$\tilde{f} : X^{n+1} \leftarrow X^{n+1}, \quad (3.23)$$

which maps the sequence of states $(x_0, x_1, x_2, \dots, x_n)$ to the sequence of states $(x_0, f x_1, f^2 x_2, \dots, f^n x_n)$. It follows that $(\tilde{f} \cdot \text{const}_n) = \gamma_n$ and thus $\gamma_n \setminus (S \cdot \text{const}_n) = (\tilde{f} \cdot \text{const}_n) \setminus (S \cdot \text{const}_n)$. Furthermore, since const_n is a function, we might use Equations 2.29 and 2.30 to get

$$(\tilde{f} \cdot \text{const}_n) \setminus (S \cdot \text{const}_n) = \text{const}_n^\circ \cdot (\tilde{f} \setminus S) \cdot \text{const}_n. \quad (3.24)$$

It follows that for \tilde{f} as defined above, $\tilde{x}' := \text{const}_n x'$, $\tilde{x} := \text{const}_n x$, and $\tilde{S} := S$, Equation (3.16) is satisfied in the case of the n -step static transitional resilience relation. Note that in the case of \tilde{f} we consider trajectories as states. We have that the system f in state x is resilient to the change in state to x' with respect to the tolerance relation S and the n -step transitional static resilience relation if and only if the system \tilde{f} in state $\text{const}_n x$ is resilient to the change in state to $\text{const}_n x'$ with respect to the tolerance relation S and the static resilience relation.

n-Step Transitional Dynamic Resilience Relation

In this case we want that

$$\tilde{x}' \tilde{f} \setminus \tilde{S} \tilde{x} \equiv x' \gamma_n \setminus (S \cdot \gamma_n) x. \quad (3.25)$$

This case is similar to the case of the n -step transitional static resilience relation above. Using the same \tilde{f} and again the Equations 2.29 and 2.30 we have that

$$(\tilde{f} \cdot \text{const}_n) \setminus (S \cdot \gamma_n) \equiv \text{const}_n^\circ \cdot (\tilde{f} \setminus S) \cdot \gamma_n. \quad (3.26)$$

Thus if we further define $\tilde{x}' := \text{const}_n x'$, $\tilde{x} := \gamma_n x$ and leave S unchanged, i.e. $\tilde{S} := S$, Equation (3.16) is satisfied in the case of the n -step transitional

dynamic resilience relation. The interpretation is that the system f in state x is resilient to the change in state to x' with respect to the tolerance relation S and the n -step transitional dynamic resilience relation if and only if the system \tilde{f} in state $\gamma_n x$ is resilient to the change in state to $const_n x'$ with respect to the tolerance relation S and the static resilience relation. In this case, the states of the system are again trajectories.

Why is all this useful? First of all, these transformations might be interesting from a computational point of view. Secondly, we want to raise awareness that it is important to watch out how the primitive concepts “entity”, “disturbance”, and “similarity criteria” are modelled. Using the same resilience relation alone, i.e. the same mathematical representation of the “situation before/after”, does not suffice to lead to the same conclusion about whether or not the entity is resilient to that disturbance or not: two statements which seem to be at first sight different, because they are using different definitions of the resilience relations, might be equivalent; or, two statements which seem to be at first sight equivalent because they are referring to the same definition of the resilience relation might in fact be different. Therefore it is of utmost importance to make explicit how the primitive concepts are modelled.

3.4.4 A Note on Non-Deterministic Systems

Instead of representing an entity by a discrete, deterministic dynamical system in a given state, we now consider the case of a discrete, non-deterministic dynamical system given by a transition relation

$$F : X \leftarrow X \tag{3.27}$$

where X is again the set of states. Given the current state of the system $x \in X$, the transition relation tells us that the state y is a possible next state of x if yFx . In contrast to the previous deterministic case, we do not require F to be simple. We do, however, expect it to be entire such that each state has at least one possible next state. Our interpretation of the transition relation F is again that of the “normal” behavior of the entity.

A finite or infinite sequence of states, $(x_0, x_1, x_2, \dots, x_n)$ or (x_0, x_1, x_2, \dots) respectively, is now called a *trajectory* of x_0 if $x_k F^k x_0$ for all $k \in \mathbb{N}$, where F^k denotes the k th iterate of F . In the former case the trajectory corresponds to a *possible* behavior within n transitions, in the latter it represents a possible run

of the state unbounded in time. To simplify and shorten notation, we introduce again a relation $\gamma_n : X^{n+1} \leftarrow X$ for $n \in \mathbb{N} \cup \{\infty\}$, which maps a state to its possible trajectories. Note that γ_n is a function if and only if F is one.

Example 16 We might extend the fish population model to the non-deterministic case by allowing the growth rate to take on a range of values. This time, we consider the transition relation

$$F : \mathbb{R}_+ \leftarrow \mathbb{R}_+$$

which relates a population size y to an initial size x , i.e. yFx , if

$$y \in \left\{ rx \left(1 - \frac{x}{K} \right) \mid r \in [r_1, r_2], K := 1000 \right\}$$

where $0 < r_1 \leq r_2$, for instance $r_1 = 1.9$ and $r_2 = 2.1$. It is easily checked that F is entire. If $r_1 = r_2$ it is in addition simple, hence a transition function. The range of possible r 's might be interpreted as the normally possible range of growth rates which might vary depending on environmental factors such as the temperature of the lake.

We are now going to apply the definitions of the resilience relations from the sections above (Subsections 3.4.1 and 3.4.2) to a discrete, non-deterministic dynamical system. We show that all of these definitions transfer in a meaningful way, only the representations of the “situation before/after” have to be changed so as to match the non-determinacy of the system. The interested reader might use the transition relation of Example 3.4.4 to come up with examples for the definitions in terms of our fish population.

Static Resilience Relation

In the case of the static resilience relation we have $Res := F \setminus S$. Accordingly, a change (x', x) is a resilient change for the system F with respect to the tolerance relation S and the resilience relation Res if all possible next states of the disturbed state x' are similar to the initial state x :

$$x' Res x \equiv \forall y : yFx' \Rightarrow ySx. \quad (3.28)$$

We might interpret each of the possible next states y of x' (i.e. if yFx') as a possible “situation after”. Thus, if we furthermore represent the “situation before” by the initial state x , $Res = F \setminus S$ matches our interpretation of the dictionary definition of “resilient” (“an entity is resilient to a disturbance if its situation after the disturbance occurred is similar to the situation before the disturbance”).

Dynamic Resilience Relation

If $Res := F \setminus (S \cdot F)$ then

$$x' Res x \equiv \forall y : y F x' \Rightarrow (\exists z : y S z \wedge z F x) \quad (3.29)$$

Accordingly, we would call the system F in state x resilient to the change in state to x' with respect to the tolerance relation S and the resilience relation Res if all possible next states of the disturbed state x' (“situation after”) are similar to at least one possible next state of the initial state x (“situation before”). Thus, the definition $Res := F \setminus (S \cdot F)$ seems to be meaningful in the case of a discrete, non-deterministic dynamical system as well.

n-Step Static Resilience Relation

The interpretation of $x' Res x$ for $Res := F^n \setminus S$ would be that the system F in state x is resilient to the change in state to x' if all possible n th iterates of x' (“situation after”), are similar to x (“situation before”) with respect to the tolerance relation S . In symbols:

$$x' Res x \equiv \forall y : y F^n x' \Rightarrow y S x. \quad (3.30)$$

In this sense, the definition of the resilience relation $Res := F^n \setminus S$ also corresponds to our interpretation of the dictionary definition of resilient.

n-Step Dynamic Resilience Relation

In the case of $Res := F^n \setminus (S \cdot F^n)$ we consider a change in state (x', x) to be a resilient change for the system F with respect to the tolerance relation S if each possible n th iterate of the disturbed state x' is at least similar to one possible n th iterate of the initial state x :

$$x' Res x \equiv \forall y : y F^n x' \Rightarrow (\exists z : y S z \wedge z F^n x). \quad (3.31)$$

This also reflects our interpretation of the definition of “resilient”; thereby the “situation before” is represented by one of the possible n th iterates of the initial state x , and the “situation after” is represented by each of the possible n th iterates of x' .

n-Step Transitional Static Resilience Relation

If $Res := \gamma_n \setminus (S \cdot const_n)$ we consider the system F in state x resilient to the change in state to x' with respect to the tolerance relation S and the resilience

relation Res if all possible trajectories of the disturbed state x' are similar to the constant sequence of $(n + 1)$ x 's. In symbols:

$$x' Res x \equiv \forall(y_0, y_1, \dots, y_n) : (y_0, y_1, \dots, y_n) \gamma_n x' \Rightarrow (y_0, y_1, \dots, y_n) S \underline{x},$$

where \underline{x} denotes the constant sequence of $(n + 1)$ x 's. The “situation after” is this time represented by each possible trajectory of the disturbed state, and the “situation before” by a the constant sequence of x 's.

n-Step Transitional Dynamic Resilience Relation

According to the n -step transitional dynamic resilience relation we consider the system F to be resilient to the change in state (x', x) if all possible trajectories of x' are similar to at least one possible trajectory of x . In symbols:

$$x' Res x \equiv \forall(x'_0, x'_1, \dots, x'_n) : (x'_0, x'_1, \dots, x'_n) \gamma_n x' \Rightarrow \left[\exists(x_0, x_1, \dots, x_n) : (x'_0, x'_1, \dots, x'_n) S(x_0, x_1, \dots, x_n) \wedge (x_0, x_1, \dots, x_n) \gamma_n x \right].$$

In this case, the “situation after” is represented by each of the possible trajectories of x' , and the “situation before” by one of the possible trajectories of x . In this sense, the definition of the resilience relation $Res := F^n \setminus S$ also corresponds to our interpretation of the dictionary definition of resilient.

Hence, the calculus of relations presented in Chapter 2 allows us without difficulty to consider more generally a discrete, *non-deterministic* system in a given state as a representation of the entity. The definitions of the resilience relations in terms of residuation as presented in the Subsections 3.4.1 and 3.4.2 transfer in a meaningful way. The equivalent formulations (without using residuation) of the resilience relations given in the remarks, and some of the transformations given in the Subsection 3.4.3 are not valid in the case of non-deterministic systems since they use the Equations 2.29 and 2.30 which hold only for functions.

Chapter 4

Applications

The objective of this section is to relate the formal framework thus developed to the definitions of resilience within the ecological literature, more precisely to the definitions of resilience by Pimm and by Holling, to see whether they might be captured in terms of our formal framework.

4.1 Pimm Resilience

Resilience as defined by S.L. Pimm refers to “how fast the variables return towards their equilibrium following a perturbation” ([8], p. 322), or, in other words, to “the time required to return to an equilibrium following a perturbation” ([23], p. 426). Thereby it is usually understood that a system is the more resilient to a perturbation the less time it needs to return to its original state.

In contrast to the definition of resilience in the framework, the Pimm definition does not define what “the resilience” is; instead, it only gives a criterion for deciding how resilient an ecosystem to a certain perturbation is. So whenever we mention in the following one of the definitions from Section 3.4.1 or from Section 3.4.2, we do not mean to refer to the resilience relations defined in them but rather to the criterion given in these definitions as to when we consider a change in state resilient.

We now identify the primitive concepts and interpret them in terms of our framework. First of all, the term “variables” is just another name for the “state of a system”. The definition thus refers to a dynamical system and its states (the entity). In the following, we consider again, as in the framework, a discrete,

deterministic dynamical system given by a transition function $f : X \leftarrow X$ to illustrate and interpret the definition in terms of our framework. We note that instead of referring to a state for which $f x = x$ as an “equilibrium”, it is in the case of a discrete system more common to say that x is a “fixed point”. The name “equilibrium” is more common in the case of continuous dynamical systems. Secondly, a “perturbation” is just another name for “disturbance”, which we represented as a change in the system state. Accordingly, the definition now reads: “how many transitions n are required such that the system f returns to the fixed point x after a change in state from the fixed point x to another state x' ”.

From this follows that the definition of resilience in the sense of Pimm is in terms of our framework a ∞ -step transitional type of resilience: in order to say after how many transitions we consider the system returned, we compare each state of the trajectory of x' to the fixed state x , or, in other words, we compare the trajectories of x' and x statewise. The “situation” is therefore represented by a trajectory. The “situation after” is the trajectory of the disturbed state x' . We cannot deduce from the definition whether it relates to the static or dynamic case, i.e. if the “situation before” is a constant sequence of the initial state x , or if it is the trajectory of x , since in the case of a fixed point the trajectory of a fixed point is the same as a sequence of the fixed point (of the same length as the trajectory).

Furthermore, the definition of Pimm relates fixed points and changes in that fixed point to levels of similarity (“*how many* transitions are needed ...”). It is with regard to these levels of similarity that the statement about being “more resilient to ...” in the sense of Pimm is made (“a system is the more resilient to a perturbation the less time it needs to return ...”). This statement does not, however, correspond to the definitions of “being more resilient” in terms of the presented formal framework (Definitions 4 and 5). Note that with the representation of the similarity criteria as a tolerance relation as used in this framework, it is not possible to take into account levels of similarity: we only dichotomize states or sequences of states into those which we do and those which we do not consider similar. We cannot say *how* similar two states or two trajectories are. Thus, all we might express in terms of our framework is either “whether or not the system f , after a change in state from the fixed point x to another state x' , returns to the fixed point within a certain number of transitions m ”, or “whether or not the system f returns at all (i.e. within some number of transitions) to the fixed point after a change in state from the fixed point x to another state x' ”.

We now define tolerance relations on trajectories which represent the similarity criteria used in these two expressions, i.e. which come close to the similarity criteria used in the Pimm definition. Suppose that $S^* : X \leftarrow X$ is a tolerance relation on the set of states representing the similarity criteria on the states. We would say that the system f returns after m transitions to the fixed point x after a change in state from the fixed point x to x' if for all $k \geq m$ we have that $f^k x'$ is similar to the fixed point x with respect to S^* , i.e. $(f^k x') S^* x$ for all $k \geq m$. We define a tolerance relation $S_1 : X^\infty \leftarrow X^\infty$ on the set of infinite sequences of states by

$$(y_0, y_1, y_2, \dots) S_1 (x_0, x_1, x_2, \dots) \equiv \forall k \geq m : y_k S^* x_k \quad (4.1)$$

which considers two sequences of states to be similar if corresponding states are similar from the m th position on. It might easily be checked that S_1 is a tolerance relation. Using the tolerance relation S_1 when comparing the trajectory of the disturbed state x' to the fixed point x enables us to decide “whether or not the system f , after a change in state from the fixed point x to another state x' , returns within m transitions to the fixed point”.

In order to decide “whether or not the system f returns at all (i.e. within some number of transitions) to the fixed point after a change in state from the fixed point x to another state x' ”, we would use the tolerance relation $S_2 : X^\infty \times X^\infty$ defined by

$$(y_0, y_1, y_2, \dots) S_2 (x_0, x_1, x_2, \dots) \equiv \exists m \in \mathbb{N}. \forall k \geq m : y_k S^* x_k \quad (4.2)$$

which considers two sequences of states to be similar if there is a natural number m such that from this position onwards corresponding states are similar with respect to S^* . It might easily be checked that S_2 is a tolerance relation as well.

In short, it is possible to identify the framework’s primitive concepts in the Pimm definition of resilience. The definition corresponds to the criterion used to decide whether a change in state is resilient of the ∞ -step transitional resilience relation definitions (Definitions 13 and 14). Since the Pimm definition considers only fixed points, it is not possible to decide between static and the dynamic case. Furthermore, it is not possible, with our representation of the similarity criteria as a tolerance relation, to take into account levels of similarity as used in the Pimm definition.

4.2 Holling Resilience

Resilience in the sense of C.S. Holling is the “ability of ... systems to absorb changes ..., and still persist” ([5]). It can be measured by “the magnitude of disturbance that can be absorbed before the system redefines its structure” ([24], Chapter ‘Suprise, Resilience, and Flexibility’), whereby it is understood that the system changes its structure if it does not return to the same equilibrium as before the disturbance [23]. The more the system can be disturbed and still remains its structure, the higher its resilience.

We try again to identify the primitive concepts of our framework in the Holling definition of resilience and give mathematical representations for them. First of all, like the Pimm definition, the entity in the definition is a “system”, which we represent again as a discrete, deterministic dynamical system given by a transition function $f : X \leftarrow X$. From the last remark above (“it is understood that the system changes its structure if it does not return to the same equilibrium as before the disturbance”) we deduce that, in fact, the definition does not only refer to the system but also to the system in a specific state. Secondly, the Holling definition explicitly names “disturbances”, which are represented in our framework by a change in state (x', x) . According to the definition, the initial state x is, as in the case of the Pimm definition, considered to be a fixed point of the system f . We would read the criterion used in the Holling definition to decide whether or not a change is a resilient change in terms of our terminology: “the system f in a state x , which is a fixed point, is resilient to a change in state to x' if the system returns to x ”. Obviously, this criterion corresponds to the criterion given by the Pimm definition which might be captured by our framework (i.e. without levels of similarity): we compare the trajectory (“situation”) of x' (“situation after”) statewise to the trajectory of x (“situation before”) either via the tolerance relation $S_1 : X^\infty \leftarrow X^\infty$ or $S_2 : X^\infty \leftarrow X^\infty$ defined above. In the case of S_1 , we consider two sequences of states to be similar if corresponding states are similar from the m th position on, where we might interpret m as a biological sensible time scale. As in the case of the Pimm definition, we might not specify further if the definition corresponds to our transitional static or dynamic resilience since it considers only fixed points.

In contrast to the Pimm definition of resilience, the Holling definition does not only give a criterion to decide when we consider a change to be resilient change for a system, but in fact defines the resilience of a state x extensionally (“the magnitude of disturbance that can be absorbed”) as we do in Definition 1; it

differs to our definition only in that it assumes in addition a metric with which it measures the magnitude of the changes. In this way, also the Holling definition above about “being more resilient” (that is, “the more the system can be disturbed and still remains its structure, the higher its resilience”) corresponds to our definition of “being more resilient” (Definitions 4 and 5).

In summary we can say that the both definitions may be captured in terms of the ∞ -step transitional static or dynamic resilience relation; both of the definitions consider only fixed points, in which case the static and dynamic types of resilience relations coincide. The analysis shows that the criteria, which the definitions of Pimm and Holling use to decide whether or how much a system in a state is resilient to a change in state, are similar; they only differ in that the Pimm definition relates the number of transitions needed to return to levels of similarity. These levels of similarity, however, cannot be represented by a tolerance relation. In contrast to the Pimm definition, the Holling definition defines resilience extensionally, as we do in the framework.

Chapter 5

Conclusion

The aim of this thesis was to propose a formal framework of resilience, i.e. to define the concept of resilience in a general, yet concise way by using mathematical notions. Our approach was to first analyse the definitions of resilience as used in everyday language and within the ecological literature in order to identify the building blocks of the term and their underlying structure. In a next step we translated these building blocks into the mathematical language and combined them in order to arrive at a formal definition of resilience.

From the analysis of the definition of resilience provided by the Oxford Dictionary of English followed that resilience is a relative concept: it expresses a certain relation between *entities* and *disturbances* of these entities (“an entity *is resilient to* a disturbance”) with respect to what we consider to be the *situation before* and *after* the disturbance, and with respect to certain *similarity criteria* that are used to compare both situations. From the representation of the entity and of the disturbance as a discrete, deterministic dynamical system $f : X \leftarrow X$ in a given state x and by a change in state from the initial state x to a state x' denoted by (x', x) respectively, it followed that the calculus of relations provides a natural setting for the formalization of resilience: we might represent the *resilience of a system* as a relation Res on the set of states X , which we called the *resilience relation* of the system f , and which maps initial states x to states x' in such a way that $x' Res x$ means that we consider the system f in state x to be resilient to the change in state to x' , i.e. we consider (x', x) to be a *resilient change* of the system f . In set-theoretic terms, Res is thus the set of all resilient changes. In this way, the *resilience of a state* x might be represented by the set of all states to which the initial state x can be changed to such that (x', x) is a resilient change. Moreover, we showed that the

representation of resilience by a relation allows to introduce further terminology belonging to the realm of resilience (e.g. “being resilient to a set of changes”, “comparing states and systems with respect to their resilience”).

We then proposed several definitions of a resilience relation, which differ first of all in the interpretation of the “situation”, i.e. whether we represent a situation to be a state or a trajectory, and accordingly whether we represent the similarity criteria as a tolerance relation on the set of states or on the set of trajectories respectively; secondly in the interpretation of the “situation before” the change in state, i.e. if we focus on staying similar to the initial state (the static case), or if we take into account the evolving character of the system (the dynamic case); and thirdly they differ in the time scale we use in which we require the system to be recovered (the n -step cases). In all of the proposed definitions, the mathematical notion of *residuation* proved useful. We showed furthermore that the proposed definitions are also meaningful if we represent the entity by a discrete, non-deterministic dynamical system in a given state (instead of a deterministic one). We pointed out that it is not only important to name the type of resilience relation one is considering, but also to give the representations of the primitive concepts, because using the same resilience relation alone, i.e. the same mathematical representation of the “situation before/after”, does not suffice to lead to the same conclusion about whether or not the entity is resilient to that disturbance or not. Two statements which seem to be at first sight different, because they are using different definitions of the resilience relations, might be equivalent; or, two statements which seem at first sight to be equivalent because they are referring to the same definition of the resilience relation might in fact be different.

In the last chapter we related our formal framework back to the definitions of resilience by S.L. Pimm and C.S. Holling. The analysis showed that both definitions might be captured in terms of the ∞ -step transitional (static or dynamic) resilience relation of our framework. Only the similarity criteria of the Pimm definition, which consider levels of similarity, cannot be fully reflected by our representation of the similarity criteria as a tolerance relation. The possible tolerance relations which might represent the similarity criteria of the Pimm definition as closely as possible correspond to the tolerance relation with which we would represent the similarity criteria used in the Holling definition of resilience. This might seem surprising knowing that both definitions are treated to be incompatible in the literature. However, the Pimm definition of resilience provides only a condition under which we might decide when an entity is re-

silient to a disturbance, and does not provide a definition of “the resilience”. The Holling definition on the other hand defines “the resilience” in an extensional way, as we do in the framework.

We suggest several points of departure for future work. Theoretical developments might include the structural refinement of the our framework. For instance, the notion of a *v-valued* or *fuzzy* relation may be appropriate to take into account levels of similarity such as the ones used in the Pimm definition of resilience; or, the notion of an *allegory*, as the generalization of a relation, may provide a basis for extending the framework to other types of systems such as continuous-time and stochastic ones. Furthermore, it would be interesting to try to relate this formal framework of resilience to the one of vulnerability suggested by Ionescu *et al.* [10], which was one of the motivations for developing this framework. These theoretical developments should simultaneously be accompanied by practical applications, to test the framework against reality as well as to be aware of the needs of the users.

Instead of being of operational use, the multitude of definitions of resilience, general, overlapping and contradicting as they are, have rendered the term so vague and confusing as to be almost meaningless. However, the case need not to be past all hope. The first step to work the way through the thicket of definitions has been to become aware of the thicket itself. The next step should be the systematic analysis and discussion of the concept and its definitions in order to make the underlying ideas of the various definitions explicit, and to differentiate these ideas from the nucleus of the term; the final aim would be an agreed upon operational definition. Hopefully this thesis provides some guidance for this process.

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