Fuzzy Complex Dynamical Networks and Its Synchronization

Nariman Mahdavi, Mohammad Bagher Menhaj, Jürgen Kurths, and Jianquan Lu

***Abstract*—In this paper, the robust synchronization problem of fuzzy complex dynamical networks is investigated. A fuzzy complex dynamical network is an extension to an uncertain complex dynamical network in which all sources of parametric uncertainties are modeled with fuzzy numbers, i.e., all nodes' dynamics are described by fuzzy differential equations (FDEs) that permit a better description of a real process occurring in the presence of inaccuracy. To resolve the synchronization problem, this paper introduces new adaptive and impulsive controllers in which globally exponential synchronization of fuzzy dynamical networks under easily verified conditions is guaranteed. Moreover, we propose an efficient method that helps to find certain suitable nodes to be impulsively controlled via pinning, noting that these nodes, in general, vary at distinct impulsive time instants. Therefore, by using adaptive controllers and applying impulsive controllers to only a small portion of nodes, the whole network will completely be synchronized to a certain objective state. Finally, two numerical examples are given to illustrate the effectiveness of the proposed controllers.**

***Index Terms*—Adaptive control, complex dynamical networks, fuzzy differential equations (FDEs), impulsive control, pinning control, synchronization.**

I. Introduction

An INCREASING number of systems in nature, engineering, or socioeconomic systems can be modeled as networks of interconnected dynamical nodes. Typically, each node of the network is a nonlinear dynamical system interacting with the others via a topology defined on the network edges. Biological neural networks, ecosystems, social groups, the Internet, World Wide Web, or electrical power grids are only some typical examples (see [1]–[4] and references therein).

One of the most interesting and significant phenomena in complex dynamical networks is synchronization among all network's dynamical nodes. In fact, synchronization is one of the most typical collective behavior and basic motions in nature [5], [6]. Specifically, adaptive rules [7]–[10], impulsive methods [11]–[16], or a combination of them [17], [18] have been used for getting synchronization in networks of coupled chaotic systems.

Most of the existing literature considered only complete synchronization when both the network structure and the coupling strength are exactly known. Moreover, the network nodes' dynamics are assumed to be exactly identical, and there is no mismatch between them. However, it should be noted that due to noise and/or uncertainties, the nodes' dynamics do not obey precise state equations, and in most practical situations, only some estimations of them are available [17]. Furthermore, exact prior knowledge about coupling structures and different weightings are usually unavailable [7]. Therefore, a method to completely synchronize the whole network in spite of these uncertainties would be necessary. To deal with unknown network couplings, numerous robust and adaptive methods have been developed [7]–[27], whereas only a few papers addressed the problem of nodes' dynamics uncertainty [10], [14], [15], [26].

All of the above papers only studied networks whose dynamics are described by ordinary differential equations (ODEs). Knowing the fact that, in the mathematical modeling of real-world phenomena, as the complexity of the system being modeled increases, we are either not able to formulate an appropriate mathematical model or the model is too complicated to be useful in practice. Moreover, knowledge about dynamical systems modeled by ODEs is often incomplete or vague [28]–[34]. For example, parameter values, functional relationships, or initial conditions may not be known precisely. It is therefore necessary to have some mathematical apparatus to describe vague notions of existing components of uncertainty. Furthermore, uncertainties might not be of a probabilistic type [28]. In this case, fuzzy differential equations (FDEs) become natural ways to model uncertain dynamical systems. Uncertainty in the nonlinear model of manufacture dynamics [35], biological treatment of waste water [36], dynamics of the liquid level in an oil tank [37], service composition described with business process execution language [38], turbulence dynamics [39], and atmospheric and medical cybernetics [40] were taken into consideration by using FDEs.

On the other hand, a pinning control strategy was proposed to control a complex network by controlling a small fraction of all nodes [41]–[48]. Therefore, the control action is directly exerted into these pinned nodes, and then it propagates to the

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uncontrolled nodes through the connections among the nodes. A main question in pinning control is how many and what type of nodes should be controlled to force the whole network to the desired state [44]–[46]. To deal with this problem, pinning adaptive control and adaptive tuning of the coupling strength have been proposed [47], [48]. However, in spite of a simple structure and wide use of impulsive controllers in synchronization of chaotic systems [49], [50], stabilization of hybrid neural networks [51], and their effectiveness and robustness in synchronization of complex dynamical networks [11]–[18], pinning impulsive controllers are not yet studied for the synchronization of complex dynamical networks. Moreover, to the best of our knowledge, fuzzy complex dynamical networks and impulsive or adaptive control scheme for their synchronization have not also been studied.

In the line of the above discussions, this paper further considers the issue of robust synchronization of complex dynamical networks with different parameter uncertainties, where all sources of parametric uncertainty are modeled with fuzzy numbers. After that, each node's dynamics is described by FDEs that enables us to refine the description of a real process occurring in the presence of inaccuracy, by analyzing an equation or a system of differential equations defined on a space of fuzzy sets. By utilizing the stability results for FDEs and impulsive systems established in [34] and [50], respectively, we derive sufficient conditions for the existence of adaptive and impulsive controllers under which globally exponential synchronization of fuzzy dynamical networks can be guaranteed. In addition, we propose a method for finding certain suitable nodes to be impulsively controlled via pinning, noting that these nodes vary at distinct impulsive time instants. The proposed controller needs small controlling cost and is very useful in practice.

The remaining of this paper is organized as follows. In Section II, some basic definitions and terminology are introduced, and a basic result of [34] for stability of FDEs is recalled. In Section III, the problem statements with necessary assumptions are given. Section IV studies synchronization of fuzzy complex dynamical networks and their exponential convergence based on adaptive–impulsive and adaptive–pinning-impulsive controllers. The effectiveness of the proposed methods is demonstrated through numerical simulations on a directed network and a small-world network in Section V. Conclusions and future directions are finally drawn in Section VI.

II. Basic Concepts

Denote by κ^n the set of all nonempty compact subsets of the n-dimensional Euclidean space R^n and by κ^n_C the subspace of κ^n consisting of nonempty convex compact sets. If $A, B \in \kappa^n$, the Hausdorff metric d_H on κ^n is defined by

$$d_H(A,B) = \max\{\rho(A,B), \rho(B,A)\}$$

with $\rho(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|$, where $\|.\|$ is a norm in linear spaces.

Let D^n denote the set of upper semicontinuous normal fuzzy sets on R^n with compact support. That is, fuzzy set u belongs to D^n if and only if the α-level set $[u]^\alpha$ is a nonempty compact subset of R^n for all $0 \le \alpha \le 1$. The α-level set of u, $0 \le \alpha \le 1$, is defined as $[u]^\alpha = \{x \in R^n | u(x) \ge \alpha\}$. Clearly, for $\alpha \le \beta$, we get $[u]^\alpha \supseteq [u]^\beta$. Element $u \in D^n$ is said to be fuzzy convex if

$$u(\lambda x + (1-\lambda)y) \ge \min\{u(x), u(y)\} \tag{1}$$

for all $x, y \in [u]^0$ and all $\lambda \in [0,1]$. It is easy to verify that if u is fuzzy convex, then each $[u]^\alpha$ is convex. Denote by E^n the subset of fuzzy convex elements of D^n. It is possible to define metric d_∞ on D^n (*or* E^n) as

$$d_\infty(u,v) = \sup_{0\le\alpha\le1} d_H([u]^\alpha,[v]^\alpha), \quad u,v \in D^n(\text{or} \quad E^n). \tag{2}$$

Let 0 be the fuzzy singleton of D^n (or E^n); for each $u \in D^n$ (*or* E^n), write $\|u\| = d_\infty(u,0)$. All of the metric spaces (κ^n, d_H), $(R^n, \|.\|)$, (D^n, d_∞), and (E^n, d_∞) are complete. A more detailed description of these spaces can be found in [52].

Consider the following FDE:

$$\dot{x}(t) = F(t, x(t)), \qquad x(0) = X_0 \in E^n \tag{3}$$

where $F: R_+ \times E^n \to E^n$. There are many attempts to define a fuzzy derivative and then study FDEs. One of the earliest was a generalization of the Hukuhara derivative of a set-valued function [32]. However, soon it appeared that the corresponding solution has a drawback of becoming fuzzier as time goes by. Consequently, this formulation cannot reflect any of the behavior of ODEs such as stability and bifurcation. To resolve the problem, Hüllermeier [28] interpreted FDEs as a family of differential inclusions at each α-level, where $0 \le \alpha \le 1$, as follows:

$$\dot{x}(t) \in [F(t,x(t))]^\alpha, \qquad x_\alpha(0) = x_0 \in X_\alpha := [X_0]^\alpha \tag{4}$$

where $[F(.,.)]^\alpha : R_+ \times R^n \to \kappa^n_C$. Afterward, Diamond [33] proved that the attainable solution sets $A_\alpha(X_\alpha, t)$, $\alpha \in [0,1]$ of the family of inclusions (4) on $[0,t]$ are the level sets of fuzzy set $A(X_0,t) \in D^n$ and then extended existing results of stability and periodicity to time-dependent differential inclusions [34]. The idea was to solve these differential inclusions and using the stacking theorem of Negoita and Ralescu [53] to bunch these solutions into a fuzzy solution.

Stacking Theorem [53]: Let $\{Y_\alpha \subset R^n : 0 \le \alpha \le 1\}$ be a family of compact subsets satisfying the following:

– $Y_\alpha \in \kappa^n$ for all $0 \le \alpha \le 1$;
– $Y_\beta \subseteq Y_\alpha$ for $0 \le \alpha \le \beta \le 1$;
– $Y_\alpha = \bigcap_{i=1}^\infty Y_{\alpha_i}$ for any nondecreasing sequence $\alpha_i \to \alpha$ in [0, 1].

Then, there is fuzzy set $u \in D^n$ such that $[u]^\alpha = Y_\alpha$. In particular, if Y_α is also convex, then $u \in E^n$. Conversely, the level sets $[u]^\alpha$ of any $u \in D^n$ satisfy these conditions, i.e., $[u]^\alpha$'s become also convex if $u \in E^n$.

An impulsive effect is introduced to the FDE (3) as

$$\begin{cases} \dot{x}(t) = F\left(t, x(t)\right), & t \neq t_k \\ x\left(t_k^+\right) = x\left(t_k^-\right) + I\left(x\left(t_k^-\right)\right), & t = t_k \\ x\left(t_0^+\right) = X_0 \in E^n, & k = 1, 2, \ldots, \end{cases} \tag{5}$$

where $I : E^n \to E^n$, $x(t_k^+)$ is the right limit of $x(t)$ at $t = t_k$, $x(t_k^-)$ is the left limit, and for each $x \in E^n$, $\lim\ F(t, y) = F(t_k^+, x)$ exists as $(t, y) \to (t_k^+, x)$. For future references, we need the following definitions and theorems.

Definition 1: Let $V : R_+ \times R^n \to R_+$; then, V is said to belong to class ν_0 if [50]

1) V is continuous in $(t_k, t_{k+1}] \times R^n$, and for each $x \in R^n$, $k = 1, 2, \ldots, \lim_{(t,y)\to(t_k^+,x)} V(t, y) = V(t_k^+, x)$ exists;
2) V satisfies Lipschitz condition $|V(t, y) - V(t, x)| \leq L\|y - x\|$, where $L \geq 0$.

Definition 2: A function $V : R_+ \times K \to R_+$ is a Lyapunov function of the inclusion (4) on $K \subseteq R^n$ if [34]

1) $V(t, x) \geq \varphi(\|x\|)$ on $K \cap (rB^n)$ for some $r > 0$ and some continuous strictly increasing $\varphi : [0, r) \to R_+$. B^n is the unit ball of R^n;
2)

$$\frac{\partial}{\partial t} V(t, x) + \nabla V(t, x) u \leq -W(x) \leq 0 \tag{6}$$

for all $t \geq t_0$, all $x \in K$ and every $u \in [F(t, x)]^\alpha$, where $W : K \to R_+$ is continuous on K. If V is a Lyapunov function for (4) on K, define $E = \{x | W(x) = 0, x \in K\}$.

Definition 3 [50]: For $(t, x) \in (t_k, t_{k+1}] \times R^n$, we define the upper right derivative of Lyapunov function $V(t, x) \in \nu_0$ as

$$D^+ V(t, x) = \lim_{h\to 0}\ sup \frac{1}{h} \left[V\left(t + h, x + hf(t, x)\right) - V(t, x)\right].$$

Definition 4 [34]: If $u \in D^n$ is a fuzzy set and U and $W \subset D^n$ are closed subsets of D^n, we define the distance from W and Hausdorff separation, respectively, by

$$\rho_*(u, W) = \inf_{w\in W} d_\infty(u, w)$$
$$\rho_D(U, W) = \sup_{u\in U} \rho_*(u, W).$$

Lemma 1: Let $A(t) \in R^{n\times n}$ be a positive definite matrix, $B(t) \in R^{n\times n}$ be a symmetric matrix, and $\lambda_{\max} B$ denotes the maximum eigenvalue of matrix B. Then, for any $x \in R^n$ and $t \in R_+$, we have

$$x^T B(t) x \leq \lambda_{\max}\left(A(t)^{-1} B(t)\right) \cdot x^T A(t) x.$$

Recall that any crisp set $X \subset R^n$ is also a fuzzy set, with the membership function being the characteristic function χ_X.

Theorem 1: Let V be a Lyapunov function for (4) on K, and its derivative is bounded above according to (6) almost everywhere on [34, Th. 4.1]. If, for each solution $x(t)$ of (4), $W(x(t))$ is absolutely continuous, then $\rho_D(A(x_0, t), \chi_E) \to 0$ as $t \to \infty$. In particular, if $V = V(x)$ does not explicitly depend on t, then writing $E(c) = E \cap \{x | V(x) \leq c\}$, $\rho_D(A(x_0, t), \chi_{E(c)}) \to 0$ for some $c > 0$ as $t \to \infty$.

III. Problem Formulation

Consider the following uncertain complex dynamical network consisting of N nonidentical coupled nodes (each of them is an n-dimensional dynamical system) with parametric uncertainties as well as nonlinear couplings, which is described by

$$\dot{\underline{x}}_i(t) = (A + \Delta A_i)\underline{x}_i(t) + \varphi(t,_i) + g_i(_1, \ldots,_N) \qquad t \geq 0 \tag{7}$$

where $i = 1, 2, \ldots, N$, $_i = (_{i1},_{i2}, \ldots,_{in})^T \in R^n$ is the state vector of node i, A is an $n \times n$ matrix, ΔA_i are norm-bounded parametric uncertainties, $\varphi : R_+ \times R^n \to R^n$ is a smooth nonlinear vector function, and $g_i : R^{nN} \to R^n$ values are smooth but unknown nonlinear coupling functions.

Now, consider the case that our information about the parameters' uncertainty and/or initial condition are vague, and it is only possible to describe them with fuzzy linguistic variables, e.g., "near zero," "approximately one," or "more than two." Within a fuzzy context, these linguistic variables are usually represented by proper fuzzy numbers and could be considered as a generalization of the crisp case. Therefore, the fuzzy interpretation of (7) leads to the following fuzzy complex dynamical network described by FDEs:

$$\dot{x}_i(t) = \tilde{A}_i x_i(t) + \tilde{\varphi}(t, x_i) + \tilde{g}_i(x_1, \ldots, x_N) \qquad t \geq 0 \tag{8}$$

where fuzzy states $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in E^n$, $i = 1, 2, \ldots, N$, $\tilde{A}_i$ are $n \times n$ matrices with fuzzy number elements, and $\tilde{g}_i$, $\tilde{\varphi}$ satisfy the subsequent assumptions.

Assumption 1: A1: There exist nonnegative constants $r_{ij} \geq 0$, i, and $j = 1, 2, \ldots, N$ such that $\tilde{g}_i : E^{nN} \to E^n$ satisfies the following Lipschitz condition for any x_j, $y_j \in E^n$:

$$d_\infty\left[\tilde{g}_i(x_1, \ldots, x_N), \tilde{g}_i(y_1, \ldots, y_N)\right] \leq \sum_{j=1}^{N} r_{ij} d_\infty(x_j, y_j). \tag{9}$$

Moreover, for any $y \in E^n$, $\tilde{g}_i(y, \ldots, y) = 0$ to guarantee the vanishing of diffusive couplings due to complete synchronization.

Assumption 2: A2: There exist positive constants η_k such that $\tilde{\varphi} : R_+ \times E^n \to E^n$ satisfies the following Lipschitz condition for any $t \in (t_{k-1}, t_k]$ and x, $y \in E^n$:

$$d_\infty\left[\tilde{\varphi}(t, x), \tilde{\varphi}(t, y)\right] \leq \eta_k d_\infty(x, y). \tag{10}$$

It is assumed that when the network achieves synchronization, all the states of all nodes become identical, namely, $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ as $t \to \infty$, where $s(t)$ is the synchronous solution of the isolated node of the system, i.e.,

$$\dot{s}(t) = As(t) + \tilde{\varphi}(t, s) \qquad t \geq 0. \tag{11}$$

Here, $s(t) \in E^n$ is a fuzzy singleton at each t and could be a fixed point, periodic, quasi-periodic, or chaotic orbit, and A is an $n \times n$ matrix with fuzzy singletons as its elements, namely, χ_a, where $a \in R$.

The main objective of this paper is to design appropriate adaptive and impulsive controllers such that the states of the nodes $x_i(t)$, where $i = 1, 2, \ldots, N$, completely synchronize with the state $s(t)$ of the isolated system (11), in spite of unknown parameter uncertainties, i.e.,

$$\lim_{t\to\infty} \|e_i(t)\| = 0 \qquad i = 1, 2, \ldots, N \tag{12}$$

where $e_i(t) = x_i(t) - s(t)$ is the synchronization error.

For the fuzzy complex dynamical network (8), impulsive controllers $I_i(t)$ are designed as

$$I_i(t) = \sum_{k=1}^{\infty} B_{ik} e_i(t)\delta(t - t_k), \qquad i = 1, 2, \ldots, N \tag{13}$$

where B_{ik} are $n \times n$ constant (fuzzy singleton) matrices to be designed later; the impulsive time instants t_k satisfy $0 \le t_0 < t_1 < t_2 < \ldots$, with $\lim_{k\to\infty} t_k = \infty$; and $\delta(\cdot)$ is the Dirac impulse function. Then, the fuzzy adaptive–impulsive controlled network is given by

$$\begin{cases} \dot{x}_i(t) = \tilde{A}_i x_i(t) + \tilde{\varphi}(t, x_i) \\ \qquad + \tilde{g}_i(x_1, \ldots, x_N) + u_i(t) & t \ne t_k \\ x_i\left(t_k^+\right) = x_i\left(t_k^-\right) + B_{ik} e_i\left(t_k^-\right), & t = t_k \\ x_i\left(t_0^+\right) = X_{i0} \in E^n, \quad i = 1, 2, \ldots, N, & k = 1, 2, \ldots \end{cases} \tag{14}$$

where $u_i(t) \in E^n$ are fuzzy adaptive controllers to be designed later. After that, the fuzzy error dynamical systems can be obtained as

$$\begin{cases} \dot{e}_i(t) = Ae_i(t) + \tilde{\tilde{A}}_i x_i(t) + \tilde{\tilde{\varphi}}(t, x_i, s) \\ \qquad + \tilde{\tilde{g}}_i(x, s) + u_i(t), & t \ne t_k \\ e_i\left(t_k^+\right) = e_i\left(t_k^-\right) + B_{ik} e_i\left(t_k^-\right), & t = t_k \\ e_i\left(t_0^+\right) = E_{i0} \in E^n, \quad i = 1, 2, \ldots, N, & k = 1, 2, \ldots \end{cases} \tag{15}$$

where $\tilde{\tilde{A}}_i = \tilde{A}_i - A$, $\tilde{\tilde{g}}_i(x, s) = \tilde{g}_i(x_1, \ldots, x_N) - \tilde{g}_i(s, \ldots, s)$, and $\tilde{\tilde{\varphi}}(t, x_i, s) = \tilde{\varphi}(t, x_i) - \tilde{\varphi}(t, s)$.

As an example, let $(a, b, c)_{\mathrm{tri}}$ be a triangular fuzzy number, whose support is the compact interval $[a, c]$, and the height is equal to 1 at b, i.e., $\mathrm{hgt}(b) = 1$. In E^2, consider $\tilde{A}$ and A as follows:

$$\tilde{A} = \begin{bmatrix} (0.8, 1, 1.2)_{\mathrm{tri}} & (-0.1, 0, 0.2)_{\mathrm{tri}} \\ \chi_0 & (-1.1, -1, -0.9)_{\mathrm{tri}} \end{bmatrix}$$

$$A = \begin{bmatrix} \chi_1 & \chi_0 \\ \chi_0 & \chi_{-1} \end{bmatrix}.$$

The matrix A shows the desired value of the system parameters, whereas, $\tilde{A}$ reflects the uncertainty on the system parameters usually described by triangular fuzzy numbers around the desired (center) value. Therefore, $\tilde{A}_{11}$, $\tilde{A}_{12}$, and $\tilde{A}_{22}$, indicate approximately one, approximately zero, and approximately minus one, respectively. It also shows that there is no uncertainty on A_{21}. We also have

$$\tilde{\tilde{A}} = \tilde{A} - A = \begin{bmatrix} (-0.2, 0, 0.2)_{\mathrm{tri}} & (-0.1, 0, 0.2)_{\mathrm{tri}} \\ \chi_0 & ;(-0.1, 0, 0.1)_{\mathrm{tri}} \end{bmatrix}$$

where $\tilde{\tilde{A}}$ indicates approximately zero matrix.

IV. Synchronization of Fuzzy Complex Dynamical Networks

In this section, we will propose adaptive–impulsive controllers for complete synchronization of the fuzzy complex dynamical network (8). In the first subsection, the impulsive effect will be applied to all nodes, whereas in the second subsection, the pinning impulsive control strategy will be investigated. In other words, only a small portion of nodes is selected to be controlled via the impulsive controller. These controllers will be designed based on the stability results for impulsive systems as established in [50] and the stability of FDEs [34].

The fuzzy adaptive controllers $u_i(t)$ are chosen as follows:

$$u_i(t) = -\hat{A}_i(t) x_i(t), \qquad i = 1, 2, \ldots, N \tag{16}$$

where $\hat{A}_i(t)$ are the estimators of the approximately zero fuzzy matrices $\tilde{\tilde{A}}_i$. The update laws of these fuzzy parameters are chosen as

$$\begin{cases} \hat{A}_i(t) = \left\langle \hat{a}_{jk}^i(t) \right\rangle, & j, k = 1, 2, \ldots, n \\ \dot{\hat{a}}_{jk}^i(t) = k_{ijk} e_{ij}(t) x_{ik}(t), & t \ne t_k \end{cases} \tag{17}$$

where $i = 1, 2, \ldots, N$, $\hat{a}_{jk}^i(t) \in E^1$, and k_{ijk} are positive constants determining the rate of adaptation. Moreover, there is no change in the value of $\hat{a}_{jk}^i(t)$ when $t = t_k$.

A. Adaptive–Impulsive Method

Theorem 2: Suppose that Assumptions A1 and A2 hold, and the adaptive controllers (16) with the update laws (17) along with impulsive controllers (13) are applied to the fuzzy complex dynamical network (8). Then, the controlled dynamical network (14) becomes globally exponentially synchronized with isolated system (11) if there exist constants $\gamma_k > 0$, $\alpha_{ik} \ge 0$, $i = 1, 2, \ldots, N$, and $k = 1, 2, \ldots$, such that the following conditions are satisfied:

1) For all $t \in (t_{k-1}, t_k]$

$$A^T + A + \left(2\eta_k + \sum_{j=1}^{N} r_{ij} + \sum_{j=1}^{N} r_{ji}\right) I_n \le \alpha_{ik} I_n. \tag{18}$$

2) For all $k = 1, 2, \ldots$

$$\alpha_k(t_k - t_{k-1}) + \ln \beta_k \le -\gamma_k \tag{19}$$

where $\beta_k = \max_{1\le i\le N} \lambda_{\max}\{(I_n + B_{ik})^T (I_n + B_{ik})\}$, and $\alpha_k = \max_{1\le i\le N} \alpha_{ik}$.

Proof: Consider the following family of differential inclusions at each α-level, $0 \le \alpha \le 1$ for the fuzzy error dynamical systems (14) and the update laws (17), respectively as follows:

$$\begin{cases} \dot{e}_i(t) \in \Big[Ae_i(t) + \tilde{\tilde{A}}_i x_i(t) + \tilde{\tilde{\varphi}}(t, x_i, s) \\ \qquad + \tilde{\tilde{g}}_i(x, s) + u_i(t)\Big]^{\alpha}, & t \ne t_k \\ e_i\left(t_k^+\right) = e_i\left(t_k^-\right) + B_{ik} e_i\left(t_k^-\right), & t = t_k \\ e_{i\alpha}\left(t_0^+\right) = e_{i0} \in [E_{i0}]^{\alpha}, \quad i = 1, 2, \ldots, N, & k = 1, 2, \ldots \end{cases} \tag{20}$$

$$\dot{\hat{a}}_{jk}^i(t) \in k_{ijk}\left[e_{ij}(t) x_{ik}(t)\right]^{\alpha}, \quad t \ne t_k. \tag{21}$$

Let V be a Lyapunov function for (20) as follows:

$$V_\alpha = \sum_{i=1}^{N} e_i(t)^T e_i(t) + \sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\left(\hat{a}_{jk}^i(t) - a_{jk}^i(\alpha)\right)^2}{k_{ijk}} \tag{22}$$

where $\tilde{\tilde{A}}_i = \langle a_{jk}^i \rangle$, $a_{jk}^i \in E^1$, and $a_{jk}^i(\alpha) \in [a_{jk}^i]^\alpha$.

For $t \neq t_k$, by using the fuzzy adaptive controllers (16) with the update laws (21), we have for each α the following:

$$\begin{aligned} D^+V_\alpha =& \sum_{i=1}^{N}\left(\dot{e}_i^T e_i + e_i^T \dot{e}_i\right) + 2\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\dot{\hat{a}}_{jk}^i \left(\hat{a}_{jk}^i - a_{jk}^i(\alpha)\right)}{k_{ijk}} \\ =& \sum_{i=1}^{N}\Big[e_i^T(A^T + A)e_i + 2e_i^T \tilde{\tilde{\varphi}}^\alpha(t, x_i, s) + 2e_i^T \tilde{\tilde{g}}_i^\alpha(x, s) \\ &+ 2e_i^T\left(\tilde{\tilde{A}}_i^\alpha - \hat{A}_i^\alpha(t)\right)x_i\Big] \\ &+ 2\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{k=1}^{n} e_{ij}x_{ik}\left(\hat{a}_{jk}^i(t) - a_{jk}^i(\alpha)\right). \end{aligned} \tag{23}$$

It can be easily verified that

$$e_i^T\left(\tilde{\tilde{A}}_i^\alpha - \hat{A}_i^\alpha(t)\right)x_i = \sum_{j=1}^{n}\sum_{k=1}^{n} e_{ij}x_{ik}\left(a_{jk}^i(\alpha) - \hat{a}_{jk}^i(t)\right). \tag{24}$$

This leads to

$$D^+V = \sum_{i=1}^{N}\left[e_i^T(A^T + A)e_i + 2e_i^T \tilde{\tilde{\varphi}}^\alpha(t, x_i, s) + 2e_i^T \tilde{\tilde{g}}_i^\alpha(x, s)\right]. \tag{25}$$

Recall that both $(R^n, \|.\|)$ and (E^n, d_∞) are complete metric spaces, and by interpreting FDEs as a family of differential inclusions, space E^n is changed to $\kappa_C^n \subset R^n$. Therefore, the Lipschitz constants in A1 and A2 remain true. Now, by using the inequality $2x^T y \leq x^T x + y^T y$ that holds for any $x, y \in R^n$ and assumption A1, we get

$$\begin{aligned} \sum_{i=1}^{N} 2e_i^T \tilde{\tilde{g}}_i^\alpha(x, s) \leq& \sum_{i=1}^{N} 2\left\|e_i^T\right\| \sum_{j=1}^{N} r_{ij}\|e_j\| \\ \leq& 2\sum_{i=1}^{N}\sum_{j=1}^{N} r_{ij}\|e_i\|\|e_j\| \\ \leq& \sum_{i=1}^{N}\sum_{j=1}^{N} r_{ij}\left(e_i^T e_i + e_j^T e_j\right) \\ =& \sum_{i=1}^{N} e_i^T\left(\sum_{j=1}^{N} r_{ij} + \sum_{j=1}^{N} r_{ji}\right)e_i \end{aligned} \tag{26}$$

and it follows from assumption A2 that

$$2e_i^T \tilde{\tilde{\varphi}}^\alpha(t, x_i, s) \leq 2\eta_k e_i^T e_i. \tag{27}$$

Since the inequalities (26) and (27) hold, by replacing them in (25) and considering the first condition of Theorem 2, we obtain

$$\begin{aligned} D^+V_\alpha \leq& \sum_{i=1}^{N} e_i^T\left(A^T + A + 2\eta_k I_n + \left(\sum_{j=1}^{N} r_{ij} + \sum_{j=1}^{N} r_{ji}\right)I_n\right)e_i \\ \leq& \sum_{i=1}^{N} \alpha_{ik} e_i^T e_i \leq \alpha_k \sum_{i=1}^{N} e_i^T e_i \leq \alpha_k V_\alpha \end{aligned} \tag{28}$$

where $\alpha_k = (\max_{1\leq i\leq N} \alpha_{ik}) \geq 0$. On the other hand, when $t = t_k$, the second term of the Lyapunov function does not change, and we can write

$$\begin{aligned} &V_\alpha\left(t_k^+\right) - V_\alpha\left(t_k^-\right) \\ &= \sum_{I=1}^{N} e_i^T\left(t_k^+\right)e_i\left(t_k^+\right) - \sum_{I=1}^{N} e_i^T\left(t_k^-\right)e_i\left(t_k^-\right) \\ &= \sum_{I=1}^{N} e_i^T(I_n + B_{ik})^T(I_n + B_{ik})e_i - \sum_{I=1}^{N} e_i^T e_i \\ &= \sum_{I=1}^{N} e_i^T\left[(I_n + B_{ik})^T(I_n + B_{ik}) - I_n\right]e_i \end{aligned}$$

where $e_i = e_i(t_k^-)$.

From Lemma 1, we get

$$\begin{aligned} V_\alpha\left(t_k^+\right) - V_\alpha\left(t_k^-\right) \leq& \sum_{i=1}^{N} \lambda_{\max}\left((I_n + B_{ik})^T(I_n + B_{ik}) - I_n\right)e_i^T e_i \\ \leq& \sum_{i=1}^{N}(\beta_k - 1)e_i^T e_i \leq (\beta_k - 1)V_\alpha\left(t_k^-\right) \end{aligned} \tag{29}$$

where $\beta_k = \max_{1\leq i\leq N} \lambda_{\max}((I_n + B_{ik})^T(I_n + B_{ik}))$. We can also rewrite inequality (29) as $V_\alpha(t_k^+) \leq \beta_k V_\alpha(t_k^-)$.

Now, the globally exponential stability of the error dynamical system (20) can be obtained. By using (28), one can easily show that

$$V_\alpha(t) \leq V_\alpha\left(t_{k-1}^+\right)\exp\left(\alpha_k(t - t_{k-1})\right) \quad t \in (t_{k-1}, t_k]. \tag{30}$$

Then, for $t = t_1^+$, we have

$$V_\alpha\left(t_1^+\right) \leq \beta_1 V_\alpha(t_1) \leq \beta_1 \exp\left(\alpha_1(t_1 - t_0)\right)V_\alpha(t_0).$$

Similarly, $t = t_2^+$ yields that

$$\begin{aligned} V_\alpha\left(t_2^+\right) \leq& \beta_2 V_\alpha(t_2) \leq \beta_2 \exp\left(\alpha_2(t_2 - t_1)\right)V_\alpha\left(t_1^+\right) \\ \leq& \beta_1\beta_2 \exp\left(\alpha_1(t_1 - t_0) + \alpha_2(t_2 - t_1)\right)V_\alpha(t_0). \end{aligned}$$

In general

$$\begin{aligned} V_\alpha\left(t_k^+\right) \leq& \prod_{i=1}^{k} \beta_i \exp\left(\sum_{i=1}^{k} \alpha_i(t_i - t_{i-1})\right)V_\alpha(t_0) \\ =& \exp\left(\sum_{i=1}^{k} \ln\beta_i + \alpha_i(t_i - t_{i-1})\right)V_\alpha(t_0). \end{aligned}$$

Now, according to the second condition of Theorem 2, we get

$$V_\alpha\left(t_k^+\right) \leq \exp\left(-\sum_{i=1}^{k}\gamma_i\right) V_\alpha(t_0). \tag{31}$$

Consequently, if $\gamma_k > 0$, where $k = 1, 2, \ldots$, then the fuzzy attainable set $A(E_{i0}, t)$ converges to χ_0 according to Theorem 1, i.e., $\rho_D(A(E_{i0}, t), \chi_0) \to 0$as $t \to \infty$, which means that $\|e_i(t)\| \to 0$ as $t \to \infty$, and the globally exponential stability of the fuzzy error dynamical system is then proved. It means that the fuzzy complex dynamical network (8) is exponentially synchronized with the isolated system (11) under the impulsive controllers I_i and the adaptive controllers u_i. The proof is hence completed.

B. Adaptive-Pining Impulsive Method

In the previous section, when impulsive controllers are designed for the synchronization of fuzzy dynamical networks, all of the nodes should be controlled, leading to a high controlling cost. To resolve this problem, we will extend the so-called pinning control strategy to the impulsive control method. Therefore, the impulsive effect is directly exerted only into a small fraction of all nodes, and a successful synchronization of the whole network will be achieved.

Consider the following impulsive controllers:

$$I_i(t) = \sum_{k=1}^{\infty} b_k e_i(t)\delta(t - t_k), \quad i = 1, 2, \ldots, l \tag{32}$$

where the constants $b_k \in (-2, -1) \cup (-1, 0)$, which means that the impulsive effects are stabilizing for the stabilization of error dynamical networks; the set of l controlled nodes is selected at each time instant $t = t_k^-$ by sorting all synchronization errors in a descending order and by choosing the first l nodes that have higher norm values, i.e., $\|e_1(t)\| \geq \|e_2(t)\| \geq \cdots \geq \|e_l(t)\| \geq \|e_{l+1}(t)\| \geq \cdots \geq \|e_N(t)\|$.

After adding the pinning impulsive controllers (32) to the fuzzy dynamical network (8), the fuzzy error dynamical network can be rewritten as (33), which is shown at the bottom of the page.

Theorem 3: Suppose that Assumptions A1 and A2 hold, and the adaptive controllers (16) with the update laws (17) along with the pinning impulsive controllers (32) are applied to the fuzzy complex dynamical network (8). Then, the fuzzy error dynamical network (33) is globally exponentially stable if there exist constants $\gamma_k > 0$, $\alpha_k \geq 0$, and $0 < \beta_k < 1$, where $k = 1, 2, \ldots$, such that the following conditions are met:

1) For all $t \in (t_{k-1}, t_k]$

$$A^T + A + \left(2\eta_k + \sum_{j=1}^{N} r_{ij} + \sum_{j=1}^{N} r_{ji}\right) I_n \leq \alpha_{ik} I_n. \tag{34}$$

2) For all $k = 1, 2, \ldots$

$$\alpha_k(t_k - t_{k-1}) + \ln\beta_k \leq -\gamma_k. \tag{35}$$

3) At each time instant $t = t_k^-$

$$\beta_k > (1 + b_k)^2 \tag{36}$$

$$\sum_{i=1}^{l_k} \|e_i(t)\|^2 \geq \frac{1 - \beta_k}{1 - (1 + b_k)^2} \sum_{i=1}^{N} \|e_i(t)\|^2 \tag{37}$$

where $\alpha_k = \max_{1 \leq i \leq N} \alpha_{ik}$.

Proof: By choosing the same Lyapunov function as the one discussed in the proof of Theorem 2 and following a similar proof approach when $t \neq t_k$, we get

$$D^+V \leq \alpha_k V, \alpha_k = \max_{1 \leq i \leq N} \alpha_{ik} \tag{38}$$

where α_{ik} satisfies condition (34).

Now, we want to prove that by selecting the appropriate pinning nodes according to the norm descending order, the following inequality is always guaranteed at the instant $t = t_k$:

$$V\left(t_k^+\right) \quad \leq \beta_k V(t_k), \quad 0 < \beta_k < 1. \tag{39}$$

When $t = t_k$, the second term of the Lyapunov function does not change, and we can write

$$V\left(t_k^+\right) - V\left(t_k^-\right) = \sum_{i=1}^{l}(1 + b_k)^2 e_i^T e_i + \sum_{i=l+1}^{N} e_i^T e_i - \sum_{i=1}^{N} e_i^T e_i \tag{40}$$

where $e_i = e_i(t_k^-)$. Since $0 < \beta_k < 1$ and $\beta_k > (1 + b_k)^2$, inequality (37) becomes

$$\left(1 - (1 + b_k)^2\right) \sum_{i=1}^{l} \|e_i\|^2 \geq (1 - \beta_k) \sum_{i=1}^{N} \|e_i\|^2 \tag{41}$$

and after some simplifications, we have

$$(1 + b_k)^2 \sum_{i=1}^{l} \|e_i\|^2 + \sum_{i=l+1}^{N} \|e_i\|^2 \leq \beta_k \sum_{i=1}^{N} \|e_i\|^2 \tag{42}$$

$$\begin{cases} \dot{e}_i(t) = Ae_i(t) + \tilde{\tilde{A}}_i x_i(t) + \tilde{\tilde{\varphi}}(t, x_i, s) + \tilde{\tilde{g}}_i(x, s) + u_i(t), & t \neq t_k, i = 1, 2, \ldots, N \\ e_i\left(t_k^+\right) = e_i\left(t_k^-\right) + b_k e_i\left(t_k^-\right), & t = t_k, i = 1, 2, \ldots, l \\ e_i\left(t_k^+\right) = e_i\left(t_k^-\right), & t = t_k, i = l + 1, \ldots, N \\ e_i\left(t_0^+\right) = E_{i0} \in E^n, & k = 1, 2, \ldots \end{cases} \tag{33}$$

which can be rewritten as

$$V\left(t_k^+\right)-V\left(t_k^-\right)\leq(\beta_k-1)\sum_{i=1}^{N}\|e_i\|^2\leq(\beta_k-1)V\left(t_k^-\right) \quad (43)$$

and this is exactly inequality (39).

After that, the globally exponential stability of the fuzzy error dynamical system (33) can be obtained by doing the same approach discussed in the proof of Theorem 2. This completes the proof.

Remark 1: The inequality $\beta_k > (1+b_k)^2$ along with the norm descending order of synchronization errors ensures that the minimum number of nodes is selected at each time instant for the pinning control.

Remark 2: By an appropriate selection of b_k, β_k, and Δt_k, one can tune the convergence rate or the number of pinned nodes.

Remark 3: Condition (37) can be approved in two different ways: 1) fixing the number of pinned nodes l_k and finding the appropriate impulses strength b_k; or 2) fixing the value of impulses strength b_k and finding the appropriate number of pinned nodes l_k at each time instant. Now, we are in the position to give an algorithm for the practical implementation of adaptive–pinning impulsive control to synchronize the complex dynamical networks. The algorithm is presented as follows and should be done at each time instant $t = t_k^-$, where $k = 1, 2, \ldots, k_s$:

1) Calculate parameter α_k.
2) Select the suitable impulsive strengths b_k and $0 < \beta_k < 1$, and $\Delta t_k = t_k - t_{k-1}$ such that (35) and (36) are satisfied.
3) Sort norm values of all synchronization errors in the descending order.
4) If the inequality (37) is satisfied, then go to step (6); else, go to the next step. The initial value for the number of pinned nodes is $l_k = 1$.
5) Increase the number of pinned nodes by one unit, i. e., $l_k = l_k + 1$, and go back to step (4).
6) Apply the impulsive controller to the l_k pinned nodes found in step (4).
7) (Stopping criterion) Repeat the above steps for the next impulsive time instant until the norm of all synchronization errors satisfy $\|e_i(t_{k_s})\| \leq E$, with $i = 1, 2, \ldots, N$, where E determines the closeness of errors to zero, and k_s is the required impulsive time instants for the synchronization of the whole network.

V. Simulation Results

In this section, the effectiveness of the proposed method is demonstrated by two examples, in which the node dynamics is represented by a chaotic Chua circuit, and the network has a directed nearest neighbor and small-world structure. The Chua's circuit can be easily implemented by simple electronic circuits and hence widely used in complex dynamical networks analysis [3], [7], [11]–[14]. The numerical simulations are done in MATLAB environment.

Example 5.1: Consider the complex network (8) consisting of $N = 10$ coupled nonidentical Chua's chaotic circuits. The equation governing each node is

$$\dot{x}_i = (A + \tilde{\tilde{A}}_i)x_i + \tilde{\varphi}(x_i) + \tilde{g}_i(x_1, x_2, \ldots, x_N) \quad (44)$$

where $i = 1, 2, \ldots, N$, $x_i = (x_{i1}, x_{i2}, x_{i3})^T$, and the matrices $A, \tilde{\tilde{A}}_i$, and the function $\tilde{\varphi}$ are

$$A = \begin{bmatrix} \chi_0 & \chi_{-p} & \chi_0 \\ \chi_1 & \chi_{-1} & \chi_1 \\ \chi_0 & \chi_q & \chi_0 \end{bmatrix}$$

$$\tilde{\tilde{A}}_i = \begin{bmatrix} \chi_0 & \chi_0 & \chi_0 \\ \chi_0 & \chi_0 & \chi_0 \\ \chi_0 & (-0.2i, 0, 0.4i)_{\text{tri}} & \chi_0 \end{bmatrix}$$

$$\tilde{\varphi}(x_i) = \begin{bmatrix} \chi_{-p} f(x_{i1}) \\ \chi_0 \\ \chi_0 \end{bmatrix}$$

where $f(x_{i1}) = \chi_{m_0} x_{i1} + \chi_{0.5(m_1 - m_0)}(|x_{i1} + \chi_1| - |x_{i1} - \chi_1|)$. To ensure the chaotic behavior of the system, the parameters taken from [54] are chosen as $p = 0.59/0.12$, $q = 0.59/0.162$, $m_0 = -0.07$, and $m_1 = 1.5$.

In this example, there is only a parameter uncertainty on q; hence, only $\tilde{\tilde{a}}^i_{3,2}$ is modeled by a triangular fuzzy number around zero, but the other elements of $\tilde{\tilde{A}}_i$ are χ_0. The triangular fuzzy number $(-0.2i, 0, 0.4i)_{\text{tri}}$ shows that the uncertainty of the parameter q is increased by changing i from 1 to 10. Moreover, the interaction is considered as follows:

$$\tilde{g}_i(x_1, \ldots, x_N) = \begin{bmatrix} \chi_1 & \chi_{-2} & \chi_1 \\ \chi_0 & \chi_0 & \chi_0 \\ \chi_{-1} & \chi_2 & \chi_{-1} \end{bmatrix} \begin{pmatrix} x_{i1} \\ x_{i+1,1} \\ x_{i+2,1} \end{pmatrix} \quad (45)$$

where $i = 1, 2, \ldots, N-2$, and $\tilde{g}_{N-1}(x) = \tilde{g}_N(x) = 0$. One can write (45) in the following form:

$$\tilde{g}_i(x_1, \ldots, x_N) = \sum_{j=1}^{N} c_{ij}\Gamma x_j, \quad i = 1, 2, \ldots, N-2$$

where $c_{ii} = c_{i,i+2} = \chi_1$ and $c_{i,i+1} = \chi_{-2}$; otherwise, $c_{ij} = \chi_0$. Matrix Γ is

$$\Gamma = \begin{bmatrix} \chi_1 & \chi_0 & \chi_0 \\ \chi_0 & \chi_0 & \chi_0 \\ \chi_{-1} & \chi_0 & \chi_0 \end{bmatrix}.$$

Furthermore, we obtain $r_{ij} = \|\Gamma\| \times \|c_{ij}\| = \sqrt{2}\|c_{ij}\|$, $\max_i(\sum_{j=1}^{N} r_{ji}) = 4\sqrt{2}$, and $\sum_{j=1}^{N} r_{ij} = 4\sqrt{2}$. It is also easy to verify that the function $\tilde{\tilde{\varphi}}$ satisfies assumption A2 with $\eta_k = |-pm_0| = 0.34$.

All of the existing studies mentioned in the introduction are not able to completely synchronize the network even if a little

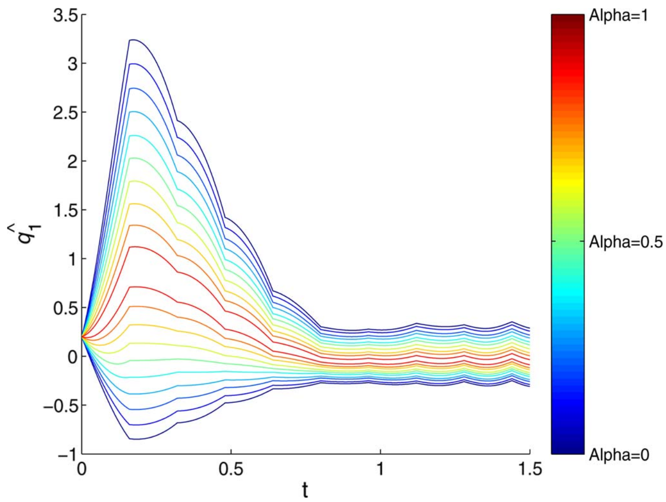

Fig. 1. Trajectory of $\hat{q}_1$ in the adaptive controller (46) with the update laws (47) at each α-level, where $0 \leq \alpha \leq 1$.

amount of uncertainty exists on the dynamics of each node. Now, according to Theorem 2, for $N = 10$, we get

$$A^T + A + \left(2\eta_k + \sum_{j=1}^{N} r_{ij} + \sum_{j=1}^{N} r_{ji}\right) I_3 \leq \alpha_k I_3$$

where $\alpha_k = \lambda_{\max}(A^T + A) + 0.68 + 4\sqrt{2} + 4\sqrt{2} = 17.15$ for each α-level, where $0 \leq \alpha \leq 1$. By considering the following impulsive controllers:

$$t_0 = 0, \quad t_k - t_{k-1} = 0.16, k = 1, 2, \ldots$$
$$B_{ik} = \text{diag}(-0.9051, -0.9051, -0.9051)$$

and using condition 2 of Theorem 2, $\beta_k = 0.009$, and the inequality $\alpha_k(t_k - t_{k-1}) + \ln \beta_k \leq -\gamma_k$ will be fulfilled by setting $\gamma_k = 1.9$.

The adaptive controllers and the update laws are designed as

$$u_i = -\begin{bmatrix} \chi_0 & \chi_0 & \chi_0 \\ \chi_0 & \chi_0 & \chi_0 \\ \chi_0 & \hat{q}_i(t) & \chi_0 \end{bmatrix} x_i(t), \quad i = 1, 2, \ldots, N \quad (46)$$

$$\dot{\hat{q}}_i(t) = 50 e_{i3}(t) x_{i2}(t). \quad (47)$$

The initial values for X_{i0}, $s_0 \in E^3$, and $\hat{q}_{i0} \in E^1$ are chosen as

$$X_{i0} = (i - 0.2, i, i + 0.2)_{\text{tri}} [\chi_1 \quad \chi_1 \quad \chi_1]^T$$
$$s_0 = [\chi_{0.3} \quad \chi_{0.6} \quad \chi_1]^T$$
$$\hat{q}_{i0} = \chi_{0.2}, \qquad i = 1, \ldots, N$$

and then, $E_{i0} \in E^3$ becomes

$$E_{i0} = \begin{bmatrix} (i - 0.5, i - 0.3, i - 0.1)_{\text{tri}} \\ (i - 0.8, i - 0.6, i - 0.4)_{\text{tri}} \\ (i - 1.2, i - 1, i - 0.8)_{\text{tri}} \end{bmatrix}^T.$$

It is illustrated in Fig. 1 that the trajectory of $\hat{q}_1$ does converge to the triangular fuzzy number $(-0.2, 0, 0.4)_{\text{tri}}$. Moreover, one can observe in Fig. 2 that a combination of adaptive and impulsive controllers efficiently synchronizes the network in spite of the parameter uncertainties since all trajectories of the error system exponentially approach zero.

Remark 4: Example 5.1, without considering any parametric uncertainties, was discussed in [7], [11], and [14], where it takes approximately seven impulsive time instants to achieve synchronization. In our example, the uncertain complex fuzzy dynamical networks are synchronized within the same time and the α-levels, where $0 \leq \alpha \leq 1$, of uncertainty in the synchronization errors are also displayed.

Example 5.2: Consider again the fuzzy complex dynamical network in Example 5.1 with $N = 30$ as follows:

$$\dot{x}_i = (A + \tilde{\tilde{A}}_i) x_i + \tilde{\varphi}(x_i) + \tilde{g}_i(x_1, x_2, \ldots, x_N), \quad i = 1, 2, \ldots, N \quad (48)$$

but the matrix $\tilde{\tilde{A}}_i$ and the interaction $\tilde{g}_i(x)$ are

$$\tilde{\tilde{A}}_i = \begin{bmatrix} \chi_0 & \chi_0 & \chi_0 \\ \chi_0 & \chi_0 & \chi_0 \\ \chi_0 & (-0.2 - 0.04i, 0, 0.4 + 0.08i)_{\text{tri}} & \chi_0 \end{bmatrix}$$

$$\tilde{g}_i(x_1, \ldots, x_N) = \sum_{j=1}^{N} c_{ij} \Gamma x_j$$

where $C = (c_{ij})_{30 \times 30}$ is the coupling configuration matrix representing the topological structure of the network, and the elements c_{ij} are defined as follows. If there exists a connection between nodes j and i $(i \neq j)$, then $c_{ij} = c_{ji} > 0$; otherwise, $c_{ij} = c_{ji} = 0$, and the diagonal elements are defined by $c_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij}$, which ensures the diffusivity condition $\sum_{j=1}^{N} c_{ij} = 0$.

In this example, C is the coupling matrix of a small-world network generated by using the algorithm in [5]. The initial degree of the nodes and the rewiring probability of the edges are chosen as $k = 4$ and $p = 0.3$, respectively. Here, we apply the adaptive–pinning impulsive method according to Theorem 3 with the impulsive constants $b_k = -0.9$ and $\beta_k = 0.1$, and the impulse intervals as $t_0 = 0$ and $t_k - t_{k-1} = 0.04$, where $k = 1, 2, \ldots$ and then try to find an appropriate number of pinned nodes l_k at each time instant according to the algorithm given in Remark 3. The adaptive controllers and the update laws are the same as in the previous example.

The initial values for X_{i0}, E_{i0}, $s_0 \in E^3$, and $\hat{q}_{i0} \in E^1$ are chosen as

$$X_{i0} = (0.5i + 0.3, 0.5i + 0.5, 0.5i + 0.7)_{\text{tri}} [\chi_1 \quad \chi_1 \quad \chi_1]^T$$
$$s_0 = [\chi_{0.3} \quad \chi_{0.6} \quad \chi_1]^T$$
$$E_{i0} = \begin{bmatrix} (0.5i, 0.5i + 0.2, 0.5i + 0.4)_{\text{tri}} \\ (0.5i - 0.3, 0.5i - 0.1, 0.5i + 0.1)_{\text{tri}} \\ (0.5i - 0.7, 0.5i - 0.5, 0.5i - 0.3)_{\text{tri}} \end{bmatrix}^T$$
$$\hat{q}_{i0} = \chi_{0.2}, \qquad i = 1, \ldots, N.$$

At the first time instant, the parameter $\alpha_k = 5.83$ is numerically computed, and the inequality $\alpha_k(t_k - t_{k-1}) + \ln \beta_k \leq$

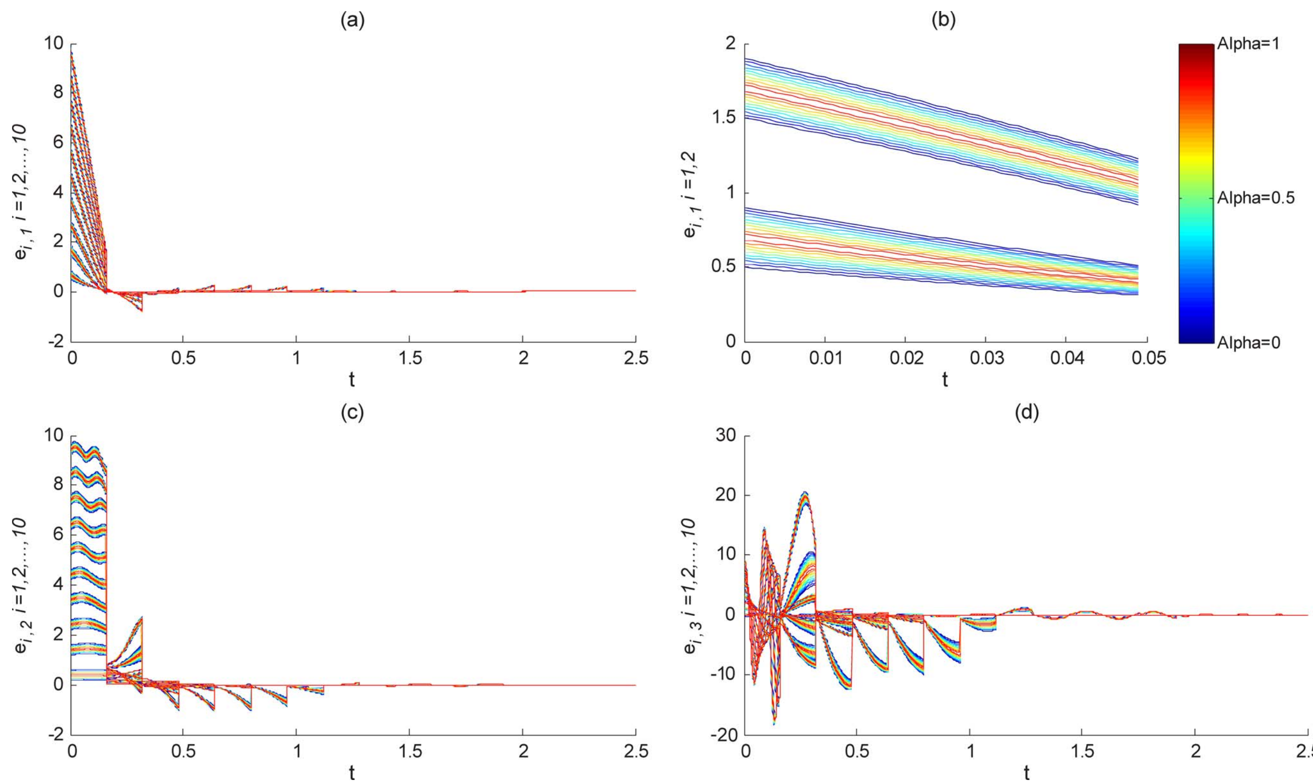

Fig. 2. Synchronization errors of fuzzy complex dynamical network (44) at each α-level, where $0 \leq \alpha \leq 1$, by adaptive–impulsive method. (a) $e_{i1}(t)$, where $i = 1, \ldots, 10$. (b) Closer look at $e_{i1}(t)$, where $i = 1, 2$. (c) $e_{i2}(t)$, where $i = 1, \ldots, 10$. (d) $e_{i3}(t)$, where $i = 1, \ldots, 10$.

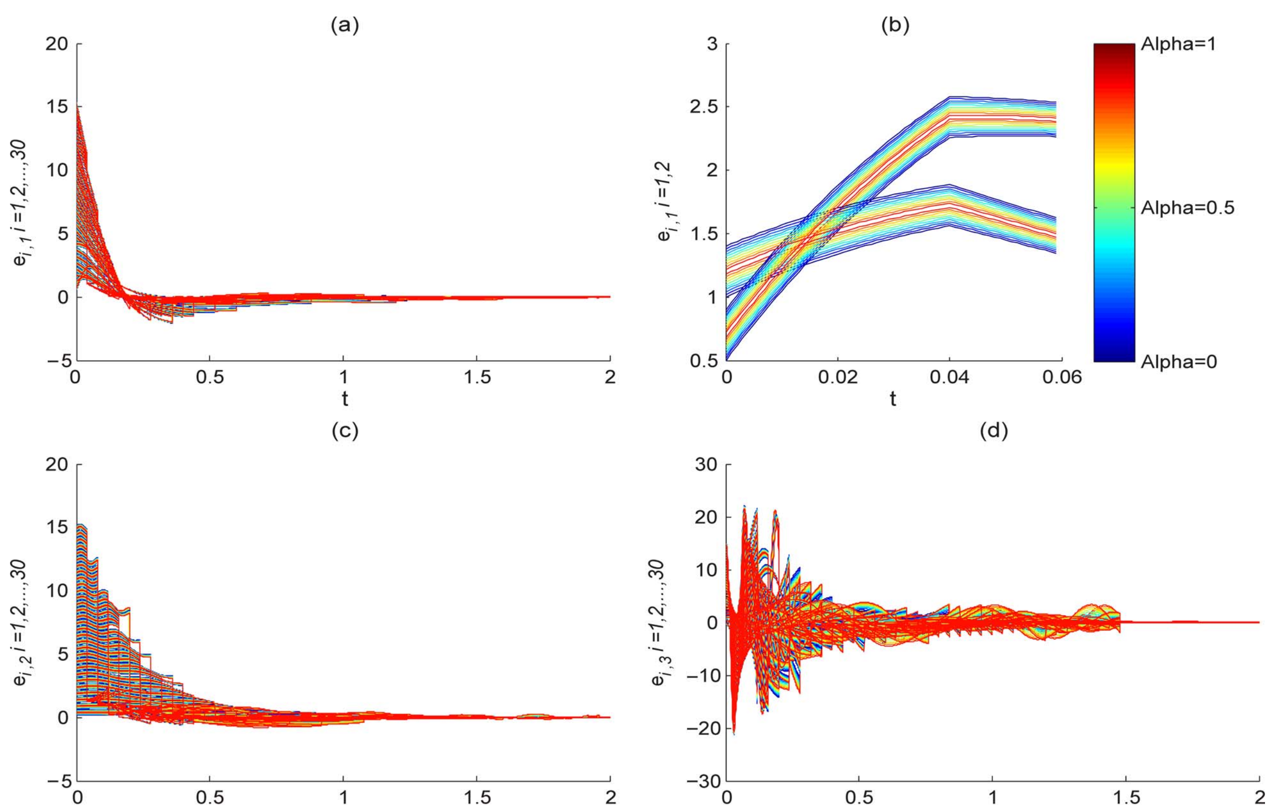

Fig. 3. Synchronization errors of fuzzy complex dynamical network (48) at each α-level, where $0 \leq \alpha \leq 1$ by adaptive–pinning impulsive method with $\beta = 0.1$. (a) $e_{i1}(t)$, where $i = 1, \ldots, 30$. (b) Closer look at $e_{i1}(t)$, where $i = 1, 2$. (c) $e_{i2}(t)$, where $i = 1, \ldots, 30$. (d) $e_{i3}(t)$, where $i = 1, \ldots, 30$.

$-\gamma_k$ will be followed by setting $\gamma_k = 2$. After sorting all synchronization errors, the nodes of 27, 28, 26, 29, and 30 are selected for pinning at the first time instant, i.e., $l_1 = 5$. The above procedure is repeated until the norm of all synchronization errors satisfies $\|e_i\| \leq 0.01$, where $i = 1, \ldots, N$. In this simulation, the required impulsive time instants for the

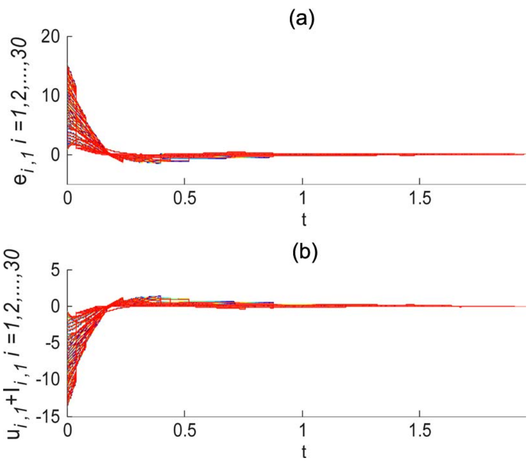

Fig. 4. (a) Synchronization errors $e_{i1}(t)$ and (b) control inputs $u_{i1}(t) + I_{i1}(t), i = 1, \ldots, 30$, for the first state of fuzzy complex dynamical network (48) at each α-level, where $0 \leq \alpha \leq 1$ by adaptive–pinning impulsive method with $\beta = 0.05$.

synchronization of the whole network k_s is equal to 50, the average number of pinned nodes at each impulsive time instant $1/k_s \sum_k \; l_k$ is equal to 4.8, and each node is pinned eight times in average until the occurrence of synchronization, i.e., $1/N \sum_k \; l_k = 8$. Whereas, in the conventional impulsive controllers [11]–[18], all nodes should be pinned at each time instant; therefore, we have $1/k_s \sum_k \; l_k = N$, and $1/N \sum_k \; l_k = k_s$.

The synchronization errors of all nodes in the network are shown in Fig. 3. It is interesting to note that, in Fig. 3(b), the impulsive controllers are not applied on nodes 1 and 2 at the first time instant, but they are affected implicitly via the interconnections of the network; hence, their errors are reduced at $t = 0.04$.

For justification of the statement given in Remark 2, the value of β_k is decreased from 0.1 to 0.05. Therefore, we have expected an increment in the value of l_k at each time instant since the right-hand side of (37) increases. The other parameters are set as before, i.e., $b_k = -0.9$, $t_0 = 0$, and $t_k - t_{k-1} = 0.04$, where $k = 1, 2, \ldots$, leading to $\gamma_k = 2.7$ instead of 2 in the previous case. Knowing the fact that γ_k determines the exponential rate of convergence, from the proof of theorem 2, we have expected less time for the occurrence of synchronization. Simulation results are shown in Fig. 4 that illustrates the first state synchronization errors of all nodes and the trajectory of the control inputs for this state. In this case, $k_s = 25$, and $1/k_s \sum_k \; l_k = 5$.

VI. Conclusion and Future Work

Synchronization of fuzzy complex dynamical networks has been studied in this paper. We have modeled all sources of parametric uncertainty in a typical complex dynamical network with fuzzy numbers; therefore, all node's dynamics are described by FDEs. After that, by using stability results for FDEs and impulsive systems, we design effective adaptive–impulsive controllers that guarantee global exponential synchronization of the fuzzy network. Moreover, adaptive–pinning impulsive controllers are designed in which impulsive effects are only applied to a small fraction of nodes. The latter controllers are more useful in practice because they need less controlling costs. Two numerical examples are given to demonstrate the effectiveness of the proposed controllers.

Although this paper is promising, there are additional sources of uncertainty that incorporate into our proposed method, such as disturbances. Consideration of both parametric and disturbance uncertainties would be an interesting future research work. Moreover, we only employed type-1 fuzzy set theory that requires the developer to describe the membership values by crisp numbers. Therefore, by taking into account the imprecision of membership functions, we may think of using type-2 fuzzy sets. Finally, what we have done in this paper is a binary stability study of fuzzy systems using Lyapunov analysis. In contrast, human-generated statements would involve degrees of stability that are articulated linguistically by terms such as “weakly stable,” “more or less stable,” and “strongly stable,” each of them being described by some appropriate fuzzy numbers [37]. We hope that this paper provides a new framework for future research on the fuzzy stability of fuzzy systems.

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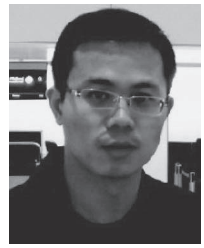

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